Graph Searching Games with Multiple Robbers and Games with Imperfect Information

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GAMES 2011, Paris
September 1, 2011
Complexity Measures

What is a simple graph?

- a small graph
- can be described by certain formalisms
  - in certain logics (e.g., FO, LTL)
  - by graph grammar
  - by an inductive construction (e.g., clique-decomposition)
  - algebraically (e.g., (bi-)rank-width)
  - ...
- with few intertwined cycles
- ...

Rabinovich – Graph Searching with Multiple Robbers, Imperfect Information
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Cyclicity Measures

- tree-width and path-width [Robertson, Seymour]
- directed tree-width [Johnson, Robertson, Seymour, Thomas]
- DAG-width [Berwanger, Dawar, Hunter, Kreutzer; Obdržálek]
- directed path-width [Reed, Seymour, Thomas]
- Kelly-width [Hunter, Kreutzer]
- entanglement [Berwanger, Grädel]
- ...
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Tree-width: Game Theoretial Definition

Game rules:
- **k** Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber
Tree-width: Game Theoretial Definition

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Path-width: the same, but robber invisible.

Tree-width = minimal number cops monotonously capturing Robber - 1
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Path-width: the same, but robber invisible.
DAG-width

- straightforward generalisation to directed graphs (\(=\) tree-width on undirected)
- the robber can run only along directed paths
DAG-width

- straightforward generalisation to directed graphs
  (= tree-width on undirected)
- the robber can run only along directed paths

Compare to other generalisations:
- Kelly-width: the robber is invisible and inert
- directed tree-width: the robber locked in his component
- bounded DAG-width: only parity games
Bounded and Unbounded DAG-width

Bounded: only few paths in one direction
- DAGs (1)
- trees with short back-edges

Unbounded:
- Grids
- trees with many long back-edges
Bounded and Unbounded DAG-width

Bounded: only few paths in one direction
- DAGs (1)
- trees with short back-edges and horizontal connections

Unbounded:
- Grids
- trees with many long back-edges
Generalisation to Multiple Robbers

Same rules, but multiple robbers:

- all $r$ robbers must be captured
- robbers can jump to each other
- global monotonicity: no robber can access a vertex that was inaccessible to all robbers

![Diagram showing multiple robbers and vertices]
Main Result

Theorem

one robber captured by $k$ cops

$\Rightarrow$

$r$ robbers captured by $k \cdot r$ cops

Rough proof idea:

- a team of $k$ cops against each robber
- play independently with each team

1. This works for tree-width...
2. This needs improvement for directed case...
1. This works for tree-width...

   Stronger assumption, stronger conclusion: more paths for robber(s)
1. This works for tree-width...

Stronger assumption, stronger conclusion: more paths for robber(s)

Theorem

For tree-width:

\[
\text{one robber captured by } k \text{ cops} \quad \Rightarrow \quad \text{r robbers captured by } k \cdot r \text{ cops}
\]
1. This works for tree-width...

Stronger assumption, stronger conclusion: more paths for robber(s)

Theorem

For tree-width:

one robber captured by $k$ cops

$\Rightarrow$

$r$ robbers captured by $k \cdot r$ cops
2. Problem with the Directed Case
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Offhanded Cops

Never place a cop outside robber’s component.
Solving Monotonicity Problem for $r$ Robbers
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Maintaining and linearizing several plays
Corollaries

**Theorem**

*Parity games of bounded imperfect information on graphs of bounded DAG-width can be solved in polynomial time.*
Corollaries

Theorem
Parity games of **bounded** imperfect information on graphs of bounded DAG-width can be solved in polynomial time.

Theorem (Robbers Hierarchy)
There are classes of graphs on which every new robber demands new cops until directed path-width is reached.