Characterising Choiceless Polynomial Time with First-Order Interpretations

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What is Choiceless Polynomial Time?

The most important problem of Finite Model Theory

Is there a logic that captures PTIME?

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Informal definition: A logic L captures PTIME if it defines precisely those properties of finite structures that are decidable in polynomial time:

1. For every sentence $\psi \in L$, the set of finite models of ψ is decidable in polynomial time.
2. For every PTIME-property S of finite structures, there is a sentence $\psi \in L$ such that $S = \{\mathfrak{A} \in \mathrm{Fin} : \mathfrak{A} \models \psi\}$.

The most important problem of Finite Model Theory

Is there a logic that captures $\textsc{Ptime}$?

Informal definition: A logic L captures $\textsc{Ptime}$ if it defines precisely those properties of finite structures that are decidable in polynomial time:

1. For every sentence $\psi \in L$, the set of finite models of ψ is decidable in polynomial time.
2. For every $\textsc{Ptime}$-property S of finite structures, there is a sentence $\psi \in L$ such that $S = \{\mathfrak{A} \in \mathsf{Fin} : \mathfrak{A} \models \psi\}$.

If there is no logic capturing $\textsc{Ptime}$, then $P \neq NP$.

What is Choiceless Polynomial Time?

- Candidate for a logic capturing PTIME

Candidates

PTIME

$\cup\!\!\!\!\not\;$

FPC

$\cup\!\!\!\!\not\;$

FP

Candidates

PTIME

$\subsetneq$

FPC

$\subsetneq$

FP captures PTIME on ordered structures

Candidates

PTIME

$\subsetneq$

FPC captures PTIME on many interesting classes of structures

$\subsetneq$

FP captures PTIME on ordered structures

Candidates

$\mathrm{P}\textsc{time}$

$\subseteq \qquad \supseteq$

$\mathrm{FP} + \mathsf{rk} \qquad \mathrm{CPT} + \mathrm{C}$

$\supsetneq \qquad \subsetneq$

FPC captures $\mathrm{P}\textsc{time}$ on many interesting classes of structures

$\subsetneq$

FP captures $\mathrm{P}\textsc{time}$ on ordered structures

Candidates

$\textsc{Ptime}$

$\subseteq \quad \subseteq$

$\mathrm{FP} + \mathsf{rk}$ $\quad$ $\mathrm{CPT} + \mathrm{C}$ $\quad$ captures $\textsc{Ptime}$ on even more classes

$\subsetneq \quad \subsetneq$

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FP $\quad$ captures $\textsc{Ptime}$ on ordered structures

Candidates

PTIME

$=?$

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$\not\supseteq$ $\quad$ $\not\subseteq$

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$\subsetneq$

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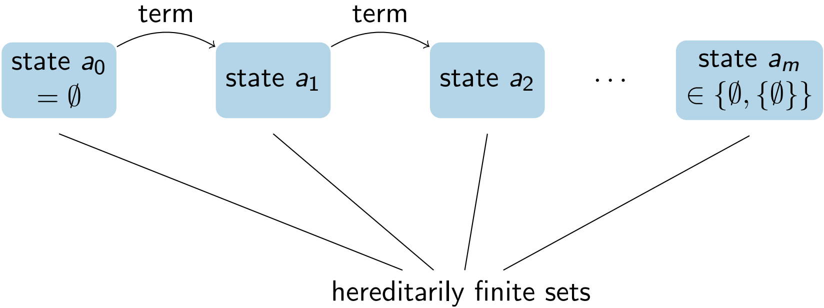
What is Choiceless Polynomial Time?

- Candidate for a logic capturing PTIME
- Iterated creation of sets

Choiceless Polynomial Time

Idea: Set-theoretic "computation" over input structure



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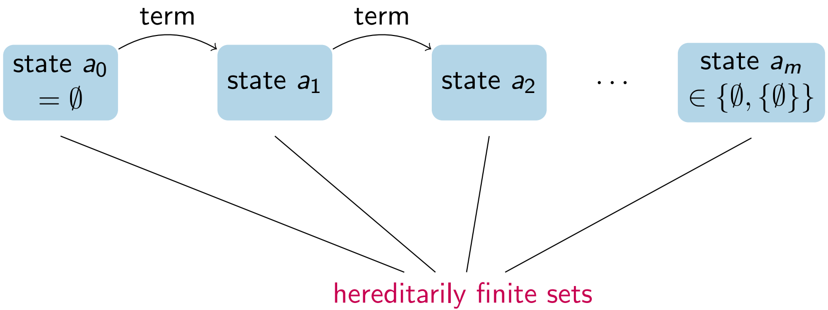
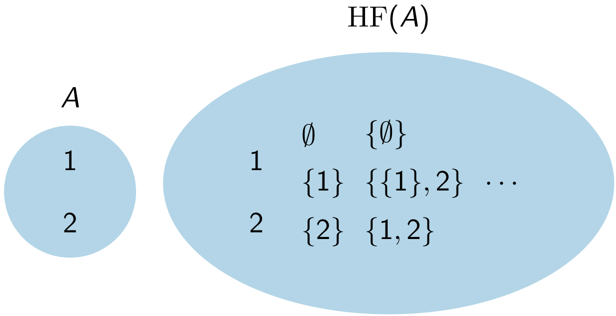


Hereditarily Finite Sets



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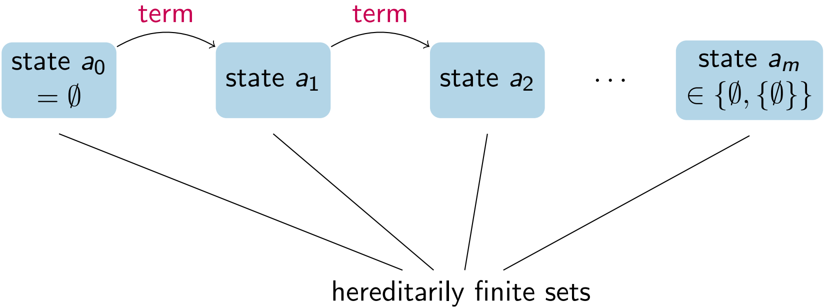

Terms

$$\{\{y : y \in \mathrm{Atoms} : Exy\} : x \in \mathrm{Atoms} : Px\}$$

$$\mathrm{Union}(\mathrm{Pair}(x, y))$$

- Constants $\mathrm{Atoms}, \emptyset$
- Relations from the input structure
- Element relation $\in$
- Comprehension terms
- Functions $\mathrm{Union}, \mathrm{TheUnique}, \mathrm{Pair}$
- Boolean connectives, equality

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Choiceless Polynomial Time

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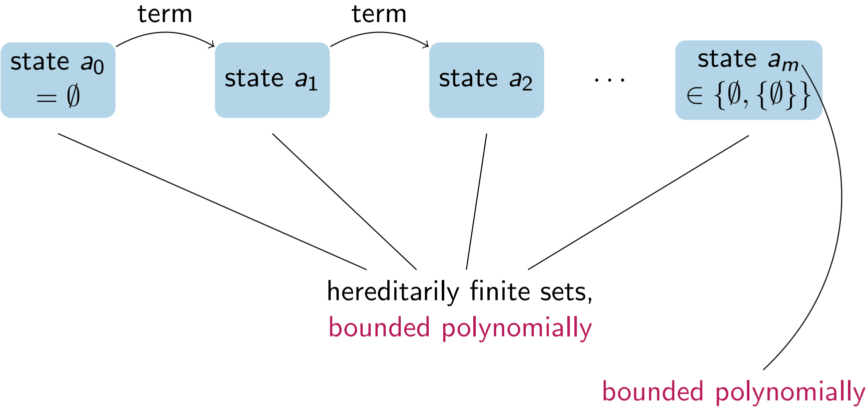

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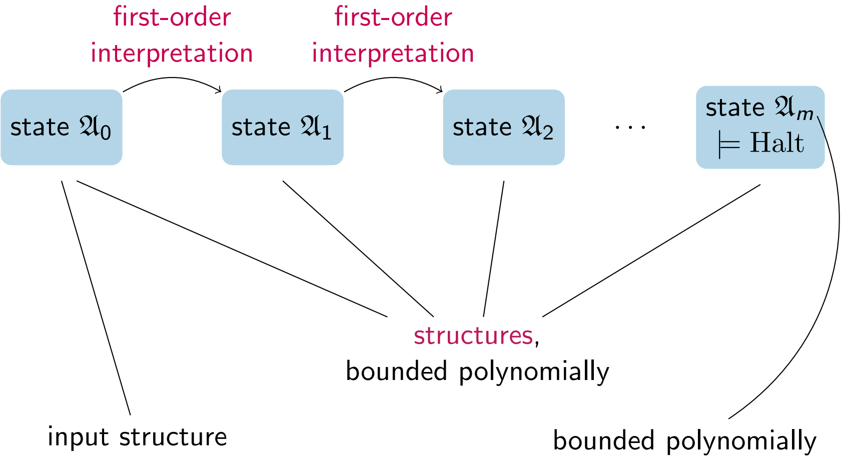
- Iterated creation of sets
- Iterated first-order interpretations

Polynomial Time Interpretation Logic



First-Order Interpretations

$$\mathcal{I} = (\varphi_{\mathsf{dom}}, \varphi_{\approx}, (\varphi_R)_{R \in \sigma})$$

$$(A, \tau) \xrightarrow{\mathcal{I}} (B, \sigma)$$

First-Order Interpretations

$$\mathcal{I} = (\varphi_{\mathsf{dom}}, \varphi_{\approx}, (\varphi_R)_{R \in \sigma})$$

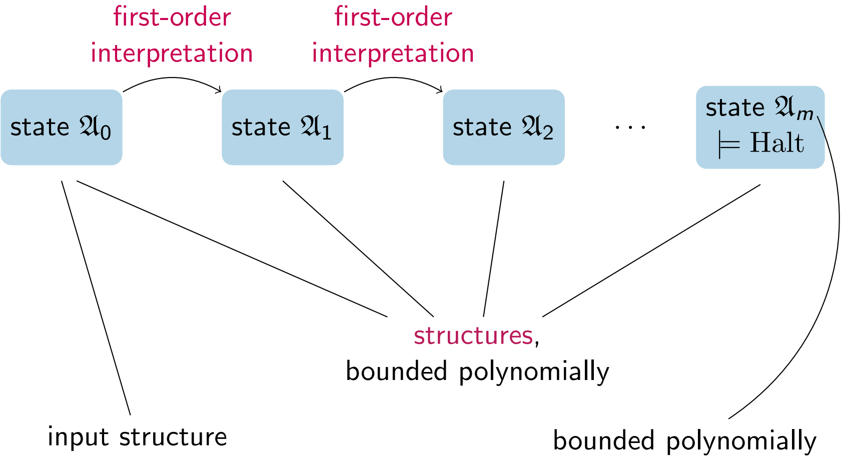
$$(\mathbb{Z}, \cdot) \xrightarrow{\mathcal{I}} (\mathbb{Q}, \cdot)$$

$$\varphi_{\mathsf{dom}}(x, y) = \text{“}y \neq 0\text{”}$$

$$\varphi_{\approx}(x_1, x_2, y_1, y_2) = \text{“}\tfrac{x_1}{y_1} = \tfrac{x_2}{y_2}\text{”}$$

$$\varphi_{\cdot}(x_1, x_2, y_1, y_2, z_1, z_2) = \text{“}\tfrac{x_1}{y_1} \cdot \tfrac{x_2}{y_2} = \tfrac{z_1}{z_2}\text{”}$$

Polynomial Time Interpretation Logic



Counting

$\mathrm{CPT} + \mathrm{C}$

$\equiv$

$\mathrm{CPT} + \mathrm{EqCard}$

$\mathrm{Card}(x)$

$\mathrm{EqCard}(x, y)$

$\mathrm{CPT} + \mathrm{C}$	$\mathrm{Card}(x)$
$\equiv$	
$\mathrm{CPT} + \mathrm{EqCard}$	$\mathrm{EqCard}(x, y)$
$\rightsquigarrow$	
Härtig quantifier	$\mathsf{H}\, xy(\phi(x), \psi(y))$
$\rightsquigarrow$	
$\mathrm{PIL} + \mathrm{H}$	PIL over $\mathrm{FO} + \mathsf{H}$ interpretations

Theorem

$\mathrm{CPT} + \mathrm{C} \equiv \mathrm{CPT} + \mathrm{EqCard} \equiv \mathrm{PIL} + \mathrm{H}$

Theorem

$\mathrm{CPT} \equiv \mathrm{PIL}$

Fragments of Interpretation Logic

$$
\begin{array}{ccc}
\mathrm{CPT} + \mathrm{C} & \equiv & \mathrm{PIL} + \mathrm{H} \\
 & & \vee \\
\mathrm{CPT} & \equiv & \mathrm{PIL}
\end{array}
$$

Fragments of Interpretation Logic

$$\text{CPT} + \text{C} \equiv \text{PIL} + \text{H}$$

$$\vee$$

$$\text{CPT} \equiv \text{PIL}$$

$$\vee$$

$$\approx\text{-free PIL} + \text{H}$$

$$\vee$$

$$\approx\text{-free PIL} \equiv \text{while}_{\mathsf{new}}|_{\text{PTIME}}$$

Fragments of Interpretation Logic

$\mathrm{CPT} + \mathrm{C} \equiv \mathrm{PIL} + \mathrm{H}$

$\vee$

$\mathrm{CPT} \equiv \mathrm{PIL} \equiv \mathrm{while}^{\mathsf{sets}}_{\mathsf{new}}|_{\mathrm{PTIME}}$

$\vee$

$\approx$ -free $\mathrm{PIL} + \mathrm{H}$

$\vee$

$\approx$ -free $\mathrm{PIL} \equiv \mathrm{while}_{\mathsf{new}}|_{\mathrm{PTIME}}$

Fragments of Interpretation Logic

$\mathrm{CPT} + \mathrm{C} \equiv \mathrm{PIL} + \mathrm{H}$

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$\mathrm{CPT} \equiv \mathrm{PIL} \equiv \mathrm{while}^{\mathsf{sets}}_{\mathsf{new}}|_{\textsc{Ptime}}$

$\vee$

$\approx$-free $\mathrm{PIL} + \mathrm{H}$

$\vee$

$\approx$-free $\mathrm{PIL} \equiv \mathrm{while}_{\mathsf{new}}|_{\textsc{Ptime}}$

1-dim. $(\mathrm{PIL} + \mathrm{H})^* \equiv \mathrm{PFPC}|_{\textsc{Ptime}} \equiv \mathrm{FPC}$

$\vee$ iff $\textsc{Ptime} = \textsc{Pspace}$ $\downarrow$

1-dim. $\mathrm{PIL} \equiv \mathrm{PFP}|_{\textsc{Ptime}} \equiv \mathrm{FP}$

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- Candidate for a logic capturing PTIME
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- "Multi-dimensional" generalisation of FP

Future Work

- Expressive power of CPT
- Comparison to Rank Logic
- Model comparison for (fragments of) PIL