Knowledge and Cooperation in Infinite Games

Bernd Puchala

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Reasoning about knowledge originates in Philosophy. Epistemology analyzes the nature of knowledge and tries to answer questions like: What is knowledge? How is knowledge acquired? What do people know? How do we know what we know?
Reasoning about knowledge originates in Philosophy.
Epistemology

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Epistemology analyzes the nature of knowledge and tries to answer questions like:

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- How is knowledge acquired?
- What do people know?
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What is knowledge?

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What do people know?

How do we know what we know?
Reasoning About Knowledge

Epistemic Logic

Idea: Put the analysis of these questions on a formal logical ground

Jaakko Hintikka, 1962: Knowledge and Belief: An Introduction to the Logic of the Two Notions

Hintikka tried to capture inherent properties of knowledge by formal logical rules and used Modal Logic with Possible World Semantics

Modal Logic already used by Aristotle

Possible Worlds Semantics developed by Carnap, Hintikka, Kripke, . . .

Current form: Kripke structures
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A Kripke-structure has the form
\[ K = (V, \text{Prop}, (E_i)_{i \in I}) \]
where
- \( V \) is a (finite) set (of possible worlds)
- \( \text{Prop} \) is a set of unary relations (atomic propositions)
- each \( E_i \) is a binary relation (alternative relation)

Epistemic Logic is defined by
\[
\phi ::= P | \phi \land \phi | \neg \phi | K_i \phi \]
\[
K_i \phi \iff \forall w \in V. (v,w) \in E_i \rightarrow \phi
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- \( \mathcal{K}, v \models K_i \varphi \) iff \( \mathcal{K}, w \models \varphi \) for all \( w \in V \) with \( (v, w) \in E_i \)
Reasoning About Knowledge

This formal treatment of knowledge has many applications:

- **Artificial Intelligence**
  - A robot should not only complete his task but he should also know when his task is completed.
- **Synthesis of systems with partial observation**
  - A safety critical action should only be performed by a system, when the controller knows that the current state of the system ensures a safe execution.
- **Security protocols often involve requirements like “no component of the system will ever know the value of any internal variable of some other component”**

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**Economics**
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- $K_i \varphi \rightarrow \varphi$ (Knowledge Axiom)
- $K_i \varphi \rightarrow K_i K_i \varphi$ (Positive Introspection Axiom)

Positive introspection and (even more) negative introspection are highly controversial among philosophers! Also questionable in some applications For our concerns justified!
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Agent $i$ knows a fact $\varphi$ about the system, if $\varphi$ holds in all states of the system which agent $i$ cannot distinguish from the current state.
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- Computing systems evolve over time.
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Consider logics which allow to express temporal statements about the knowledge of the agents.
A multi-agent system has the form 

\[ E = (R, \text{Prop}, \zeta, (\sim_i)_{i \in n}) \]

where 

- \( R \) is a set of runs
- \( \text{Prop} \) is a set of atomic propositions
- \( \zeta : R \times N \rightarrow 2^{\text{Prop}} \)

is the propositional labelling.

\( \sim_i \) is an equivalence relation on \( R \times N \).

The knowledge of agent \( i \) at some point \((\pi, n)\) is given by \( \sim_i \).

Equivalent points are indistinguishable for agent \( i \), i.e., if 

\[ (\pi, n) \sim_i (\rho, m) \]

then at point \((\pi, n)\) and \((\rho, m)\), agent \( i \) has exactly the same information, so he cannot distinguish one situation from the other.
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The *knowledge* of agent $i$ at some point $(\pi, n)$ is given by $\sim_i$

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Epistemic Temporal Logic
Epistemic Temporal Logic ETL is defined by

\[ \varphi ::= P \mid \varphi \land \varphi \mid \neg \varphi \mid X \varphi \mid \varphi U \varphi \mid K_i \varphi \]

where \( P \in \text{Prop} \) and \( i \in \{1, \ldots, n\} \)
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- $P$ iff $P \in \zeta(\pi, n)$
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\[ E \models \varphi \] iff \( E, (\pi, 0) \models \varphi \) for all \( \pi \in R \)
Epistemic Temporal Logic

Epistemic Temporal Logic $ETL + C$ with common knowledge is defined as $ETL$ with the additional rule $\varphi ::= C_B \varphi$, where $B \subseteq \{1, \ldots, n\}$
Epistemic Temporal Logic

Epistemic Temporal Logic ETL+C with common knowledge is defined as ETL with the additional rule \( \varphi ::= C_B \varphi \), where
\[ B \subseteq \{1, \ldots, n\} \]
\[ \mathcal{E}, (\pi, n) \models C_B \varphi \text{ iff for all } (\rho, m) \text{ with } (\pi, n) \sim_B (\rho, m) \text{ we have } \mathcal{E}, (\rho, m) \models \varphi \]

where \( \sim_B \) is the transitive closure of \( \bigcup_{i \in B} \sim_i \).
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Common Knowledge:
$\mathcal{E}, (\pi, n) \models C_B \varphi$ iff
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$\mathcal{E}, (\pi, n) \models \mathbf{C}_B \varphi$ iff for all $(\rho, m)$ with $(\pi, n) \sim_B (\rho, m)$ we have $\mathcal{E}, (\rho, m) \models \varphi$

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- $\mathcal{E}, (\pi, n) \models K_i \varphi$ for all $i \in B$: Everyone in $B$ knows $\varphi$
- $\mathcal{E}, (\pi, n) \models K_i K_j \varphi$ for all $i, j \in B$: Everyone in $B$ knows that everyone in $B$ knows $\varphi$
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- ...
We are mainly interested in nonterminating finite state systems with partial observation, so we consider systems which are generated by finite systems where the agents have uncertainties about the states and actions.

A finite multi-agent system has the form

\[ E = (V, \text{Prop}, \Delta, (\sim V_i)_{i \in \mathbb{N}}, (\sim A_i)_{i \in \mathbb{N}}) \]

where

- \( V \) is a finite set of states
- \( \text{Prop} \) is a finite set of atomic propositions
- \( \Delta \subseteq V \times A \times V \) is the move relation
- \( \sim V_i \) and \( \sim A_i \) are equivalence relations on \( V \) and \( A \) respectively

Run: infinite sequence \( \pi = v_0 a_1 v_1 ... \in (AV)^\omega \) such that

\[ (v_i, a_{i+1}, v_{i+1}) \in \Delta \text{ for each } i < \omega \]
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where
Epistemic Temporal Logic

We are mainly interested in nonterminating finite state systems with partial observation, so we consider systems which are generated by finite systems where the agents have uncertainties about the states and actions.

A *finite multi-agent system* has the form

$$\mathcal{E} = (V, \text{Prop}, \Delta, (\sim^V_i)_{i \in n}, (\sim^A_i)_{i \in n})$$

where

- $V$ is a finite set of states
- $\text{Prop}$ is a finite set of atomic propositions
- $\Delta \subseteq V \times A \times V$ is the move relation
- $\sim^V_i$ and $\sim^A_i$ are equivalence relations on $V$ and $A$ respectively
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**Run:** infinite sequence \( \pi = v_0a_1v_1 \ldots \in V(AV)^\omega \) such that \( (v_i, a_{i+1}, v_{i+1}) \in \Delta \) for each \( i < \omega \)
Epistemic Temporal Logic

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Here we focus on synchronous perfect recall:

$$(\pi, n) \sim^*_i (\rho, m) :\iff m = n \text{ and } a_j \sim^A_i b_j, \ v_j \sim^V_i w_j$$
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We also consider an instance of asynchronous perfect recall:

$$(\pi, n) \leftarrow_i^* (\rho, m) :\iff (\leftarrow_i \pi, n) \sim_i^* (\leftarrow_i \rho, m)$$

where $\leftarrow_i \pi$ is obtained from $\pi$ by contracting each maximal sequence $v_r a_{r+1} v_{r+1} \ldots a_s v_s$ with $v_j \sim^V_i v_{j+1}$ to $v_r$
Epistemic Temporal Logic

Observational:
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Clock:
Epistemic Temporal Logic

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Given a finite system $\mathcal{E} = (V, \text{Prop}, \Delta, (\sim_{i}^{V})_{i \in n}, (\sim_{i}^{A})_{i \in n})$, a state $v$ and a formula $\varphi \in \text{ETL}$
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$$\mathcal{E}, v \models \varphi?$$
Epistemic Temporal Logic

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**Model-Checking Knowledge and Time!**

$\mathcal{E}, v \models \varphi$ is defined as $\mathcal{R}(\mathcal{E}) \models \varphi$

where $\mathcal{R}(\mathcal{E}) = (R, \text{Prop}, \zeta, (\sim_i)_{i \in n})$ is the unravelling of $\mathcal{E}$ from $v$
Using this formalism, we want to check finite systems for epistemic properties:
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- $R$ is the set of all runs of $\mathcal{E}$ from $v$
- $\zeta(\pi, n) = \{P \in \text{Prop} \mid v_n \in P\}$
- $\sim_i$ is either $\sim^*_i$ or $\overline{\sim}^*_i$
Model Checking Knowledge

Given a finite Kripke structure $K$, a state $v$ and an epistemic formula $\phi$, does $K, v \models \phi$ hold?

Time $O(||K|| \cdot |\phi|)$ (Labeling Algorithm) for epistemic formulas with common knowledge still polynomial time.
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Given a finite system $E$, a state $v$ and an LTL formula $\phi$, does $E, v \models \phi$ hold?

Polynomial Space (Translation of LTL-formulas into Büchi automata)
Given a finite system $E$, a state $v$ and an LTL formula $\varphi$, does $E, v \models \varphi$ hold?
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For synchronous systems decidable, but non-elementary complexity (van der Meyden, Shilov '99)
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(1) Reduction to chain logic with equal level predicate
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Model Checking Knowledge and Time

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(1) Reduction to chain logic with equal level predicate

(2) Use $k$-trees to interpret formulas of knowledge-depth $k$

- For fixed $k$, infinite sequences of $k$-trees can be recognized by automata
- Also involves a factorization of formulas into temporal and knowledge components
- $\sim$ space polynomial in $|\varphi| \cdot \exp(\text{depth}(\varphi), O(|\mathcal{E}|))$
Model Checking Knowledge and Time

- For synchronous systems and formulas with common knowledge undecidable (van der Meyden, Shilov, ’99)
  Until and Common Knowledge allow an arbitrary reach through two orthogonal dimensions of the semantic structures
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  Until and Common Knowledge allow an arbitrary reach through two orthogonal dimensions of the semantic structures

- For synchronous systems and formulas with common knowledge but \textit{without until} PSPACE-complete (van der Meyden, Shilov, ’99)
  Without until, the temporal operators can only look $|\varphi|$ steps into the system
We are not only interested in model checking closed systems but in synthesizing reactive systems. Reactive systems interact with an environment. The desired behavior of the system is given by a formal specification, for example a temporal formula $\phi \in LTL$. Question: Can we ensure a faultless interaction of the components of the system, independently of the behavior of the environment? Natural Model: Games on Graphs
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Games with Partial Observation

Multi-player games on graphs:
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- A multi-player game is basically a multi-agent system, where each position is controlled either by one of the cooperating players or by the environment.
- We view the environment as an additional player.
- Cooperating player represent components of the system.
- Question: Does this coalition have a winning strategy for the game?
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- Cooperating player represent components of the system.
- Question: Does this coalition have a winning strategy for the game?
- The winning condition for the coalition is defined by the specification $\varphi$.
- We often consider systems with partial observation.
- It is then very desirable to be able to refer to the knowledge of the components in the specification: ETL.
- The environment may also have partial information about the system.
A game with $n$ players has the form

$$\mathcal{G} = (V, (V_i)_{i \in \mathbb{N}}, \text{Prop}, \Delta, (\sim^V_i)_{i \in \mathbb{N}}, (\sim^A_i)_{i \in \mathbb{N}})$$

where
Games with Partial Observation

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1. $v \sim_i w \Rightarrow v, w \in V_i$ or $v, w \notin V_i$
2. $v \sim_i w \Rightarrow \text{act}(v) = \text{act}(w)$
3. $a, b \in A_i$ and $a \neq b \Rightarrow a \not\sim^A_i b$
Games with Partial Observation

- *Play:* Run
Games with Partial Observation

- **Play**: Run
- **Strategy for player i**: function $\sigma_i : (VA)^*V_i \rightarrow A$ with
  \[
  \sigma_i(\pi) = \sigma_i(\rho) \text{ for all } \pi, \rho \in (VA)^*V_i \text{ with } \pi \sim_i \rho
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- $\sim_i \in \{\sim^*_i, \leftarrow^*_i\}$
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- **Winning strategy for the coalition**: Joint strategy $\sigma$ such that each play $\pi$ which is consistent with $\sigma$ is won by the coalition
Synthesis from ETL-specifications

An ETL-formula $\phi$ defines a winning condition $L(\phi) = \{ \pi \in (VA) \mid (\pi, 0)\|= \phi \}$.
An ETL-formula $\varphi$ defines a winning condition

$$L(\varphi) = \{ \pi \in (VA)^\omega \mid (\pi, 0) \models \varphi \}$$
Synthesis from ETL-specifications: Two Players
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Given a two-player game $G$ and an ETL-formula $\varphi$, is there a strategy $\sigma_0$ for player 0 such that for all plays $\pi$ which are consistent with $\sigma_0$ we have $\pi \in L(\varphi)$?
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- Clearly we should assume player 0 to know his own strategy $\sigma_0$
- So the evaluation of the knowledge operator $K_0$ should be relative to histories (points) which are consistent with $\sigma_0$
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- Clearly we should assume player 0 to know his own strategy \( \sigma_0 \).
- So the evaluation of the knowledge operator \( K_0 \) should be relative to histories (points) which are consistent with \( \sigma_0 \).
- However: if \((\pi, n) \sim_0 (\rho, m)\) for \( \sim_0 \in \{\sim_*^0, \overset{\leftarrow}{\sim}_0^*\} \), then \((\pi, n)\) is consistent with \( \sigma_0 \) if, and only if, \((\rho, m)\) is consistent with \( \sigma_0 \).
- Due to the fact that player 0 can distinguish any two of his own actions.
For the knowledge operator $K_1$ to make sense, we have to assume that player 1 does not know the strategy of player 0. Otherwise player 1 has full information about the history.
For the knowledge operator $K_1$ to make sense, we have to assume that player 1 does not know the strategy of player 0. Otherwise, player 1 has full information about the history. If player 0 models a controller of an environment which is not actually antagonistic but merely unpredictable, then given any strategy $\sigma_0$ for player 0, the joint system is constrained by $\sigma_0$, so the knowledge of both players should be relative to $\sigma_0$. If player 0 models a network server which might interact with a user, then the protocol of the server may be off-limits to the user, so player 1 does not know $\sigma_0$. One requirement for the protocol might be that the user is never able to learn the value of some internal variables of the server.
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Otherwise player 1 has full information about the history.

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One requirement for the protocol might be that the user is never able to learn the value of some internal variables of the server.

This can be expressed using $K_1$. 
Consider a parity game $G$ with coloring $\text{col} : V \rightarrow \{1, \ldots, r\}$
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\[
\text{parity} := \bigvee_{c \text{ even}} GF \text{col}^{-1}(c) \land FG \bigwedge_{c' < c} \neg \text{col}^{-1}(c')
\]
defines the parity objective for player 0.

Additionally, requires that player 1 never knows whether player 0 knows the recent color.
Consider a parity game \( G \) with coloring \( \text{col} : V \rightarrow \{1, \ldots, r\} \)

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\( K^\text{col}_0 := \bigvee_{c \in C} K_0 \text{col}^{-1}(c) \) says that player 0 knows the color of the recent position
Consider a parity game $G$ with coloring $col : V \rightarrow \{1, \ldots, r\}$

- $\text{parity} := \bigvee_{c \text{ even}} GF \col^{-1}(c) \land FG \land_{c' < c} \neg \col^{-1}(c')$ defines the parity objective for player 0

- $\text{col}^{-1} := \bigvee_{c \in C} K_0 \col^{-1}(c)$ says that player 0 knows the color of the recent position

- So $\varphi = \text{parity} \land G(\neg K_1 \text{col} \land \neg K_1 \neg \text{col})$ additionally requires that player 1 never knows whether player 0 knows the recent color
Synthesis from ETL-specifications: Two Players
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- For synchronous finite two-player games with ETL winning conditions which use only the knowledge operator $K_0$, 2EXPTIME-complete (van der Meyden, Vardi ’98)
Synthesis from ETL-specifications: Two Players

- For synchronous finite two-player games with ETL winning conditions which use only the knowledge operator $K_0$
  2EXPTIME-complete (van der Meyden, Vardi ’98)
- Uses tree automata: A tree automaton can process the tree-representation of a strategy, viewed as a function
  $\sigma_0 : \text{Obs}_i^* \rightarrow A$
- Check that all plays are won by player 0: Universal part
- Evaluate $K_0$: Knowledge sets
Synthesis from ETL-specifications: Two Players
Knowledge set of player $i$ at point $\langle \pi, n \rangle$:

$$\mathcal{K}_i(\pi, n) = \{ \text{last}(\rho, m) \mid (\rho, m) \sim_i (\pi, n) \} \subseteq V$$

(Set of positions that player $i$ considers possible at point $\langle \pi, n \rangle$)
Synthesis from ETL-specifications: Two Players

Knowledge set of player $i$ at point $(\pi, n)$:

$$\mathcal{K}_i(\pi, n) = \{\text{last}(\rho, m) | (\rho, m) \sim_i (\pi, n)\} \subseteq V$$

(Set of positions that player $i$ considers possible at point $(\pi, n)$)

Can be computed iteratively:

$$\mathcal{K}_i(\pi, n + 1) = \text{Post}_{a_{n+1}}(\mathcal{K}_i(\pi, n)) \cap [v_{n+1}]$$

$\sim$ **Powerset Construction** (Reif ’84)

Transforms a two-player game with partial information into a two-player game with full information
Synthesis from ETL-specifications: Two Players
Powerset Construction can be generalized to arbitrary $\omega$-regular winning conditions both in the synchronous and in the asynchronous case (’10)
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**Theorem**

*Any ETL definable winning condition is $\omega$-regular. ('10)*

More precisely: For any game $G$ and any ETL-formula $\varphi$ with knowledge-operators $K_0$ and $K_1$ which have either $\sim_i^*$ or $\preceq_i^*$ semantics, we can effectively construct an S1S-formula $\psi(x)$ such that for any play $\pi$ and any $n \in \mathbb{N}$ we have

$$(\pi, n) \models \varphi \iff \pi \models \psi(n)$$

S1S: Monadic Second Order Logic interpreted in word structures $(\mathbb{N}, (P_a)_{a \in \Sigma}, <)$
Synthesis from ETL-specifications: Two Players
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- Highlights the view of $L(\varphi)$ as a winning condition (set of plays)
Synthesis from ETL Specifications: Two Players

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- Makes the powerset construction applicable to ETL winning conditions
Synthesis from ETL-specifications: Two Players

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- Makes the powerset construction applicable to ETL winning conditions
- Also holds for the asynchronous case
Synthesis from ETL specifications: Two Players

Proof.
Synthesis from ETL-specifications: Two Players

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- By induction on the structure of $\varphi$
- Interesting case: $K\varphi$
Proof.

- By induction on the structure of \( \varphi \)
- Interesting case: \( K \varphi \)
- \( (\pi, n) \models \neg K \varphi \) iff there is some \( (\rho, m) \sim (\pi, n) \): \( (\rho, m) \models \neg \varphi \)
Proof.

- By induction on the structure of $\varphi$
- Interesting case: $K\varphi$
- $(\pi, n) \models \neg K\varphi$ iff there is some $(\rho, m) \sim (\pi, n): (\rho, m) \models \neg \varphi$

\[-\exists X_{va} \left[ \forall y \left( \bigvee_{va} (X_{va}y \land \bigwedge_{wb \neq va} \neg X_{wb}y) \right) \right. \]
\[\land \left. \forall y \forall z (Sy = z \rightarrow \bigvee_{(v, w) \in E_a, b \in A} X_{va}y \land X_{wb}z) \right. \]
\[\land \left. \forall (y \leq x) (\bigwedge_{va} (P_{va}y \rightarrow \bigvee_{wb \sim va} X_{wb}y)) \right. \]
\[\land \left. \neg \psi (P_{va}/X_{va}) \right. \]
Synthesis from ETL-specifications: Two Players

In the asynchronous case?
Use automata!
A nondeterministic $\omega$-automaton can guess such a $(\rho, m)$
in this particular asynchronous case the automaton can be constructed effectively.
Synthesis from ETL-specifications: Two Players

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Synthesis from ETL-specifications: Two Players

- In the asynchronous case?
- Use automata!
- A nondeterministic $\omega$-automaton can guess such a $(\rho, m)$ asynchronously
- In this particular asynchronous case the automaton can be constructed effectively
Synthesis from ETL-specifications: Two Players

Corollary

The asynchronous synthesis problem for ETL specifications with knowledge operators $K_0$ and $K_1$ is decidable.
Given a game $G$ and an ETL-formula $\varphi$, is there a joint strategy $\sigma$ for the coalition such that for all plays $\pi$ which are consistent with $\sigma$ we have $\pi \in L(\varphi)$?
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- For synchronous 3-player games with LTL winning conditions undecidable (Pnueli, Rosner ’90, Reif ’01)
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- For synchronous hierarchical $n$-player games with LTL winning conditions decidable (Pnueli, Rosner ’90, van der Meyden, Wilke ’05)
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- For synchronous 3-player games with LTL winning conditions undecidable (Pnueli, Rosner ’90, Reif ’01)
- For synchronous hierarchical $n$-player games with LTL winning conditions decidable (Pnueli, Rosner ’90, van der Meyden, Wilke ’05)
- For synchronous 3-player games with ETL winning conditions where only player 0 has partial observation undecidable (van der Meyden, Wilke ’05)
Synthesis from ETL-specifications

On the other hand:

- Omega-Regularity of ETL winning conditions also holds for $n$ agents (same proof)
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Synthesis from ETL specifications

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- Omega-Regularity of ETL winning conditions also holds for $n$ agents (same proof)
- Omega-Regular $n$ player games with can be transformed into parity games
- such that the hierarchy of knowledge if preserved
- Parity Objectives are LTL-definable
On the other hand:

- Omega-Regularity of ETL winning conditions also holds for $n$ agents (same proof)
- Omega-Regular $n$ player games with can be transformed into parity games such that the hierarchy of knowledge if preserved
- Parity Objectives are LTL-definable
- Hierarchical games with ETL winning conditions are decidable
Reasoning About Knowledge

Given a game $G$ and an ETL specification $\varphi$, is there a joint strategy $\sigma$ for the coalition such that $R(G, \sigma) | = \varphi$?

Where $R(G, \sigma)$ is the unravelling of $G$ with respect to $\sigma$!

There, the evaluation of knowledge operators is relative to plays which are consistent with $\sigma$. So all players know the strategy of any other cooperating player. In the two-player case this is equivalent but in the multiplayer case it makes a difference.
Reasoning About Knowledge
Reasoning About Knowledge in Games

Synthesis from ETL-specifications

Reason:
The actual question van der Meyden, Vardi and Wilke ask is

Given a game $G$ and an ETL specification $\varphi$, is there a joint strategy $\sigma$ for the coalition such that $R(G, \sigma) \models \varphi$?

where $R(G, \sigma)$ is the unravelling of $G$ with respect to $\sigma$!
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- There, the evaluation of knowledge operators is relative to plays which are consistent with $\sigma$
- So all players know the strategy of any other cooperating player
- In the two-player case this is equivalent but in the multiplayer case it makes a difference
Knowing the strategies of your companions is different from relying on their strategies:
Knowing the strategies of your companions is different from relying on their strategies:

$\mathcal{K}_1 = \{\text{left, right}\} : \neg \mathcal{K}_1 P_{\text{left}}$
Observing the actions of your companions is not enough for knowing their strategies:
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Observing the actions of your companions is not enough for knowing their strategies:

$$\mathcal{K} = \{1, 2\} : \neg K_1 P_1$$
Your knowledge-set does not provide you sufficient information, even if you *do know* the strategies of your companions:
Cooperation in Multi-Player Games

Your knowledge-set does not provide you sufficient information, even if you *do know* the strategies of your companions:

Diagram:
- Nodes: 1 to 4, x, y, z
- Edges: a, b
- Formulas: $K_0 K_1 \neg P_z$

Explanation:
- The diagram illustrates a game scenario with multiple paths and knowledge states.
- Node x has two outgoing edges labeled a and b, leading to nodes 1 and 2, respectively.
- Node y has two outgoing edges labeled a, leading to nodes 2 and 3.
- Node z has two outgoing edges labeled b, leading to nodes 3 and 4.
- The formula $K_0 K_1 \neg P_z$ represents knowledge states and negations in the game context.
Cooperation in Multi-Player Games

Your knowledge-set does not provide you sufficient information, even if you *do know* the strategies of your companions:
Cooperation in Multi-Player Games

Your knowledge-set does not provide you sufficient information, even if you do know the strategies of your companions:

\[ K_0 = \{y\} \]

\[ K_0 K_1 \neg P_z \]

\[ K_0 K_1 \neg P_x \]