Parity Games with Imperfect Information and Complexity Measures

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Parity Games with Imperfect Information

Parity games: \((V, V_0, v, (E_a)_{a \in A}, \Omega)\)

Indistinguishable vertices build an information set.
Parity Games with Imperfect Information

- Parity games:
  \((V, V_0, v, (E_a)_{a \in A}, \Omega)\)
- **Indistinguishable** vertices build an information set.
Powerset Construction (Reif 1984)

Information tracking with perfect recall:

\[ \text{Theorem (Reif)} \]

Player 0 wins from \( v \) in \( G \) \( \implies \) Player 0 wins from \( \{ v \} \) in \( G_{\text{perf}} \).
Powerset Construction (Reif 1984)

Information tracking with perfect recall:

\[ \text{Player 0 wins from } v \text{ in } G^{\text{imp}} \iff \text{Player 0 wins from } \{v\} \text{ in } G^{\text{perf}}. \]
Motivation:

- solve parity games in P despite imperfect information,
- but:
  - not known whether PARITY is in P
  - the powerset graph can be exponentially larger

Hope: on **simple** graphs PARITY in P (like without imperfect information)

⇒ Need to measure **complexity** of a graph.
Tree-width: Game Theoretical Definition

Game rules:
- $k$ Cops, one Robber
- Robber runs along cop free paths
- Cops fly
- Cops want to capture Robber

Tree-width = minimal number of cops monotonously capturing Robber - 1

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On Directed Graphs

- **tree-width** (forget directions of edges) ⊕ (Obdržálek 2003)
- directed tree-width ?
- **DAG-width** ⊕ (Berwanger et al. ’06; Obdržálek ’06)
- Kelly-width ⊕ (Kreutzer, Hunter 2008)
- entanglement ⊕ (Berwanger, Grädel 2005)
- ...
- no game characterisation:
  - clique-width
  - rank-width
  - bi-rank-width
- ...

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DAG Game

- DAG-width game (Berwanger et al. 2006; Obdržálek 2006)
  - like before, but Robber runs along directed paths (+ monotonicity)
  - monotonicity costs: positive (Kreutzer, Ordyniak, 2008)
  - DAG-width bounded $\Rightarrow$ PARITY in P

Interesting questions (asked a year ago):
- Monotonicity bounded? (most interesting) — Weak DAG games, Kaiser, Puchala, R.
- Many more offhanded cops? — Yes, Kaiser, Puchala, R.
- Many more cops if many Robbers? — Very strong conjecture: Yes, Puchala, R., in this talk.
Unbounded Imperfect Information

(General Case)
Measures Grow Exponentially

Theorem (Puchala, R.)

 Exists $G^{imp}$ of small complexity, but $G^{perf}$ of exponential complexity (w.r.t. all our measures).

- very large information sets

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Reachability: EXPTIME-hard even if entanglement $\leq 2$ and directed path-width $\leq 2$. (Based on original idea for hardness by Reif.)
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- further restrictions needed
- natural approach: bound size of information sets
Bounded Imperfect Information
General Procedure

- show for appropriate $\oplus$ measures:

**Lemma**

$\text{measure}(G^{imp}) \leq k$, $|\text{information sets}| \leq r$

$\Rightarrow \text{measure}(G^{perf}) \leq f(k, r)$

- then
  - if $\text{measure}(G^{imp}) \leq k$ and $|\text{information sets}| \leq r$
  - $(\Rightarrow |G^{perf}| \text{ polynomial in } |G^{imp}|)$

- $\Rightarrow \text{PARITY in P}$
How Measures Behave (Puchala, R.)

- **tree-width**: \( \Theta \) \( \text{treewidth}(G^{imp}) = 2 \), but \( \text{treewidth}(G^{perf}) \) unbounded
  - **still**: \( \Theta \) \( \text{treewidth}(G^{imp}) \) bounded \( \Rightarrow \) \( \text{DAG}(G^{perf}) \) bounded
- **entanglement**: \( \Theta \) \( \text{entanglement}(G^{imp}) = 2 \), but \( \text{entanglement}(G^{perf}) \) unbounded
- **DAG-width**:
  - **non-monotone** (not appropriate): \( \Theta \) \( f(k, r) = k \cdot r \cdot 2^{r-1} \)
  - if every information set is an SCC: \( \Theta \), \( f(k, r) = k \cdot r^2 \cdot 2^{r-1} \)
  - if every \(|\text{information set}| \leq 2\): \( \Theta \), \( f(k, r) \) bounded
  - **in general**: \( \Theta \) a newer result: \( f(k, r) = kr \cdot 2^{r-1} \)
- **Kelly-width**: ?
- **directed tree-width**: ? (idea doesn’t work)
General Proof Idea for Boundedness of Measures

\[ \{v\} \]

\[ \{v_1, \ldots, v_r\} \]

\[ \text{nm-DAG-width: } f(k, r) = k \cdot r \cdot 2^{r-1} \]
General Proof Idea for Boundedness of Measures

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General Proof Idea for Boundedness of Measures

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\[ \{v_1, \ldots, v_r\} \]

\[ \text{nm-DAG-width: } f(k, r) = k \cdot r \cdot 2^r - 1 \]
General Proof Idea for Boundedness of Measures
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\[
m\text{-DAG-width: }\quad f(k, r) = k \cdot r \cdot 2^{r-1}
\]
New DAG-Game: Multiple Robbers

- Cops must capture \( r \) Robbers.
- Robbers can jump:

- Monotonicity:
  no robber can access a vertex that was inaccessible to all robbers
For Tree-width

$k$ cops win Tree-width Game

$\iff$ forget directions of edges and win

Stronger assumption: robber has more paths to run!
For Tree-width

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\[ \iff \text{forget directions of edges and win} \]

Stronger assumption: robber has more paths to run!

Theorem

For tree-width:

If \( k \) cops \textit{monotonously} win against one robber

then \( k \cdot r \) cops \textit{monotonously} win against \( r \) robbers.
For Tree-width

\( k \) cops win Tree-width Game

\[\Leftrightarrow \text{forget directions of edges and win}\]

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Summing up for Tree-width

Theorem

If $|\text{information set}| \leq r$ and tree-width of $G \leq k$ then PARITY is efficiently solvable on $G$.

What doesn’t work for DAG-width?
Summing up for Tree-width

Theorem

If $|\text{information set}| \leq r$ and tree-width of $G \leq k$ then PARITY is efficiently solvable on $G$.

What doesn’t work for DAG-width?
Problem with Monotonicity
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Place Cops only Inside?

Theorem (Kaiser, Puchala, R.)

There is a family of graphs such that

- Four cops capture the robber.
- Unboundedly many cops needed if only inside the component.
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Solving Monotonicity Problem for $r$ Robbers
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Thank you for your attention!