Parity Games with Partial Information Played on Graphs of Bounded Complexity [1] [2]

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Abstract. We address the strategy problem for parity games with partial information and observable colors, played on finite graphs of bounded complexity. We consider several measures for the complexity of graphs and analyze in which cases, bounding the measure decreases the complexity of the strategy problem on the corresponding class of graphs. We prove or disprove that the usual powerset construction for eliminating partial information preserves boundedness of the graph complexity.
For the case where the partial information is unbounded we prove that this is not the case for any measure we consider. We also prove that the strategy problem is Exptime-hard on graphs with directed path-width at most 2 and Pspace-complete on acyclic graphs. For games with bounded partial information we obtain that the powerset construction, while neither preserving boundedness of entanglement nor of (undirected) tree-width, does preserve boundedness of directed path-width. Hence, parity games with bounded partial information, played on graphs with bounded directed path-width can be solved in polynomial time.

1 Introduction

Parity games are played by two players moving a token along the edges of a labeled graph by choosing appropriate edge labels, called actions. The vertices of the graph, called positions, have priorities and the winner of an infinite play of the game is determined by the parity of the least priority which occurs infinitely often. Parity games play a key role in modern approaches to verification and synthesis of state-based systems. They are the model-checking games for the modal μ-calculus, a powerful specification formalism for verification problems. Moreover, parity objectives can express all ω-regular objectives and therefore capture fundamental properties of non-terminating reactive systems, cf. [19]. Such a system can be modeled as a two-player game where changes of the system state correspond to changes of the game position. Situations where the change of the system can be controlled correspond to positions of player 0, uncontrollable situations correspond to positions of player 1. A *winning strategy* for player 0 then yields a controller that guarantees satisfaction of some ω-regular specification.

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[2] Full version of [15]. Additional material on preservation of DAG-width formerly appearing as Section 5 is now included in [16].

The problem to determine, for a given parity game G and a position v, whether player 0 has a winning strategy for G from v, is called the strategy problem. The algorithmic theory of parity games with full information has received much attention during the past years, cf. [10]. The most important property of parity games with full information is the memoryless determinacy which proves that the strategy problem for parity games is in NP $\cap$ co-NP.

However, assuming that both players have full information about the history of events in a parity game is not always realistic. For example, if the information about the system state is acquired by imprecise sensors or the system encapsulates private states which cannot be read from outside, then a controller for this system must rely on the information about the state and the change of the system to which it has access. I.e. in the game model, player 0 has uncertainties about the positions and actions in the game, so we have to add partial information to parity games in order to model this kind of problems. The uncertainties are represented by equivalence relations on the positions and actions in the game graph meaning that equivalent positions respectively actions are indistinguishable for player 0. Solving the strategy problem for such games is much harder than solving parity games with full information, since we have to keep track of the knowledge of player 0 during a play of the game. For this we compute, for any finite history, the set of positions that player 0 considers possible in this situation. This procedure is often referred to as *powerset construction.*

Such a knowledge tracking is inherently unavoidable and leads to an exponential lower bound for the time complexity of the strategy problem for reachability games with partial information [17] and a super-polynomial lower bound for the memory needed to implement winning strategies in reachability games [3,14]. Therefore, it is expedient to look for classes of games with partial information, where the strategy problem has a lower complexity. A simple while effective approach is to bound the partial information in the game, i.e. the size of the equivalence classes of positions which model the uncertainties of player 0 about the current position. This is appropriate in situation where, e.g., the imprecision of the sensors or the amount of private information of the system does not grow if the system grows. Then, the game which results from the powerset construction has polynomial size, so partial information parity games with a *bounded* number of observable priorities can be solved in polynomial time. Hereby *observable* means, that the priorities are constant over equivalence classes. However, if the number of priorities is not bounded, we cannot prove this approach to be efficient, since the question whether full information parity games with arbitrarily many priorities can be solved in polynomial time is still open.

To obtain a class of parity games with partial information that can be solved in polynomial time, one has to bound certain other parameters. A natural approach is to bound the complexity of the game graphs with respect to appropriate measures. Such graph complexity measures have proven enormous usefulness in algorithmic graph theory. Several problems which are intractable in general can be solved efficiently on classes of graphs where such measures are bounded. The key note here is that bounded complexity with respect to appropriate measures allows to decompose the graph into small parts which are only sparsely related within the graph in a certain sense. One can then solve the problem on these small parts which requires, for each part, only a fixed amount of time, and combine the partial solutions in an efficient way. This has proven to be applicable to a large number of graph theoretic decision problems, e.g., all MSO-definable graph properties [5]. More recently, it has also been applied to the strategy problem for (full information) parity games. It has been shown that parity games played on graphs with bounded tree-width or bounded (monotone) DAG-width or bounded entanglement can be solved in polynomial time [2,4,12]. The natural question is whether such results can also be obtained for games with partial information.

Since the direction of the edges is inherently important when solving games and when performing the powerset construction, we primarily consider measures for directed graphs. However, we prove a negative result about (undirected) tree-width, which is the most important measure for undirected graphs, as a prototype witness for the high potential of the powerset construction to create graph complexity when the direction of edges is neglected. From the large variety of measures for directed graphs we focus on DAG-width, directed path-width and entanglement. Two other important measures are directed tree-width [9] and Kelly-width [8]. For those measures, however, our techniques cannot be applied directly, due to somewhat inconvenient conditions in the definitions.

In Section 3 we prove that in the case where the partial information is unbounded, there are classes of graphs G with complexity at most 2 such that the complexity of the corresponding powerset graphs is exponential in the *size* of G for any measure we consider. We also prove that the strategy problem for reachability games with partial information is EXPTIME-hard on graphs with entanglement at most 2 and directed path-width at most 2 and that the problem is PSPACE-complete on acyclic graphs. Notice that reachability games form a subclass of parity games. Roughly speaking, these results show that bounding the graph complexity does not decrease the complexity of the strategy problem, as long as the partial information is unbounded.

In Section 4 we consider parity games with bounded partial information. In this case, the graphs which result from the powerset construction have polynomial size, so if the construction additionally preserves boundedness of appropriate graph complexity measure, then the corresponding strategy problem is in PTIME. We obtain that the powerset construction, while neither preserving boundedness of entanglement nor of (undirected) tree-width, does preserve boundedness of directed path-width and of *non-monotone* DAG-width. So, parity games with bounded partial information, played on graphs of bounded directed path-width can be solved in polynomial time. Moreover, if DAG-width has bounded monotonicity cost, which is an open problem, the same result holds for the case of bounded DAG-width. [1]

[1] Although boundedness of monotonicity cost for DAG-width remains open, it has been proved that parity games with bounded partial information can be solved in PTIME on graphs of bounded DAG-width [16].

2 Preliminaries

Games and Strategies. A *parity game* has the form $G = (V, V_0, (f_a)_{a \in A}, \mathrm{col})$, where V is the set of positions, A is the set of actions and for each action $a \in A$, $f_a : \mathrm{dom}(f_a) \subseteq V \to V$ is a function. We write $v \xrightarrow{a} w$ if $f_a(v) = w$. Furthermore, $V_0 \subseteq V$ are the positions of player 0 and $\mathrm{col} : V \to C$ is a function into a finite set $C \subseteq \mathbb{N}$ of colors (also called priorities). We define $V_1 := V \setminus V_0$ and $A_i := \bigcup\{\mathrm{act}(v) \mid v \in V_i\}$ for $i = 0, 1$. The directed graph (V, E) with $E = \bigcup\{E_a \mid a \in A\}$ where $E_a = \{(u, v) \in V \times V \mid\ u \in \mathrm{dom}(f_a) \text{ and } f_a(u) = v\}$ for each $a \in A$ is called the *game graph* of G. Here we consider only *finite* games, i.e. games where V and A are finite.

For a finite sequence $\pi \in V(AV)^*$, by $\mathrm{last}(\pi)$ we denote the last position in π. For $v \in V$, a *play* in G from v is a *maximal* finite or infinite sequence $\pi = v_0 a_0 v_1 \ldots \in v(AV)^* \cup v(AV)^\omega$ such that $v_i \in \mathrm{dom}(f_{a_i})$ and $f_{a_i}(v_i) = v_{i+1}$ for each i. A finite play π is won by player 0 if $\mathrm{last}(\pi) \in V_1$. An infinite play π is won by player 0 if $\min\{c \in C \mid \mathrm{col}(v_i) = c \text{ for infinitely many } i < \omega\}$ is even. A play π is won by player 1 if and only if it is not won by player 0. A *reachability game* is a parity game with $C = \{1\}$, i.e. player 0 wins only finite plays which end in positions $v \in V_1$. Now let $\mathcal{H}_{\mathrm{fin}}$ be the set of all finite histories $\pi \in V(AV)^*$ of plays in G from v. A *strategy* for player i for G is a function $g : \{\pi \in \mathcal{H}_{\mathrm{fin}} \mid \mathrm{last}(\pi) \in V_i\} \to A$ such that $g(\pi) \in \mathrm{act}(\mathrm{last}(\pi))$ for all $\pi \in \mathrm{dom}(g)$. A history $\pi = v_0 a_0 v_1 a_1 v_2 \ldots$ is called *compatible* with g if for all j such that $v_j \in V_i$ we have $a_j = g(v_0 a_0 \ldots a_{j-1} v_j)$. We call a strategy g for player i a *winning strategy* from v_0 if each play π in G from v_0 that is compatible with g is won by player i.

Partial Information. The *knowledge* of player i after some history $\pi \in \mathcal{H}_{\mathrm{fin}}$ is given by an equivalence relation $\sim_i \subseteq \mathcal{H}_{\mathrm{fin}} \times \mathcal{H}_{\mathrm{fin}}$ where $\pi \sim_i \pi'$ if π and π' are indistinguishable for player i by means of his given information. So, after π has been played, to the best of player i's knowledge, it is possible that instead, π' has been played. A strategy $g : \{\pi \in \mathcal{H}_{\mathrm{fin}} \mid \mathrm{last}(\pi) \in V_i\} \to A$ for player i for G is called a *partial information strategy* with respect to $\sim_i$ ($\sim_i$-strategy, for short) if $g(\pi) = g(\pi')$ for all $\pi, \pi' \in \mathcal{H}_{\mathrm{fin}}$ with $\pi \sim_i \pi'$. Notice that a $\sim_i$-strategy g for player i is winning from all positions in a set $U \subseteq V$ if and only if it is winning from a simulated initial position v_0 which belongs to player $1 - i$ and from which he can secretly choose any position $v \in U$. Moreover, any $\sim_i$-strategy g which is only defined on histories from some initial position v_0 can be extended to a $\sim_i$-strategy g' with $\mathrm{dom}(g) = \{\pi \in \mathcal{H}_{\mathrm{fin}} \mid \mathrm{last}(\pi) \in V_i\}$ by giving g' appropriate value on histories from some initial position $v_0' \neq v_0$. So in our *antagonistic* two-player setting, it suffices to consider strategies which are winning from single initial positions v_0 and only defined on histories from v_0.

Now, if we are given a game G, a position v in G and some equivalence relation $\sim_i$ on $\mathcal{H}_{\mathrm{fin}}$, then the question whether player i has a *winning $\sim_i$-strategy* for G from v is independent of the partial information of player $1 - i$. Therefore, in this work we investigate games with partial information only for player 0.

We consider games played on finite graphs where player 0 has uncertainties about the positions and actions in the game, modeled by equivalence relations. The relation $\sim_0$ is then obtained by extending these equivalence relations to an equivalence relation on $\mathcal{H}_{\text{fin}}$. In particular, $\sim_0$ is finitely represented which is necessary when considering decision problems for games with partial information. A *parity game with partial information* has the form $\mathcal{G} = (G, \sim^V, \sim^A)$, where $G = (V, V_0, (f_a)_{a\in A}, \text{col})$ is a parity game and $\sim^V \subseteq V \times V$ and $\sim^A \subseteq A \times A$ are equivalence relations such that the following conditions hold:

(1) If $u, v \in V$ with $u \sim^V v$ then $u, v \in V_0$ or $u, v \notin V_0$,
(2) If $a, b \in A_0$ with $a \neq b$ then $a \not\sim^A b$,
(3) If $u, v \in V_0$ with $u \sim^V v$, then $\text{act}(u) = \text{act}(v)$.
(4) If $u, v \in V$ with $u \sim^V v$ then $\text{col}(u) = \text{col}(v)$.

Condition (1) says that player 0 always knows when it is his turn and condition (2) says that player 0 can distinguish all the actions that are available to him at some position of the game. Condition (3) ensures that player 0 always knows which actions are available to him when it is his turn. Finally, condition (4) says that the colors of the game are observable for player 0.

We say that a game $\mathcal{G}$ has *bounded partial information*, if there is some $k \in \mathbb{N}$, such that for any position $v \in V$ the equivalence class $[v] := \{w \in V \mid v \sim^V w\}$ of v has size at most k. Notice that the equivalence classes $[a] := \{b \in A \mid a \sim^A b\}$ of actions $a \in A$ may, however, be arbitrarily large.

The equivalence relation on finite histories is defined as follows. For $\pi = v_0 a_0 \dots a_{n-1} v_n, \rho = w_0 b_0 \dots b_{m-1} w_m \in V(AV)^*$, let

$$\pi \sim^* \rho :\Longleftrightarrow n = m \text{ and } v_j \sim^V w_j \text{ and } a_j \sim^A b_j \text{ for all } j.$$

The winning region $\text{Win}_0^{\mathcal{G}}$ of player 0 in $\mathcal{G}$ is the set of all positions $v \in V$ such that player 0 has a winning $\sim^*$-strategy for $\mathcal{G}$ from v.

Remark 1. Consider the interaction between components of a system where the behavior of each component is prescribed by a controller which has to rely on the information available to this component. In such settings it might seem more appropriate to ask for a $\sim_0^*$-strategy for player 0 which is winning against all $\sim_1^*$-strategies of player 1 rather than a winning $\sim_0^*$-strategy for player 0. However, it is easy to see that in our perfect recall setting, this is equivalent.

Powerset Construction. The usual method to solve games with partial information is a powerset construction originally suggested by John H. Reif in [17]. The construction turns a game with partial information into a *nondeterministic* game with full information such that the existence of winning strategies for player 0 is preserved.

A *nondeterministic parity game* has the form $G = (V, V_0, (E_a)_{a\in A}, \text{col})$ where V, V_0, A, and col are as in a deterministic game and for $a \in A$, E_a is a binary relation on V. Plays, strategies and winning strategies are defined as before.

Nondeterministic games are not determined in general and hence not equivalent to deterministic games. However, for each nondeterministic game G and each player $i \in \{0,1\}$, we can construct a deterministic game G^i such that the existence of winning strategies for player i is preserved. We simply resolve the nondeterminism by giving player $1-i$ control of nondeterministic choices. Technically, for any $v \in V$ and any $a \in \mathrm{act}(v)$ we add a unique a-successor of v to the game graph which belongs to player $1-i$ and from which he can choose any a-successor of v in the original game graph. The coloring of such a new position is the coloring of its unique predecessor. This construction does not increase the complexity of the game graph with respect to any measure we consider here.

Now for a parity game $\mathcal{G} = (G, \sim^V, \sim^A)$, $G = (V, V_0, (f_a)_{a \in A}, \mathrm{col})$ with partial information, we construct the corresponding game $\overline{G} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a \in A}, \overline{\mathrm{col}})$ with full information as follows. First, for $S \subseteq V$ and $B \subseteq A$ we define the set $\mathrm{Post}_B(S) := \{v \in V \mid \exists\, s \in S, \exists\, b \in B \;:\; b \in \mathrm{act}(s) \wedge f_b(s) = v\}$. The components of $\overline{G}$ are defined as follows.

- $\overline{V} = \{\overline{v} \in 2^V \mid \exists v \in V : \overline{v} \subseteq [v]\}$ and $\overline{V}_0 = \overline{V} \cap 2^{V_0}$
- $\forall\, a \in A$: $(\overline{v}, \overline{w}) \in \overline{E}_a :\Longleftrightarrow \exists\, w \in \mathrm{Post}_a(\overline{v})$: $\overline{w} = \mathrm{Post}_{[a]}(\overline{v}) \cap [w]$
- $\mathrm{col}(\overline{v}) = \mathrm{col}(v)$ for some $v \in \overline{v}$.

It can be shown that this construction in fact preserves winning strategies for player 0, that means, for any $v_0 \in V$, player 0 has a winning $\sim^*$-strategy for $\mathcal{G}$ from v_0 if and only if he has a winning strategy for $\overline{G}$ from $\{v_0\}$. So when asking for a winning $\sim^*$-strategy for player 0 from a given position v_0, we are only interested in the part of the graph $\overline{G}$ which is reachable from $\{v_0\}$. We denote this subgraph of $\overline{G}$ by $\overline{G}_{v_0}$. The key-property for the correctness of the construction is given in the following lemma which is proved straightforwardly.

Lemma 2. *For each finite history $\overline{\pi} = \overline{v}_0 a_1 \overline{v}_1 \ldots a_n \overline{v}_n$ in $\overline{G}$ and all $v_n \in \overline{v}_n$, there is a finite history $\pi = v_0 a'_1 v_1 \ldots a'_n v_n$ in $\mathcal{G}$ such that $v_i \in \overline{v}_i$ for all $i \in \{0, \ldots, n\}$ and $a'_i \sim^A a_i$ for all $i \in \{1, \ldots, n\}$.*

Graph Complexity. We consider only directed graphs without multi-edges, but possibly with self-loops, i.e. a graph is a pair $G = (V, E)$ where $E \subseteq V \times V$. An undirected graph is a graph with a symmetric edge relation.

All measures we consider can be characterized in terms of *cops and robber games* (or *graph searching games*), where we have two players, a cop player and a robber player. Basically, the robber player moves a robber token along cop free paths of the graph. The cop player has a number k of cops at his disposal and he can place and move them on and between vertices. At the very moment a cop is moving he does not block any vertex. The goal of the cop player is to place a cop on the robber, the robber player's goal is to elude capture. That means, infinite plays are won by the robber and finite plays, which end in a position where the robber has no legal moves available, are won by the cops. The number k of cops is a parameter of the game, that means, for any natural number k we have a k-cops and robber game. In the following, we give a more formal definition of these games. We start with the game defining DAG-width, as DAG-width is the central measure in our considerations.

DAG-width, introduced in [2,13], is defined via the cops and directed visible robber game (DAG-width game, for short). The DAG-width game on a graph $G = (V, E)$ is defined as follows. For a set $U \subseteq V$ of vertices we say that a vertex u' is reachable from a vertex u in $G - U$ if there is a path

$$u \xrightarrow{E} u^1 \xrightarrow{E} \dots \xrightarrow{E} u^t \xrightarrow{E} u'$$

such that $v^l \notin U$ for all $l \in \{1, \dots, t\}$. For sets $U, U' \subseteq V$, the set of vertices that are reachable in $G - U$ from some vertex in U' is denoted by $\mathrm{Reach}_{G-U}(U')$. Cops' positions are of the form (U, v) where the cops occupy the set U and the robber is on vertex v. The cops can move to any position (U, U', v), which means that the next set of vertices occupied by the cops will be U. (Notice that we could equivalently allow the cops to move at most one cop in each move [2].) The robber's positions are of the form (U, U', v). The robber can move to any position (U', w) with $w \notin U'$ such that w is reachable from v in $G - (U \cap U')$. From the initial position, denoted $\perp$, the robber can go to any position $(\emptyset, v)$. So, a play of the DAG-width game with k cops has the form $\pi = (U_0, v_0)(U_0, U_1, v_0)(U_1, v_1) \dots$ where the initial position $\perp$ is omitted and $U_0 = \emptyset$ such that $|U_i| \leq k$ for all i.

A play is won by the cop player if it finally reaches a position (U, U', v) such that $\mathrm{Reach}_{G-U \cap U'}(v) \subseteq U'$. A play is monotone if it never reaches a position (U, U', v) such that some $u \in U \setminus U'$ is reachable from v in $G - U$. A play is monotonously won by the cop player if it is monotone and won by the cop player. Therefore, the DAG-width game has a reachability winning condition and the monotone DAG-width game has a winning condition which is a conjunction of a reachability and a safety condition. Hence, both games are (positionally) determined. A strategy f for the DAG-width game for the cop player is called monotone, if any play which is consistent with f is monotone. So monotone winning strategies for the cop player for the DAG-width game are precisely the winning strategies for the cop player for the monotone DAG-width game. Notice we have defined monotonicity as robber-monotonicity. There is also the notion of cop-monotonicity, but as for DAG-width these notions coincide [2], this definition is justified. The DAG-width of a graph $G = (V, E)$, denoted $\mathrm{dw}(G)$, is the least natural number k such that k cops have a monotone winning strategy for the DAG-width game on G.

It is easy to see that a cops' strategy only needs to depend on the strongly connected component in which the robber currently is, but not on the particular position within this component. Hence, the cops and robber game can equivalently be described as having positions of the form (U, R) and (U, U', R) where now, R is a strongly conntected component of $G - U$. From a position (U, U', R), the robber can move to any position (U', R') such that R' is a strongly connected component of $G - U'$ which is reachable from R in $G - (U \cap U')$.

Tree-width, see [18], is a measure defined for undirected graphs which has been introduced a long time before DAG-width. However, the tree-width of a directed graph $G = (V, E)$, denoted $\mathrm{tw}(G)$, can be straightforwardly defined as $\mathrm{tw}(G) = \mathrm{dw}(\overleftrightarrow{G}) - 1$ with $\overleftrightarrow{G} = (V, \overleftrightarrow{E})$, where $\overleftrightarrow{E}$ is the symmetric closure of E. (The -1 in the definition is due to the fact, that the original definition of tree-width is customized to make trees having tree-widh 1.)

The *directed path-width* of a directed graph $G = (V, E)$, denoted $\mathrm{dpw}(G)$, is the minimum natural number k such that $k + 1$ cops have a monotone winning strategy for the DAG-width game on G against an *invisible* robber. So, in this case the cops and robber game is a game with partial information for the cops. Formally, a strategy f for the cop player has to satisfy the following condition. If $\pi = \perp (\emptyset, r_0)\ ((\emptyset, U_0), r_0)\ (U_0, r_1)\ \ldots ((U_i, U_{i+1}), r_{i+1})$ and $\pi' = \perp (\emptyset, r'_0)\ ((\emptyset, U_0), r'_0)\ (U_0, r'_1)\ \ldots ((U_i, U_{i+1}), r'_{i+1})$ are finite histories of the DAG-width game on G, then $f(\pi) = f(\pi')$.

This implies that all the cops know about the possible positions of the robber is what they can deduce from their own actions, namely that the robber is not on some position of the graph which has already been searched and has henceforth been blocked for the robber. Therefore, positions in this game can be represented as $((U, U'), R)$, where $R \subseteq V$ is a set of positions such that if the cops move from this position to a position $((U', U''), R')$, then $R' = \mathrm{Reach}_{G-U\cap U'}(R) \cap (V \setminus U')$. (The difference to DAG-width is that here, R' is a *union* of *weakly* connected components of $G - (U \cap U')$.) Basically, this representation is obtained by performing the powerset construction on the original game and then choosing a succinct representation of the positions. Notice that in this representation the moves of the robber player are omitted since they do not change the current position. Strategies can be translated from one representation of the game to the other in the obvious way. Now, given a strategy f for the cop player, we obtain a *unique* play of the game in this new representation of the game which is compatible which f, namely the play $\pi_f = ((U_0, U_0), R_0)\ ((U_0, U_1), R_1)\ ((U_1, U_2), R_2) \ldots$ with $U_0 = \emptyset$, $R_0 = V$, $U_{i+1} = f(\pi(\leq i))$ and $R_{i+1} = \mathrm{Reach}_{G-U_i\cap U_{i+1}}(R_i) \setminus U_{i+1}$. In fact, this is a one player game.

Obviously, $\mathrm{dw}(G) \leq \mathrm{dpw}(G) + 1$ for any graph G, so in particular parity games with full information can be solved in polynomial time on graphs with bounded directed path-width. Moreover it is not hard to see, that the directed path-width is not bounded by the DAG-width, that means, there is a class of directed graphs such that the DAG-width is bounded and the directed path-width is unbounded on this class.

Finally, in the *entanglement* game [4], in each position, the robber is on a vertex r of the graph. In each round, the cop player may do nothing or place a cop on r, either from outside the graph if there are any cops left or from a vertex v which was previously occupied by a cop and is then freed. No matter what the cops do, the robber must go from his recent vertex r to a new vertex r', which is not occupied by a cop, along an edge $(r, r') \in E$. If the robber cannot move, he loses. So formally, a position of the entanglement game on G is a tuple (U, r) if it is the cops' turn or, if it is the robber's turn, a tuple (U, U', r)

with $U' = (U \setminus \{v\}) \cup \{r\}$ for some $v \in U$ (the cop is coming from v to r) or $U' = U \cup \{r\}$ (a new cop from outside is coming to r). From (U, r) the cops can move to a position of the form $((U, U'), r')$. On his turn, the robber can move from $((U, U'), r)$ to a position (U', r') where $(r, r') \in E$ and $r' \notin U'$. The entanglement of a graph G, denoted $\mathrm{ent}(G)$ is the minimal number k such that k cops win the entanglement game on G.

DAG-width, tree-width and directed path-width are defined in terms of monotone winning strategies. A monotone winning strategy for k cops on G yields a decomposition of G into (possibly complex) parts of size at most k which are only sparsely related among each other. (The particular measure determines what sparesely precisely means.) Such decompositions often allow for efficient dynamic solutions of hard graph problems.

On the other hand, entanglement is defined in terms of strategies which are not necessarily monotone and only for $k = 2$, a decomposition in the above sense is known [6]. Nevertheless, parity games can be solved efficiently on graph classes of bounded entanglement.

In the following, let $\mathcal{M} = \{\mathrm{tw}, \mathrm{dw}, \mathrm{dpw}, \mathrm{ent}\}$. We say that a measure $X \in \mathcal{M}$ has *monotonicity cost* at most f for a function $f : \mathbb{N} \to \mathbb{N}$ if, for any graph G such that k cops have a winning strategy for the X-game on G, $k + f(k)$ cops have a monotone winning strategy for the X-game on G. We say that X has *bounded monotonicity cost* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that X has monotonicity cost at most f. Tree-width has monotonicity cost 0 [18] and the same holds for directed path-width, [1,7]. On the contrary, DAG-width does not have monotonicity cost 0: there is a class of graphs G_n, such that $3n - 1$ cops have a winning strategy on G_n, but $\mathrm{dw}(G_n) = 4n - 2$, [11]. Whether DAG-width has bounded monotonicity cost, is an important open problem in the structure theory of directed graphs.

3 Unbounded Partial Information

First, when the partial information is unbounded, it is easy to prove that boundedness of graph complexity measures is not preserved by the powerset construction and does not prevent the size of the graph to grow exponentially. We show that even the measures themsleves grow exponentially.

Before we prove the first result of this section we note the well-known fact that, for any $n \in \mathbb{N}$, we have $X(\mathfrak{G}_\mathfrak{n}) \geq n$ for all $X \in \mathcal{M}$, where $\mathfrak{G}_n$ is full, undirected $n \times n$-grid defined as follows: $\mathfrak{G}_n = (V_n, E_n)$ with $V_n = \{(i, j) \mid 1 \leq i, j \leq n\}$ and $\big((i_1, j_1), (i_2, j_2)\big) \in E \Leftrightarrow i_1 = i_2$ and $|j_1 - j_2| = 1$, or $j_1 = j_2$ and $|i_1 - i_2| = 1$.

Proposition 3. *There are games $\mathcal{G}_n$, $n \in \mathbb{N}$ with partial information and with $X(\mathcal{G}_n) \leq 2$ for all $n \in \mathbb{N}$ and for any $X \in \mathcal{M}$, such that the powerset graphs $\overline{G}_n$ have exponential measure X in the size of $\mathcal{G}_n$ for any $X \in \mathcal{M}$.*

Proof. From a fairly simple graph, we generate a graph containing an undirected square grid of exponential size as a subgraph. This is possible because we can consider large equivalence classes of positions and actions.

Consider a disjoint union of n directed cycles of length 2 with self-loops on each vertex where any two positions are equivalent. Additionally we have an initial position such that, by applying the powerset construction from this position, we obtain a set which contains exactly one element from each cycle. Continueing, we get sets that represent binary numbers with n digits and for each digit we have an action which causes exactly this digit to flip. So, using the Gray-code, we can create all binary numbers with n digits by successively flipping each digit. If we do this independently for the first $n/2$ digits and for the last $n/2$ digits, it is easy to see that the resulting positions are connected in such a way, that they form an undirected grid $\overline{G}_n$ of size $2^{n/2} \times 2^{n/2}$, for which we have $X(\overline{G}_n) \geq 2^{n/2}$ for any measure $X \in \mathcal{M}$.

To be more precise, for even $n < \omega$, let $\mathcal{G}_n = (G_n, \sim_n^V, \sim_n^A)$, where $G_n = (V_n, \emptyset, (f_a^n)_{a \in A_n})$ is the following game graph. The set of vertices is $\{v_0\} \cup \{(i,j) \mid j \in \{0,1\}, 1 \leq i \leq n\}$ where i stands for the number of the cycle and j for the number of a vertex in the cycle. The actions are $A_n = \{a_i \mid 1 \leq i \leq n\} \cup \{\neg_i \mid 1 \leq i \leq n\}$. Here the actions a_i lead from v_0 to the cycles: we have $v_0 \xrightarrow{a_i} (0, i)$ for $1 \leq i \leq n$. Further actions make the cycles:

- $(i,j) \xrightarrow{\neg_i} (i, 1-j)$ for $1 \leq i \leq n$ and $j \in \{0,1\}$.
- $(i,j) \xrightarrow{\neg_k} (i,j)$ for $1 \leq i \leq n$ with $k \neq i$ and $j \in \{0,1\}$.

Partial information is defined by $(i,j) \sim_n^V (k,l)$ and $a_i \sim_n^A a_k$ for any $1 \leq i,j,k,l \leq n$. So each two positions from any two cycles are indistinguishable and each two of the actions a_σ^i are indistinguishable. It is clear that $X(G_n) \leq 3$ for any measure $X \in \mathcal{M}$.

In Figure 1, the graph $\mathcal{G}_2$ and the powerset graph $\overline{G}_{v_0}^n$ are depicted. The position $\{v_0\}$ of the powerset graph is omitted and a position (i,j) is represented as j. Positions $\{j_1, j_2\}$ are denoted $j_1 j_2$.

Now, performing the powerset construction on $\mathcal{G}_n$ from v_0 we obtain the graph $\overline{G}_{v_0}^n$ which obviously contains the position $\{(1,0), \ldots, (n,0)\}$. From this position, an undirected square grid of exponential size is constructed as follows. We successively apply actions $\neg_i$ for $i \in \{1, \ldots, n/2\}$ to create each vertex $\{(1, j_1), \ldots, (n/2, j_{n/2}), (1,0), \ldots, (n,0)\}$ with $j_1, \ldots, j_{n/2} \in \{0,1\}$. In each step we can change exactly one j_r to $1 - j_r$, so the creation of all these vertices from $\{(1,0), \ldots, (n/2, 0), (n/2+1, 0), \ldots, (n,0)\}$ can, for instance, be done using the usual Gray-code for binary numbers: we get the next vertex by applying $\neg_i$ to the previous vertex $\{(1, j_1), \ldots, (n/2, j_{n/2}), (n/2+1, 0), \ldots, (n, 0)\}$, which changes exactly one position (i, j_i). This undirected path forms the upper horizontal side of the grid. Analogously, by successively applying the actions $\neg^i$ for $i \in \{n/2+1, \ldots, n\}$ we can create each vertex $\{(1,0), \ldots, (n/2, 0), (n/2+1, j_{n/2+1}), \ldots, (n, j_n)\}$ with $j_{n/2+1}, \ldots, j_n \in \{0,1\}$ using the Gray-code. This undirected path forms the left vertical side of the grid. (Of course, terms like *left* and *horizontal* are used here only for convenience, they do not have any mathematical meaning in this context.)

Likewise, given any vertex $\{(1, j_1), \ldots, (n/2, j_{n/2}), (n/2+1, 0), \ldots, (n, 0)\}$ we can create any vertex $\{(1, j_1), \ldots, (n/2, j_{n/2}), (n/2+1, j_{n/2+1}), \ldots, (n, j_n)\}$ by successively applying the actions $\neg_i$ for $i \in \{n/2+1, \ldots, n\}$ in the same order as before and given any vertex $\{(1, 0), \ldots, (n/2, 0), (n/2+1, j_{n/2+1}), \ldots, (n, j_n)\}$, by successively applying the actions $\neg_i$ for $i \in \{1, \ldots, n/2\}$, we can create any vertex $\{(1, j_1), \ldots, (n/2, j_{n/2}), (n/2+1, j_{n/2+1}), \ldots, (n, j_n)\}$. All these paths form a $2^{n/2} \times 2^{n/2}$-grid and therefore, the tree-width of $\overline{G}^n_{v_0}$ is exponential in the size of $\mathcal{G}_n$. Furthermore, using that $\overline{G}^n_{v_0}$ is undirected one easily checks that for all $X \in \mathcal{M}$, $X(\overline{G}^n_{v_0}) \geq \mathrm{tw}(\overline{G}^n_{v_0})$. □

Remark 4. Notice that exponential size of the resulting graph is not needed for unbounded growth of graph complexity measures. If we consider, for example, a disjoint union of *two* undirected paths of length n with appropriate actions and self-loops on all positions, then the construction of the corresponding powerset graph yields an $n \times n$-grid.

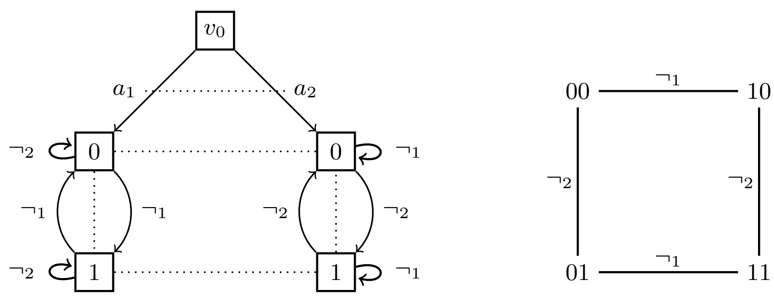

Fig. 1. Game graph $\mathcal{G}_2$ and the powerset graph $\overline{G}^2_{v_0}$.

Towards our analysis of the complexity of the strategy problem for games with partial information on graphs of bounded complexity, we first note that on trees, solving games with partial information is not harder than solving games with full information. Performing the powerset construction on a tree, we again obtain a tree, where the set of positions on each level partitions the set of positions on the corresponding level of the original tree. This new tree can therefore be computed in polynomial time and has at most as many vertices as the original tree. In the following results we prove that at soon as we consider at least DAGs which are not trees, the strategy problem for reachability games becomes intractable as long as we do not bound any other parameters.

For the proofs of the subsequent results, we need the following facts, see for example [20].

Lemma 5.

(1) APSPACE = EXPTIME.
(2) *For all* $L \in$ ASPACE$(S(n))$ *with* $S(n) \geq n$ *there is an alternating Turing machine with a single tape and space bound* $S(n)$ *which accepts* L.
(3) APTIME = PSPACE.
(4) *For all* $L \in$ ATIME$(T(n))$ *with* $T(n) \geq n$ *there is an alternating Turing machine with a single tape and time bound* $O(T^2(n))$ *which accepts* L.

Theorem 6. *The following problem is* EXPTIME*-hard. Given a partial information reachability game* $\mathcal{G} = (G, \sim^V, \sim^A)$ *with* $\text{ent}(G) \leq 2$ *and* $\text{dpw}(G) \leq 3$ *and a position* $v_0 \in V(G)$*, is* $v_0 \in \text{Win}_0^{\mathcal{G}}$*?*

Proof. By Lemma 5, for any $L \in$ EXPTIME, there is an alternating Turing-machine $M = (Q, \Gamma, \Sigma \supseteq \Gamma, q_0, \delta)$ with only one tape and space bound n^k for some $k \in \mathbb{N}$, where n is the size of the input, that recognizes L. As usual, Q is the set of states, Γ and Δ are the input and the tape alphabets, q_0 is the initial state, and δ is the transition relation. First assume that M is deterministic. We describe the necessary changes to prove the general case afterwards.

Let $\Delta = \Sigma \uplus (Q \times \Sigma) \uplus \{\#\}$. Then each configuration C of M is described by a word $C = \#w_0 \ldots w_{i-1}(qw_i)w_{i+1} \ldots w_t\# \in \Delta^*$ over Δ and since M has space bound n^k and we have $k \geq 1$, w.l.o.g. we can assume that $|C| = n^k + 2$ for all configurations C of M on inputs of length n. Moreover, for a configuration C of M and some $2 \leq i \leq n^k + 1$ the symbol number i of C' where $C' = \text{Next}(C)$ only depends on the symbols number $i-1$, i and $i+1$ of C. So there is a function $f : \Delta^3 \to \Delta$ such that for any configuration C of M and any $2 \leq i \leq n^k + 1$, if the symbols number $i-1$, i and $i+1$ of C are $a_{-1}a_0a_1$ then the symbol number i of the successor configuration $C' = \text{Next}(C)$ of C is $f(a_{-1}a_0a_1)$.

Now let $u = u_1 \ldots u_n \in \Gamma^*$. The idea for the game corresponding to u is the following. Player 0 selects symbols from Δ, such that the sequence constructed in this way forms an accepting run of M on u. In order to check the correctness of the construction that player 0 provides, player 1 may, at any point during the play but only *once*, memorize the recent position $i \in \{1, \ldots, n^k\}$ within the recent configuration and the last three symbols chosen by player 0. Then, in the next configuration, player 1 may check the i-th symbol chosen by player 0 to be correct according to the symbols which he has previously memorized and the function f. If the i-th symbol proves incorrect, player 0 loses, otherwise, player 1 loses. Player 0 must not notice when player 1 memorizes the recent position, which defines the partial information in the game. To justify the bounds on the graph complexity measures that we have claimed, we define the game formally.

We define the game $\mathcal{G}_u = (G, \sim^V, \sim^A)$ with partial information as follows. The set of positions is $V = \{v_0\} \cup \{0,1\} \times \Delta \times \{0, \ldots, n^k\} \times Q \times \{0, \ldots, n^k\} \times \Delta^3$, so a position has the form $(\sigma, \delta, i, q, j, \delta_1\delta_2\delta_3)$ where σ is the player whose turn it is, δ is the recent symbol as chosen by player 0 and i is the recent position within the recent configuration. Moreover, q is the last state $q \in Q$ chosen by player 0. Finally, j and the sequence $\delta_1\delta_2\delta_3$ represent the information which player 1 has memorized. Now we give a complete list of the moves that can be made in the game. For convenience, the actions are omitted in the description. The player whose turn it is, is given by the first component of a position except for position v_0 which belongs to player 1. Moreover, for an n-tuple $\overline{x} = (x_1, \ldots, x_n)$ we denote x_i by $\mathrm{pr}_i(\overline{x})$. The possible moves are:

- **from** v_0 **to** $(0, \#, 0, q_0, j, \delta_1\delta_2\delta_3)$ where $j \neq \emptyset$ and $\delta_1\delta_2\delta_3$ are symbols number $j-1$, j and $j+1$ of the initial configuration $C_{\mathrm{in}}(u)$ of M on u
- **from** v_0 **to** $(0, \#, 0, q_0, 0, \#\#\#)$
- **from** $(0, \delta, i, q, 0, \delta_1\delta_2\delta)$ with $i \leq n^k$ **to** $(1, \delta', i+1, q', 0, \delta_2\delta\delta')$ where $\delta' \in \Delta \setminus \{\#\}$ and $q' = q$ if $\delta' \notin Q \times \Sigma$ and $q' = \mathrm{pr}_1(\delta')$ if $\delta' \in Q \times \Sigma$
- **from** $(0, \delta, n^k+1, q, 0, \delta_1\delta_2\delta)$ **to** $(1, \#, n^k+2, q, 0, \delta_2\delta\#)$
- **from** $(0, \delta, i, q, j, \delta_1\delta_2\delta_3)$ with $i \neq n^k+1$ and $j \neq 0$ **to** $(1, \delta', i+1, q, j, \delta_1\delta_2\delta_3)$ where $\delta' \in \Delta \setminus \{\#\}$
- **from** $(0, \delta, n^k+1, q, j, \delta_1\delta_2\delta_3)$ with $j \neq 0$ **to** $(0, \#, n^k+2, q, j, \delta_1\delta_2\delta_3)$.
- **from** $(1, \delta, i, q, j, \delta_1\delta_2\delta_3)$ with $i \neq j$ **to** $(0, \delta, i, q, j, \delta_1\delta_2\delta_3)$.
- **from** $(1, \delta, i, q, 0, \delta_1\delta_2\delta)$ **to** $(0, \delta, i, q, i-1, \delta_1\delta_2\delta)$, if $i \geq 3$.
- **from** $(1, \#, n^k+2, q, 0, \delta_1\delta_2\#)$ with $q \notin Q_{\mathrm{acc}} \cup Q_{\mathrm{rej}}$, **to** $(0, \#, 0, 0, \#\#\#)$.
- **from** $(1, \delta, n^k+2, q, j, \delta_1\delta_2\delta_3)$ with $j \neq 0$ **to** $(0, \#, 0, j, \delta_1\delta_2\delta_3)$.

Of a position $(\sigma, \delta, i, q, j, \delta_1\delta_2\delta_3)$, only the first three entries are visible to player 0, that means, two positions are indistinguishable if and only if the they coincide in the first three components. Which amounts exactly to the claim that player 0 is never aware whether his construction is checked by player 1. Moreover, any two actions of player 1 are indistinguishable for player 0. Now, at positions $(1, \delta, i, q, i, \delta_1\delta_2\delta_3)$, player 0 has won if $f(\delta_1\delta_2\delta_3) = \delta$, otherwise, player 1 has won. At a position $(1, \#, n^k+2, q, 0, \delta_1\delta_2\#)$ with $q \in Q_{\mathrm{acc}} \cup Q_{\mathrm{rej}}$, player 0 has won if $q \in Q_{\mathrm{acc}}$ and player 1 has won, if $q \in Q_{\mathrm{rej}}$.

Structurally, G consists of $|\Delta|^3 \cdot n^k + 1$ augmented DAGs t_0 and $t_j^{\overline{\delta}}$, $j = 1, \ldots, n^k$, $\overline{\delta} \in \Delta^3$. Each $t_j^{\overline{\delta}}$ has a unique top node, edges from any non-bottom level only to the level below and $2 \cdot n^k + 1$ levels in total. For a fixed number $j \in \{1, \ldots, n^k\}$, we refer to the union of the $t_j^{\overline{\delta}}$ for $\overline{\delta} \in \Delta^3$ by t_j. We also have a unique root node v_0 for the whole graph from which there is an edge to the top of t_0 and, for any $0 < j \leq n^k$, there is an edge to exactly one top node of t_j. Moreover, for any $0 \leq j \leq n^k$ and any $\overline{\delta} \in \Delta^3$, from any node at level $2 \cdot n^k + 1$ of t_j, there is a back-edge to the top of t_j (which is the only cyclicity in the graph). Finally, on the i-th level of t_i for $i \geq 1$, there are no outgoing edges. So obviously $\mathrm{ent}(G) \leq 1$ and $\mathrm{dpw}(G) \leq 2$. (Notice that we are still considering the special case where M is deterministic.)

Now, as long as player 1 has not yet decided to memorize, the play takes place in t_0 and player 1 keeps track of the information which he needs in case he decides to memorize. If he decides to memorize at position i within some configuration, then this means, that he wants to check the character at position $i-1$ within the next configuration, given the characters $i-2$, $i-1$ and i of the current configuration, so he switches to the corresponding position in t_{i-1}. As we have already mentioned, player 0 never notices whether player 1 leaves t_0 or not. If player 1 lets player 0 write down characters until some character (q, a) with $q \in Q_{\mathrm{acc}} \cup Q_{\mathrm{rej}}$ is written, then the winner is determined according to the state q. (Notice that player 0 finally has to write a character (q, a) with $q \in Q_{\mathrm{acc}} \cup Q_{\mathrm{rej}}$ since he has a reachability objective.) If player 1 wants to check player 0's construction, then he can decide to do this at exactly one point during a play by moving to some t_i with $i \neq 0$ as mentioned above. If the character he wants to check is incorrect he wins, otherwise he loses.

Obviously, $\mathcal{G}$ can be constructed from a given input $u \in \Gamma^*$ in polynomial time. If the word u is accepted by M, then clearly player 0 wins the game from v_0 by simply writing down the run of M on u character by character. Now let conversely f be a winning strategy for player 0 for $\mathcal{G}$ from v_0 and let $\overline{\delta} = \delta_1\delta_2 \ldots \delta_k$ be the sequence of characters given by player 0 according to f, if player 1 plays in t_0 all the time. Assume, $\overline{\delta}$ does not represent the unique run of M on u. Then there is some $i < k$, such that, up to position i, $\overline{\delta}$ coincides with the unique run of M on u, but up to position $i+1$ it does not. So, let δ_{i+1} be the l-th position within the recent configuration. We modify the play as follows. During the construction of the previous configuration (or from v_0, if the previous configuration is the initial configuration), player 1 chooses a position of the form $(0, \delta, i+2, q, i+1, \delta_1\delta_2\delta)$, i.e. he memorizes at position $i+2$. By our assumption on $\overline{\delta}$, the resulting play is lost by player 0. However, since player 0 does not notice that player 1 memorizes and f is a partial information strategy, the resulting play is compatible with f in contradiction to the fact that f is a winning strategy for player 0 for $\mathcal{G}$ from v_0. So, in contrary to our assumption, $\overline{\delta}$ represents the unique run of M on u. Due to the definition of the winning condition of $\mathcal{G}$ this run must be accepting, i.e. $u \in L(M)$.

Now consider the general case, where M is not necessarily deterministic. W.l.o.g. we can assume that each non-terminal configuration of M has exactly two successor configurations. If there is a configuration C with just a single successor configuration then we add a default successor to C which leads to acceptance if C is universal and which leads to rejectance if C is existential. If there is a configuration with $b > 2$ successors, then we replace this b-branching configuration tree by a binary branching configuration tree of depth b by modifying the transition function of M in an appropriate way. Obviously, this construction can be done in such a way that it merely increases the state space of M and the time bound by a constant factor, but not the space bound. Now, instead of one function f, we have two functions$f_1, f_2 : \Delta^3 \to \Delta$, such that the following holds. If C is a configuration of M, $l \in \{1, 2\}$ and $2 \leq i \leq n^k + 1$, and the symbols number $i-1$, i and $i+1$ of C are $a_{-1}a_0a_1$ then the symbol number i of the successor configuration $C_l = \mathrm{Next}_l(C)$ number l of C is $f_l(a_{-1}a_0a_1)$. Thus, for each $j \in \{0, \ldots n^k\}$ we use two copies t_j^1 and t_j^2 of t_j. From v_0, for all $l \in \{0, 1\}$, an edge to exactly one top node of t_j^l exists. At a leaf node of t_j^l, if the recent configuration is existential (as determined by the recent state) then player 0 chooses whether to proceed at a top node of t_j^1 or of t_j^2. If the recent configuration if universal, then player 1 makes this choice. (Notice that the particular top node of t_i^l which is chosen is determined by the recent position, it is merely the l which is chosen by one of the players.) Partial information is defined as before with the additional condition that player 0 observes the copy of t_j in which the play currently takes place. Now, for $i > 0$ and $l \in \{1, 2\}$, in t_i^l the correctness of the construction player 0 provides is checked using the function f_l. The trick which player 1 uses to find the flaw in the construction if M does not accept some input u is exactly the same as before. Clearly these modifications merely increase the entanglement of the graph from at most 1 to at most 2. and the directed path-width from at most 2 to at most 3. □

Remark 7. It easy to see, that the tree-width of the game graphs constructed in the proof of Theorem 6 is bounded by some $k \in \mathbb{N}$ which is independent of the input u. Therefore, the strategy problem for reachability games with partial information on graphs of tree-width at most k is EXPTIME-hard.

Remark 8. Notice that the graph which we have constructed in the proof of Theorem 6 is not strongly connected and that partial information cuts through different strongly connected components. However, to make the graphs strongly connected it suffices to connect each position of player 1 via an undirected edge with some dummy position $\Diamond$ which belongs to player 0 and from which he can choose to go to a terminal position of player 1 immediately. This merely increases both entanglement and directed path width by just 1 and obviously does not harm the correctness of the construction.

The cases of entanglement and directed path-width at most 1 are still open for reachability games, while they are solved for sequence-forcing games. A sequence-forcing condition has the form (S, col) where $\mathrm{col} : V \to C$ is a coloring of V and $S \subseteq \{1, \ldots, r\}^k$ is a set of sequences of length k for some $k < \omega$. Player 0 wins an infinite play π of a sequence-forcing game if for some $i < \omega$ we have $\mathrm{col}(\pi(i))\,\mathrm{col}(\pi(i+1)) \ldots \mathrm{col}(\pi(i+k)) \in S$. It is not hard to see that if k is fixed, sequence-forcing games can be polynomially reduced to reachability games by using a memory which stores the last k colors that have occurred. (Notice that this reduction may, however, increase the complexity of the game graph.) In particular, the strategy problem for sequence-forcing games with fixed k is in PTIME. On the other side, the strategy problem for sequence-forcing games with partial information is EXPTIME-hard on graphs of entanglement and directed path-width at most 1, even for $k = 3$. This yields, roughly speaking, the following result.

Theorem 9. *Adding partial information to games played on graphs of entanglement and directed path-width at most* 1 *can cause an unavoidable exponential blow-up of the time complexity of the corresponding strategy problem.*

Proof. We modify the proof of Theorem 6 as follows. From the nodes on level $2 \cdot n^k + 1$ of t_j^1 and t_j^2 we do not allow moves directly back to the top of $(t_j^{\overline{\delta}})^1$ or $(t_j^{\overline{\delta}})^2$, but we redirect all edges to a single position $(\overline{\delta}, j)$, which belongs to player 1. From this position, player 1 may move to position $(0, \overline{\delta}, j)$ which belongs to player 0 or to position $(1, \overline{\delta}, j)$ which belongs to player 1. Furthermore, from $(0, \overline{\delta}, j)$, player 0 chooses whether to proceed in $(t_j^{\overline{\delta}})^1$ or in $(t_j^{\overline{\delta}})^2$ and from $(1, \overline{\delta}, j)$ player 1 makes this choice. Partial information is defined as before, so all the positions $(\overline{\delta}, j)$ are indistinguishable for player 0 and two positions $(\sigma, \overline{\delta}, j)$ and $(\sigma', \overline{\delta}', j')$ are distinguishable for player 0 if and only if $\sigma \neq \sigma'$. The coloring of the positions is defined as follows. The nodes on level $2 \cdot n^k + 1$ of t_j^1 and t_j^2 are colored with 0, if the recent configuration is existential (as determined by the recent state) and with 1, if the recent configuration if universal. Each position $(\overline{\delta}, j)$ is colored with 0, the positions $(0, \overline{\delta}, j)$ get the color -1 and the positions $(1, \overline{\delta}, j)$ get the color 1. All other positions are colored with 0. Now, $S = \{(0, 0, 1)\}$, that means, the unique sequence that player 0 wants to enforce is $(0, 0, 1)$. This forces player 1 into giving control back to player 0 if the last configuration that player 0 constructed has been existential. Now the proof of Theorem 6 carries over without essential modifications, showing that the strategy problem for sequence-forcing games, played on graphs of entanglement and directed path-width at most 1 is EXPTIME-hard. ◻

Finally, if the we consider acyclic game graphs, the strategy problem for partial information reachability games is PSPACE-complete. Notice that acyclic graphs are precisely those having DAG-width 1.

Theorem 10. *The strategy problem for reachability games with partial information on acyclic graphs is* PSPACE*-complete.*

Proof. Let the given game be $\mathcal{G} = (G, \sim^V, \sim^A)$ and let v_0 be the initial position. First we prove the membership in PSPACE. The idea is that carrying out the powerset construction on an acyclic graph G we again obtain an acyclic graph $\overline{G}$ where by Lemma 2, the paths in $\overline{G}$ are not longer than the paths in G, so we can solve the reachability game on $\overline{G}$ by an APTIME algorithm. We describe this algorithm informally, claiming that its correctness is obvious. Starting from $\{v_0\}$, we proceed as follows. Given a position $\overline{v} \in \overline{V}$ in the corresponding game $\overline{G}_{v_0}$ with full information, if $\overline{v} \in \overline{V}_0$, then $\exists$ guesses a successor of $\overline{v}$ and if $\overline{v} \in \overline{V}_1$, then $\forall$ chooses a successor position of $\overline{v}$. If the computation reaches a leaf-node in $\overline{V}_1$ the algorithm accepts and if the computation reaches a leaf-node in $\overline{V}_0$ the algorithm rejects. The construction of a successor position of some position $\overline{v}$ can obviously be done in polynomial time. Moreover, if $\overline{\pi} = \overline{v}_0 \to \overline{v}_1 \to \ldots \to \overline{v}_k$ is any path in $\overline{G}_{v_0}$ then, according to Lemma 2, there is a path $\pi = v_0 \to v_1 \to \ldots \to v_k$ with $v_i \in \overline{v}_i$ for $i = 0, \ldots, k$. Since G is acyclic, $k \leq n$. So, the computation stops after at most n steps.

Conversely, if $L \in$ PSPACE, then, according to Lemma 5, there is an alternating Turing machine $M = (Q, \Gamma, \Sigma \supseteq \Gamma, q_0, \delta)$ with only one tape and time bound n^k for some $k \in \mathbb{N}$ that recognizes L. Now we use the same construction as in the proof of Theorem 6. Since M has time bound n^k and only a single tape, M has obviously space bound n^k. So we can describe configurations of M in the very same way as in the proof of Theorem 6 and we can construct a game with positions as before. However, the essential difference here is that at a position $(1, i, q, j, \overline{\delta})$ with $i = n^k + 2$, the next move does not lead back to the top of $t_j^{\overline{\delta}}$, but it leads to the root of a new copy of $t_j^{\overline{\delta}}$. If some input u is accepted by M, then player 0 can prove this by constructing at most $|u|^k$ configurations, so winning strategies carry over between the game constructed in the proof of Theorem 6 and the game constructed here in the obvious way. Moreover, since the graph we have constructed is acyclic by definition, the proof is finished. □

4 Bounded Partial Information

We turn to the case where the size of the equivalence classes of positions is bounded. The first observation is that bounded tree-width may become unbounded when applying the powerset construction. Afterwards we shall see, that the same result holds for entanglement.

Proposition 11. *There are games $\mathcal{G}_n$, $n \in \mathbb{N}$ with bounded partial information and $X(\mathcal{G}_n) \leq 3$ for all $n \in \mathbb{N}$ and any $X \in \mathcal{M}$ such that the corresponding powerset graphs $\overline{G}_n$ have unbounded tree-width.*

Proof. As a first step consider partial grids, see the second graph on Figure 2. For an even $n < \omega$, let $\mathfrak{G}_n^{1/2}$ be obtained from the full, undirected $n \times n$-grid $\mathfrak{G}_n$ as follows. On each odd horizontal level number h, $h = 1, 3, \dots, n-1$ we delete each even vertical edge $(i, h) \longleftrightarrow (i, h+1)$, $i = 2, 4, \dots, n$ and on each even horizontal level number h, $h = 2, 4, \dots, n-2$ we delete each odd vertical edge $(i, h) \longleftrightarrow (i, h+1)$, $i = 1, 3, \dots, n-1$. So altogether we have $\mathfrak{G}_n^{1/2} = (V_n, E_n^{1/2})$ with $(i, j) \overset{E}{\longleftrightarrow} (i+1, j)$ for all $1 \leq i \leq n-1$ and all $1 \leq j \leq n$ and $(i, j) \overset{E}{\longleftrightarrow} (i, j+1)$ if and only if i and j are both odd or i and j are both even. Similar as for full grids, $\text{tw}(\mathfrak{G}_n^{1/2}) = n/2$: it is easy to see that $n/2+1$ cops have a winning strategy for the tree-width game on G, so $\text{tw}(\mathfrak{G}_n^{1/2}) \leq n/2$. Moreover, for each *odd* $1 \leq h \leq n$ let $H_h := \{(i, h), (i, h+1) \,|\, 1 \leq i \leq n\}$ and for each *odd* $1 \leq v \leq n$ let $V_v := \{(v, j), (v+1, j) \,|\, 1 \leq j \leq n\}$. Then the set $\mathcal{B} = \{B_{h,v} \,|\, 1 \leq h, v \leq n,$ h, v odd $\}$ with $B_{h,v} = H_h \cup V_v$ is obviously a bramble in $\mathfrak{G}_n^{1/2}$ and for any set $U \subseteq V$ of at most $n/2 - 1$ vertices we have $U \cap B = \emptyset$ for least one $B \in \mathcal{B}$. So $\mathcal{B}$ has order at least $n/2$ which shows that $\text{tw}(\mathfrak{G}_n^{1/2}) \geq n/2$.

Now we define a class of graphs G_n such that the powerset construction converts them to partial grids, see Figure 2. For any even natural number $0 < n < \omega$ let $\mathcal{G}_n = (G_n, \sim_n^V, \sim_n^A)$ where $G_n = (V_n, \emptyset, (f_a^n)_{a \in A_n})$ is the following game graph:

- $V_n = \{v_0\} \cup \{(i, j) \,|\, 1 \leq i, j \leq n\}$,
- $A_n = \{a_{i,j} \,|\, 1 \leq i, j \leq n\} \cup \{\leftarrow, \rightarrow\}$,
- $v_0 \overset{a_{ij}}{\longleftrightarrow} (i, j)$ for $1 \leq i, j \leq n$,
- $(i, j) \overset{\rightarrow}{\longrightarrow} (i+1, j)$ and $(i+1, j) \overset{\leftarrow}{\longrightarrow} (i, j)$ for $1 \leq i \leq n-1$ and $1 \leq j \leq n$.

So G_n is a union of n undirected paths, each of length n, together with the root v_0 which has a directed edge to each position (i, j). Obviously, for any measure X we have $X(G_n) \leq 2$.

Partial information is defined as follows. If $i, j \in \{1, \ldots, n\}$ are both odd, then $(i+1, j) \sim_n^V (i+1, j+1)$ and $a_{i+1,j} \sim_n^A a_{i+1,j+1}$ and if $i, j \in \{1, \ldots, n\}$ are both even, then $(i-1, j) \sim_n^V (i-1, j+1)$ and $a_{i-1,j} \sim_n^A a_{i-1,j+1}$. Notice that with this definition, each equivalence class (of positions as well as of actions) has size at most 2. Moreover, $\{(1, 1)\}$ forms a singleton $\sim_n^V$-equivalence class.

Now, performing the powerset construction on $\mathcal{G}_n$ from v_0, we obtain the graph $\overline{G}_{v_0}^n$ which contains an isomorphic copy of G_n as a subgraph, where each position (i, j) is replaced by $\{(i, j)\}$. Moreover, for any odd numbers $i, j \in \{1, \ldots, n\}$ we have the position $\{(i+1, j), (i+1, j+1)\}$ from which there are edges to $\{(i, j)\}$ and to $\{(i, j+1)\}$ and for any even number $i, j \in \{1, \ldots, n\}$ we have the position $\{(i-1, j), (i-1, j+1)\}$ from which there are edges to $\{(i, j)\}$ and to $\{(i, j+1)\}$. If $i \neq 1$ and $i \neq n$ there are also other edges from these new vertices (to $\{(i+2, j)\}$ and to $\{(i-2, j)\}$), but we do not need to consider them. It is easy to see that $\mathrm{tw}(\overline{G}_{v_0}^n) \geq \mathrm{tw}(\mathfrak{G}_n^{1/2}) = n/2$. (Remember that for tree-width we convert directed edges to undirected ones.) □

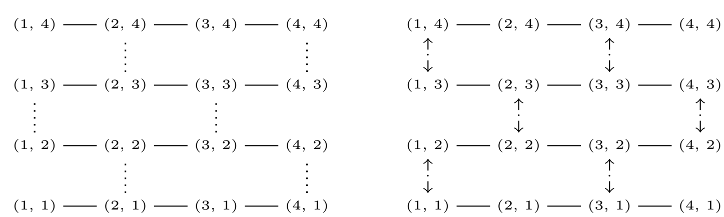

Fig. 2. Game graph $\mathcal{G}_4$ (without v_0) and a subgraph of its powerset graph $\overline{G}_{v_0}^4$

Proposition 12. *There are games $\mathcal{G}_n$, $n \in \mathbb{N}$ with bounded partial information and $X(G) \leq 2$ for all $n \in \mathbb{N}$ and any $X \in \mathcal{M} \setminus \{\mathrm{dpw}\}$ such that the corresponding powerset graphs $\overline{G}_n$ have unbounded entanglement.*

Proof. The graph G_n consists of two disjoint copies $\mathcal{T}_1$ and $\mathcal{T}_2$ of the full undirected binary tree. From a vertex in $\mathcal{T}_1$, a directed edge leads to the corresponding vertex in $\mathcal{T}_2$ and there are no edges from $\mathcal{T}_2$ to $\mathcal{T}_1$. Undirected trees have entanglement two, so $\mathrm{ent}(G) = 2$. The edges from $\mathcal{T}_1$ to $\mathcal{T}_2$ are implemented by gadgets which create, when the powerset construction is performed, a back edge while also preserving the original edge. So the graph $\overline{G}_n$ again consists of two disjoint copies of the full undirected binary tree but corresponding vertices are now connected in both directions.

To be more precise, let $n < \omega$ be an even natural number. For an alphabet Σ and $k < \omega$, by $\Sigma^{<k}$ we denote the set of all words $u \in \Sigma^*$ with $|u| < k$.

Furthermore, if Σ and Γ are alphabets and $\pi : \Sigma \to \Gamma$ is some function, then for $u = u_1 \dots u_k \in \Sigma^*$, by $\pi(u)$ we denote the word $\pi(u_1) \dots \pi(u_k) \in \Gamma^*$. Now let $\mathcal{G}_n = (G_n, \sim_n^V, \sim_n^A)$, where $G_n = (V_n, \emptyset, (f_a^n)_{a \in A_n})$ is the following game graph, see the first graph in Figure 3. By π_1 we denote the mapping $\{0,1\} \to \{a,b\}$, $0 \mapsto a$, $1 \mapsto b$ and by π_2 we denote the mapping $\{a,b\} \to \{\overline{0}, \overline{1}\}$, $a \mapsto \overline{0}$, $b \mapsto \overline{1}$.

- $V_n = 0\{0,1\}^{<n} \cup a\{a,b\}^{<n} \cup \overline{0}\{\overline{0},\overline{1}\}^{<n}$
- $A_n = \{0, 1, \overline{0}, \overline{1}, \rightarrow, \circlearrowleft\}$
- $u \overset{0}{\longleftrightarrow} u0$ and $u \overset{1}{\longleftrightarrow} u1$ for any $u \in 0\{0,1\}^{<n-1}$
- $u \overset{\circlearrowleft}{\longleftrightarrow} u$ for any $u \in 0\{0,1\}^{<n}$.
- $u \overset{\overline{0}}{\longleftrightarrow} u\overline{0}$ and $u \overset{\overline{1}}{\longleftrightarrow} u\overline{1}$ for any $u \in \overline{0}\{\overline{0},\overline{1}\}^{<n-1}$
- $u \overset{\rightarrow}{\longleftrightarrow} \pi_1(u)$ for any $u \in 0\{0,1\}^{<n}$.
- $u \overset{\rightarrow}{\longrightarrow} \pi_2(u)$ for any $u \in a\{a,b\}^{<n}$.

So structurally, G_n consists of two disjoint copies of the full undirected binary tree of depth n, together with the nodes $u \in a\{a,b\}^{<n}$ which connect the two trees in such a way, that from each $v \in 0\{0,1\}^{<n}$ there is an undirected edge to the corresponding $u \in a\{a,b\}^{<n}$ and there is a directed edge from u to the copy $\overline{v} \in \overline{0}\{\overline{0},\overline{1}\}^{<n}$ of v. It is easy to see that $X(G_n) \leq 3$ for each measure X. Partial information is defined by $u \sim_n^V \overline{u}$ for each $u \in 0\{0,1\}^{<n}$, $0 \sim_n^A \overline{0}$ and $1 \sim_n^A \overline{1}$.

The powerset construction on $\mathcal{G}_n$ from 0 (see Figure 3) yields the graph $\overline{G}_0^n$ which has $\{0\}$ as position and therefore has also $\{a\}$ and $\{0,\overline{0}\}$ as positions. $\{0\}$ has a directed $\rightarrow$-edge to $\{a\}$, $\{a\}$ has an undirected $\rightarrow$-edge to $\{0,\overline{0}\}$ and, $\{0,\overline{0}\}$ has a directed $\circlearrowleft$-edge back to $\{0\}$. Moreover, $\{0\}$ has an undirected 0-edge to $\{00\}$ and an undirected 1-edge to $\{01\}$. Likewise, $\{0,\overline{0}\}$ has an undirected $(0,\overline{0})$-edge to $\{00,\overline{00}\}$ and an undirected $(1,\overline{1})$-edge to $\{01,\overline{01}\}$. On the lower levels the graph is described completely analog, so it essentially consists of two disjoint copies of the full, undirected binary tree of depth n, where each node and its duplicate in the other copy are connected by an undirected edge. Adapting a proof from [4] for similar graphs, we now prove $\text{ent}(\overline{G}_0^n) \geq n/2 - 2$.

Assume, the robber is in some leaf node $u \in 0\{0,1\}^{<n}$ such that the unique path from its duplicate $\overline{u} \in \overline{0}\{\overline{0},\overline{1}\}^{<n}$ to the root position $\overline{0}$ is cop-free. Then, since u has $n-1$ ancestors, but only $n/2-2$ cops are available to the cop-player, there is some ancestor $v \preceq u$ of u (where $\preceq$ is the prefix order on words) such that the following holds. If w is the predecessor of v in the tree $0\{0,1\}^{<n}$, then v, $\overline{v}$, w and $\overline{w}$ are cop-free and moreover, from w there is some cop-free path to a leaf node such that also the corresponding duplicate path in the tree $\overline{0}\{\overline{0},\overline{1}\}^{<n}$ is cop-free. Now the robber moves as follows. He goes from u to $\overline{u}$ and from there, via the cop-free path in $\overline{0}\{\overline{0},\overline{1}\}^{<n}$, to $\overline{v}$. Notice that the cops can occupy only the vertex where the robber is at the moment. Then he proceeds to v, from v to w and from w he goes via the cop-free path in $0\{0,1\}^{<n}$ to a leaf node $u' \in 0\{0,1\}^{<n}$. Then, the unique path from its duplicate $\overline{u}' \in \overline{0}\{\overline{0},\overline{1}\}^{<n}$ to the root position $\overline{0}$ is cop-free, so we can use the strategy we have just described again. In this way, the robber is never captured by $n/2-2$ cops. $\square$

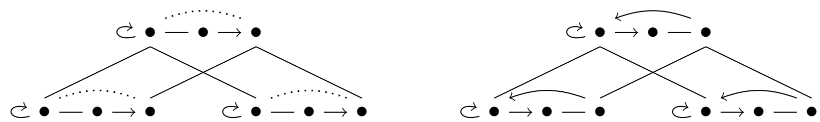

Fig. 3. Game graph $\mathcal{G}_2$ and its powerset graph $\overline{G}^2_{v_0}$.

Now we prove that in contrast to tree-width and entanglement, *non-monotone* DAG-width is preserved by the powerset construction. Throughout the remaining part of this section, let

$$\mathcal{G} = (G, \sim^V, \sim^A) \text{ with } G = (V, V_0, (f_a)_{a\in A}, \mathrm{col})$$

denote a parity game with bounded partial information, i.e.

$$\text{there is some } r \in \mathbb{N} \text{ such that } |[u]| \leq r \text{ for all } u \in V(G).$$

Moreover, let

$$\overline{G} = (\overline{V}, \overline{V}_0, (\overline{E}_a)_{a\in A}, \overline{\mathrm{col}})$$

denote the powerset graph of $\mathcal{G}$.

Proposition 13. *If k cops win the DAG-width game on G, then $k \cdot r \cdot 2^{r-1}$ cops win the DAG-width game on $\overline{G}$.*

Proof. We fisrst describe the proof idea. We translate strategies for k cops from G to $\overline{G}$ and robber's strategies in the opposite direction. Consider positions in games on both graphs. When the robber makes a move on $\overline{G}$ to a vertex $\{v_1, \dots, v_l\}$ we consider l plays in the game on G where he moves to v_1, v_2,..., v_l. For each of these moves, the strategy for the cops for the game on G supplies an answer, moving the cops from U to U'. All these moves are translated into a move in which the cops occupy precisely the vertices of $\overline{G}$ that include a vertex from some U'. These moves of the cop player on $\overline{G}$ can be realized with $k \cdot r \cdot 2^{r-1}$ cops and guarantee that moves of the robber can always be translated back to the game on G. The key argument here is that by Lemma 2, for any path $\overline{u}^0 \xrightarrow{\overline{E}} \overline{u}^1 \xrightarrow{\overline{E}} \dots \xrightarrow{\overline{E}} \overline{u}^t$ in $\overline{G}$ and for any $u^t \in \overline{u}^t$, there is a path $u^0 \xrightarrow{E} u^1 \xrightarrow{E} \dots \xrightarrow{E} u^t$ in G such that $u^i \in \overline{u}^i$ for any $i \in \{0, \dots, t\}$. So if a play continues infinitely on $\overline{G}$ then at least one corresponding play on G continues infinitely. Hence, if we start from a winning strategy for k cops for the game on G, no strategy for the robber can be winning against $k \cdot r \cdot 2^{r-1}$ cops on $\overline{G}$. By determinacy, the result follows.

To be more formal, let f be a winning strategy for k cops for the DAG-width game on G and let $\overline{g}$ be any strategy for the robber for the DAG-width game on $\overline{G}$. Basically, we translate f to the game on $\overline{\mathcal{G}}$ and $\overline{g}$ in the opposite direction. As vertices in $\overline{G}$ are sets of vertices in G, we have to trace multiple plays in

G that correspond to one play in $\overline{G}$. Formally, we construct a play $\overline{\pi}_{fg}$ on $\overline{G}$ that is consistent with $\overline{g}$ but not won by the robber. While constructing $\overline{\pi}_{fg}$ we simultaneously construct, for every finite prefix $\overline{\pi} = (\overline{U}_0, \overline{v}_0)(\overline{U}_0, \overline{U}_1, \overline{v}_0)(\overline{U}_1, \overline{v}_1) \dots (\overline{U}_i, \overline{v}_i)$ or $\overline{\pi} = (\overline{U}_0, \overline{v}_0)(\overline{U}_0, \overline{U}_1, \overline{v}_0)(\overline{U}_1, \overline{v}_1) \dots (\overline{U}_{i-1}, \overline{U}'_i, \overline{v}_{i-1})$ of $\overline{\pi}_{fg}$, a finite tree $\zeta(\overline{\pi})$ which consists of histories of length $i+1$ in the DAG-width game on G, such that the following conditions hold.

(1) Each history in $\zeta(\overline{\pi})$ is consistent with f.
(2) For all $j \leq i+1$ and all $v \in V$ we have $v \in \overline{v}_j$ if and only if there is a position (U, v) or (U, U', v) at level $j+1$ of $\zeta(\overline{\pi})$. Moreover, for each $v \in V$, on each level there is at most one position of the form (U, v) or (U, U', v).
(3) For all $j \leq i+1$ and all $\overline{u} \in \overline{V}$ we have $\overline{u} \in \overline{U}_j$ if and only if there is a position (U', v) or (U, U', v) at level $j+1$ of $\zeta(\overline{\pi})$ such that $\overline{u} \cap U' \neq \emptyset$.
(4) If $\overline{\pi}' \preceq \overline{\pi}$ then $\zeta(\overline{\pi}') \preceq \zeta(\overline{\pi})$.

Hereby $\overline{\pi}' \preceq \overline{\pi}$ means that $\overline{\pi}'$ is a prefix of $\overline{\pi}$ and $\zeta(\overline{\pi}') \preceq \zeta(\overline{\pi})$ means, if $\zeta(\overline{\pi}')$ has depth r then $\zeta(\overline{\pi})$ has depth $s \geq r$ and up to level r, $\zeta(\overline{\pi}')$ and $\zeta(\overline{\pi})$ coincide.

To begin the induction consider any history $\overline{\pi}$ of length 1, i.e. any possible initial move $(\emptyset, \overline{u})$ of the robber player. With $\overline{\pi}$ we associate the tree $\zeta(\overline{\pi})$ consisting of a root ε which has exactly the positions $(\emptyset, v)$ for $v \in \overline{u}$ as successors. Clearly, conditions $(1) - (4)$ hold. To translate the first cops' move, having a history $\pi = (\emptyset, v)$ (on G) with $v \in \overline{v}$, consider the set $U_0 = f(\pi)$ of positions chosen to be occupied by the cops in the first move by the cop player according to f. We define $\overline{U}_0 = \overline{f}(\overline{\pi})$ by $\overline{u} \in \overline{U}_0$ if and only if there is some position (U, v) such that $\overline{u} \cap U \neq \emptyset$. This yields the history $\overline{\pi}' = (\emptyset, \overline{U}_0, \overline{v})$ (on $\overline{G}$). With this history, we associate the tree $\zeta(\overline{\pi}')$ which is obtained from $\zeta(\overline{\pi})$ by extending each history $\pi = (\emptyset, v)$ in $\zeta(\overline{\pi})$ to $\pi' = (\emptyset, v)(\emptyset, f(\pi), v)$. Again, conditions $(1) - (4)$ hold.

For translating the robber's moves in the induction step, consider any history $\overline{\pi} = (\overline{U}_0, \overline{v}_0)\,(\overline{U}_0, \overline{U}_1, \overline{v}_1)\,(\overline{U}_1, \overline{v}_2) \dots (\overline{U}_{i+1}, \overline{v}_{i+1})$ with $i \geq 1$ and let, by induction hypothesis, $\zeta(\overline{\pi}(\leq i))$ be constructed. The robber has just moved from $\overline{v}_i$ to $\overline{v}_{i+1}$, so $\overline{v}_{i+1} \notin \overline{U}_i$ and $\overline{v}_{i+1}$ is reachable from $\overline{v}_i$ in the graph $\overline{G} - (\overline{U}_i \cap \overline{U}_{i-1})$. Let $\overline{v}_i = \overline{v}^0 \overset{\overline{E}}{\to} \overline{v}^1 \overset{\overline{E}}{\to} \dots \overset{\overline{E}}{\to} \overline{v}^t = \overline{v}_{i+1}$ be a path from $\overline{v}_i$ to $\overline{v}_{i+1}$ in $\overline{G} - (\overline{U}_i \cap \overline{U}_{i-1})$. Then, by Lemma 2, for any $v \in \overline{v}_{i+1}$, there is some $u \in \overline{v}_i$ such that there is a path $u = u^0 \overset{E}{\to} u^1 \overset{E}{\to} \dots \overset{E}{\to} u^t = v$ in G with $u^l \in \overline{v}^l$ for $l = 0, \dots, t$. By condition (2) for $\zeta(\overline{\pi}(\leq i))$, there is some finite history $\pi \in \zeta(\overline{\pi}(\leq i))$ which ends in a position (U, U', u), where the last move has been made by the cop player who has chosen U' as the new set of positions, occupied by the cops. So, U corresponds to $\overline{U}_{i-1}$ and U' corresponds to $\overline{U}_i = \overline{U}_{i+1}$ in the sense of condition (3) for $\zeta(\overline{\pi}(\leq i))$. We now extend π to the finite history $\pi(U', v)$. The set of all such histories extended in this way forms the tree $\zeta(\overline{\pi})$. First, we have to show that each such $\pi(U', v)$ is actually a finite history of the DAG-width game on G, i.e. we have to show that $v \notin U'$ and v is reachable from u in $G - (U \cap U')$. First, $\overline{v}_{i+1} \notin \overline{U}_i$ and therefore, by condition (3) for $\zeta(\overline{\pi}(\leq i))$ we have $\overline{v}_{i+1} \cap U' = \emptyset$ which implies $v \notin U'$. Now assume towards a contradiction, that v is not reachable from in $G - (U \cap U')$. In particular, there must be some $l \in \{1, \dots, t\}$ such that $u^l \in U \cap U'$ (notice that $u^0 = u \notin U \cap U'$). But then, since $u^l \in \overline{v}^l$, by (3) for $\zeta(\overline{\pi}(\leq i))$ we have $\overline{v}^l \in \overline{U}_i \cap \overline{U}_{i-1}$ which contradicts the fact that $\overline{v}^0 \xrightarrow{\overline{E}} \overline{v}^1 \xrightarrow{\overline{E}} \ldots \xrightarrow{\overline{E}} \overline{v}^t$ is a path in $\overline{G} - (\overline{U}_i \cap \overline{U}_{i-1})$. Moreover, since all histories in $\zeta(\overline{\pi}(\leq i))$ are compatible with f and all moves by which we have extended histories are made by the robber player, all histories in $\zeta(\overline{\pi})$ are compatible with f. Conditions (2), (3) and (4) obviously hold for $\zeta(\overline{\pi})$.

To translate the cops' answer, for any history $\pi \in \zeta(\overline{\pi})$, consider the set $U = f(\pi)$ of positions chosen to be occupied by the cops in the next move by the cop player according to f. We define $\overline{U} = \overline{f}(\overline{\pi})$ by $\overline{v} \in \overline{U}$ if and only if there is some history $\pi \in \zeta(\overline{\pi})$ such that $\overline{v} \cap f(\pi) \neq \emptyset$, i.e. the cops occupy $\overline{v}$ if in some of the plays on G they occupy some vertex in $\overline{v}$. This yields the history $\overline{\pi}' = \overline{\pi}(\overline{U}_{i+1}, \overline{U}, \overline{v}_{i+1})$. With this history, we associate the tree $\zeta(\overline{\pi}')$ which is obtained from $\zeta(\overline{\pi})$ by extending each history $\pi \in \zeta(\overline{\pi})$ with $\mathrm{last}(\pi) = (U, v)$ to $\pi' = \pi(U, f(\pi), v) = \pi(U, U', v)$. Clearly all the conditions (1) – (4) hold.

Now assume that $\overline{\pi}_{fg}$ is infinite, i.e. won by the robber. Then the tree ζ which, for any $i < \omega$, coincides up to level i with the tree $\zeta(\overline{\pi}(\leq i))$, is infinite as well. Since ζ is finitely branching, by König's Lemma there is some infinite path π through ζ. By condition (1), π is a play in the DAG-width game on G which is compatible with f. But since π is infinite, this contradicts the fact that f is a winning strategy for the cop player. It remains to count the number of cops used by the cop player in $\overline{\pi}_{fg}$. Consider any position $(\overline{U}_{i-1}, \overline{U}_i, \overline{v}_{i-1})$ occuring in $\overline{\pi}_{fg}$. By condition (2), at level $i+1$ of $\zeta(\overline{\pi})$, there occur at most $|\overline{v}_i| \leq r$ many histories. Each such history is consistent with f, so at most k vertices are occupied by the cops. Hence, by condition (3), $|\overline{U}_i| \leq k \cdot r \cdot 2^{r-1}$. Therefore, the robber does not have a winning strategy against $k \cdot r \cdot 2^{r-1}$ cops in the DAG-width game on $\overline{G}$. By determinacy, $k \cdot r \cdot 2^{r-1}$ cops have a winning strategy. □

Unfortunately, this strategy translation does not necessarily preserve monotonicity, as the following example shows.

Example 14. We give an example where the strategy translation from Proposition 13 does not preserve monotonicity of the cops' strategy. Consider the graph G depicted in Figure 4 and the following monotone (partial) strategy for the cops. First, put a cop on v_0. If the robber goes to 1, put a cop on 1 and then move the cop from 1 to 3. If the robber goes to 2, put a cop on 5 and if the robber goes to 4, put a new cop on 4. In the game on the powerset graph, consider the following play, which is consistent with the translated cops' strategy. First, the cops occupy $\{v_0\}$. Let the robber go to $\{1, 2\}$ in which case the cops occupy $\{1, 2\}$ and $\{5\}$. Now the robber goes to $\{3, 4\}$, so the cop from $\{1, 2\}$ is removed. At this moment, the vertex $\{1, 2\}$ becomes available for the robber again, so the translated strategy is non-monotone. Notice that, nevertheless, $\mathrm{dw}(\overline{G}) = 2$.

An interesting special case where an adapted translation of strategies does preserve monotonicity is given by games with strongly connected equivalence classes of positions. Intuitively this means that for any characteristic of the current state which player 0 is unsure about, it is possible for player 1 to change the value of this characteristic into any other possible value privately, i.e. without

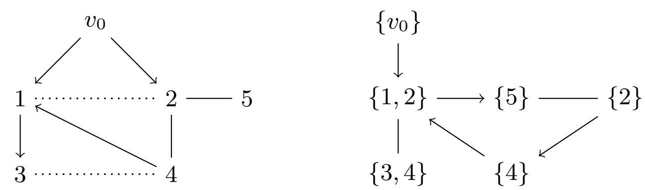

Fig. 4. Monotone strategy is translated to a non-monotone one.

changing any characteristics visible for player 0 in between. This is appropriate for situations where, e.g., the uncertainties of player 0 concern some private states of player 1 which are independent of the states visible for player 0.

Proposition 15. *If* $\mathrm{dw}(G) \le k$ *and each equivalence class of positions is strongly connected, then* $\mathrm{dw}(\overline{G}) \le k \cdot r^2 \cdot 2^{r-1}$.

Proof. First, from a *monotone* winning strategy h for the DAG-width game on G we obtain a *monotone* winning strategy f for $k \cdot r$ cops for the DAG-width game on G which is compatible with equivalence classes of positions. That means, if π is a prefix of a play which is consistent with f and $\mathrm{last}(\pi) = (U, U', v)$, then for any $u \in U'$ we have $[u] \subseteq U'$ and if $u \in U \setminus U'$ then $[u] \cap (U \setminus U') = \emptyset$. So if the cops occupy a vertex in G, then they occupy the whole equivalence class of this vertex. They remain on the whole equivalence class until they would leave every vertex in the class according to g. To see that f is monotone assume the opposite, i.e. that there are successive positions (U, v) and (U, U', v) in a play consistent with f such that there is some $w \in U \setminus U'$ which is reachable from v in $G - U$. Then all equivalence classes on the path to w (including $[w]$) in the corresponding play according to h were cop free and the robber could reach w. However, there was a vertex w' in $[w]$ occupied by the cops when playing according to h and after the cops' move in the old game that corresponds to $(U, v) \mapsto (U, U', v)$, the robber can reach w' from w in $[w]$ because $[w]$ is strongly connected. This contradicts the monotonicity of h.

Given the strategy f, we proceed exactly as in the proof of Proposition 13. It suffices to prove that for each robber's strategy g, the play $\overline{\pi}_{fg}$ is monotonously won by the cops. So assume, towards a contradiction, the opposite, i.e, there is a finite prefix $\overline{\pi} \preceq \overline{\pi}_{fg}$ of $\overline{\pi}_{fg}$ of length i for some $i < \omega$ such that $\mathrm{last}(\overline{\pi}) = (\overline{U}, \overline{U}', \overline{v})$ and such that there is some $\overline{u} \in \overline{U} \setminus \overline{U}'$ which is reachable from $\overline{v}$ in $\overline{G} - \overline{U}$. So, let $\overline{v} = \overline{v}^0 \overset{\overline{E}}{\to} \overline{v}^1 \overset{\overline{E}}{\to} \dots \overset{\overline{E}}{\to} \overline{v}^t = \overline{u}$ be a path from $\overline{v}$ to $\overline{u}$ in $\overline{G}$ with $\overline{v}^l \notin \overline{U}$ for $l = 0, \dots, t-1$. Since $\overline{u} \in \overline{U}$ and $\overline{u} \notin \overline{U}'$, according to condition (3) for $\zeta(\overline{\pi})$, there is some position (U, U', w) at level $i+1$ of $\zeta(\overline{\pi})$ such that there is some $u \in \overline{u}$ with $u \in U$ and $u \notin U'$. By Lemma 2 there is some $v \in \overline{v}$ such that there is a path $v = v^0 \overset{E}{\to} v^1 \overset{E}{\to} \dots \overset{E}{\to} v^t = u$ in G with $v^l \in \overline{v}^l$ for all $l = 0, \dots, t$. Hence $v^l \notin U$ for $l = 0, \dots, t-1$ since if there is some $l \in \{0, \dots, t-1\}$ such that $v^l \in U$, by condition (3) for $\zeta(\overline{\pi})$, we have $\overline{v}^l \in \overline{U}$ in contradiction to $\overline{v}^l \notin \overline{U}$ for $l = 0, \dots, t-1$. So u is reachable from v in $G - U$. Moreover, by condition (2) for $\zeta(\overline{\pi})$ we have $w \in \overline{v}$ and since $[w]$ is strongly connected, there is a path from w to v in G which is contained in $[w]$. Now f is compatible with equivalence classes of positions and $\overline{v} \notin \overline{U}$, so by condition (3) for $\zeta(\overline{\pi})$ we have $[w] \cap U = \emptyset$. Therefore, u is reachable from w in $G - U$. But since $u \in U \setminus U'$ and (U, U', w) is the last position of a finite history which is compatible with f, this contradicts the fact that f is monotone. □

Corollary 16. *Parity games with bounded partial information where each equivalence class of positions is strongly connected can be solved in polynomial time on graphs of bounded DAG-width.*

In a similar way as for DAG-width, strategies for the directed path-width game can be translated from the original game graph to the powerset graph. As directed path-width has monotonicity cost 0, boundedness of directed path-width of the powerset graph follows immediately, without even minding preservation of monotonicity in the strategy translation.

Proposition 17. *If* $\mathrm{dpw}(G) \leq k$*, then* $\mathrm{dpw}(\overline{G}) \leq k \cdot 2^{r-1}$.

Proof. Let f be a winning strategy for k cops for the directed path-width game on G and let $\pi = (U_0, U_0, R_0)\ (U_0, U_1, R_1) \ldots (U_{n-1}, U_n, R_n)$ be the unique play which is compatible with f. In order to define the strategy $\overline{f}$ for the cop player for the directed path-width game on $\overline{G}$ it suffices to construct a single maximal finite or infinite sequence $\overline{\pi} = (\overline{U}_0, \overline{U}_0, \overline{R}_0)\ (\overline{U}_0, \overline{U}_1, \overline{R}_1)\ (\overline{U}_1, \overline{U}_2, \overline{R}_2) \ldots$ of this game such that $\overline{U}_0 = \emptyset$, $\overline{R}_0 = \overline{V}$ and $\overline{R}_{i+1} = \mathrm{Reach}_{\overline{G}-(\overline{U}_i \cap \overline{U}_{i+1})}(\overline{R}_i) \setminus \overline{U}_{i+1}$ for all i. That means, we construct the strategy $\overline{f}$ by constructing the unique play which is compatible with $\overline{f}$. Then we prove that this play is necessarily finite which shows that the strategy which is defined by this play is a winning strategy for the cops player. For this we inductively define finite histories $\overline{\pi}_i = ((\overline{U}_0, \overline{U}_0), \overline{R}_0) \ldots ((\overline{U}_{i-1}, \overline{U}_i), \overline{R}_i)$, such that $\overline{U}_0 = \emptyset$, $\overline{R}_0 = \overline{V}$ and for all $j \leq i$, the following conditions hold.

(1) $\overline{R}_{j+1} = \left(\mathrm{Reach}_{\overline{G}-(\overline{U}_j \cap \overline{U}_{j+1})}(\overline{R}_j)\right) \setminus \overline{U}_{j+1}$.
(2) $\bigcup \overline{R}_j \subseteq R_j$.
(3) For all $\overline{v} \in \overline{V}$ we have $\overline{v} \in \overline{U}_j$ if and only if $\overline{v} \cap U_j \neq \emptyset$.

First, $\overline{\pi}_0 = (\overline{U}_0, \overline{U}_0, \overline{R}_0) = (\emptyset, \emptyset, \overline{V})$ is already defined by the conditions. Now let $\overline{\pi}_i = (\overline{U}_0, \overline{U}_0, \overline{R}_0)\ (\overline{U}_0, \overline{U}_1 \overline{R}_1) \ldots (\overline{U}_{i-1}, \overline{U}_i, \overline{R}_i)$ be constructed according to the induction hypothesis. If $\overline{R}_i = \emptyset$, then the cop player has won so assume that $\overline{R}_i \neq \emptyset$. Then $\bigcup \overline{R}_i \neq \emptyset$ and by condition (2) this yields that $R_i \neq \emptyset$. Therefore, $i < n$. We define $\overline{U}_{i+1} = \{\overline{v} \in \overline{V} \mid \overline{v} \cap U_{i+1} \neq \emptyset\}$ and $\overline{R}_{i+1} = \mathrm{Reach}_{\overline{G}-(\overline{U}_i \cap \overline{U}_{i+1})}(\overline{R}_i) \setminus \overline{U}_{i+1}$ which gives us the history $\overline{\pi}_{i+1} = \overline{\pi}_i(\overline{U}_i, \overline{U}_{i+1}, \overline{R}_{i+1})$. Conditions (1) and (3) hold by construction and condition (2) is proved with the same arguments as in the proof of Proposition 13. Now these finite histories $\overline{\pi}_i$ form a prefix-chain $\overline{\pi}_0 \preceq \overline{\pi}_1 \preceq \ldots$ which gives us the maximal finite or infinite sequence $\overline{\pi}$. We have already seen in the construction of the histories $\overline{\pi}_i$ that if $\overline{\pi}$ was infinite, then π would be infinite as well, so $\overline{\pi}$ is in fact a maximal finite history of the directed path-width game on $\overline{G}$ which gives us a winning strategy $\overline{f}$ for the cop player for this game. Moreover, this strategy uses at most $\max\{|\overline{U}_i| \mid i = 0, \dots, n\}$ cops and for any fixed i we have $|\overline{U}_i| \leq k \cdot 2^{r-1}$. Finally, since directed path-width has monotonicity cost 0 this yields $\mathrm{dpw}(\overline{G}) \leq k \cdot 2^{r-1}$. □

Corollary 18. *Parity games with bounded partial information can be solved in polynomial time on graphs of bounded directed path-width.*

Finally, we remark that our direct translation of the robber's moves back to the game on G cannot be immediately applied to the games which define Kelly-width and directed tree-width. In the Kelly-width game, the robber can only move if a cop is about to occupy his vertex. It can happen that the cops occupy a vertex $\{v_1, \dots, v_l\}$ in $\overline{G}$ but not all vertices $v_1, \dots, v_l$ in G. In the directed tree-width game, the robber is not permitted to leave the strongly connected component in which he currently is, which again obstructs a direct translation of the robber's moves from $\overline{G}$ back to G.

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