Choiceless Polynomial Time on structures with small Abelian colour classes

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Is there a logic for Polynomial Time?

Properties of finite relational structures

- connectedness, 3-colourability of graphs
- winning regions in games

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L captures PTIME on K

property $\mathcal{P}$ is L-definable on K

$$\Longleftrightarrow$$

property $\mathcal{P}$ is PTIME-decidable on K

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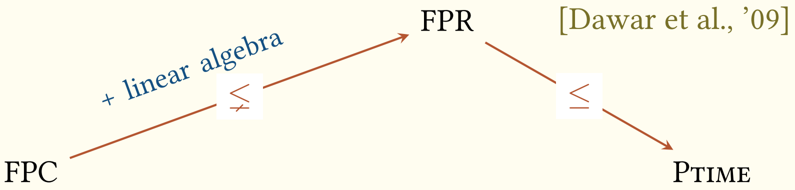

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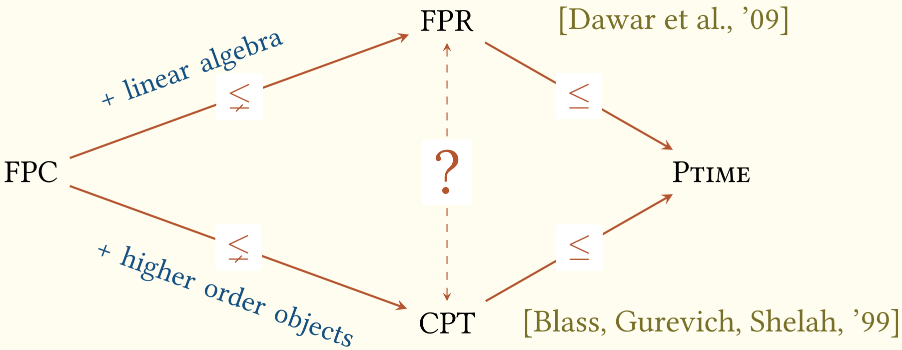

Choiceless Polynomial Time

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- isomorphism-invariance
- **choiceless** procedures (cf. "pick *some* vertex v")

PTIME-Turing machines

- string encoding
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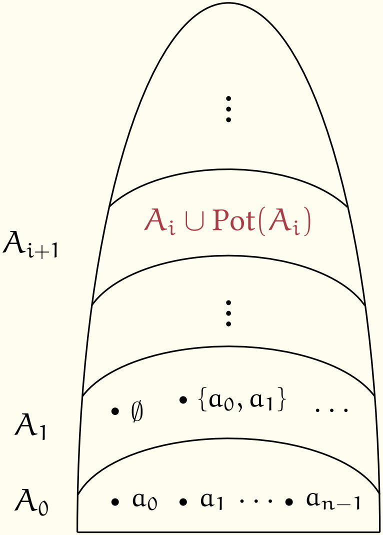
+ (polynomial-time representable) isomorphism-invariant higher-order objects

FPC ⟶ CPT

Heriditarily finite sets

$$\mathrm{HF}(A) = A_0 \cup A_1 \cup A_2 \cup \cdots$$



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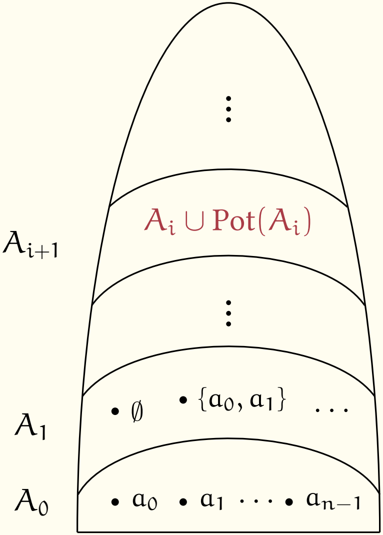

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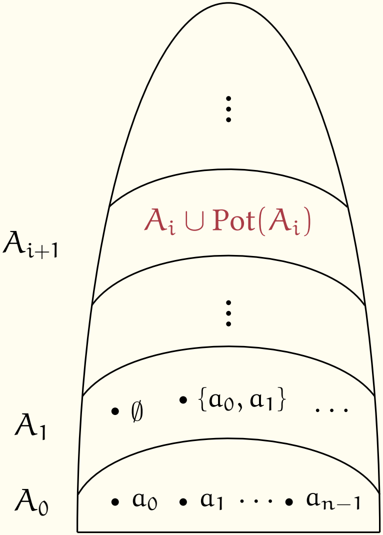

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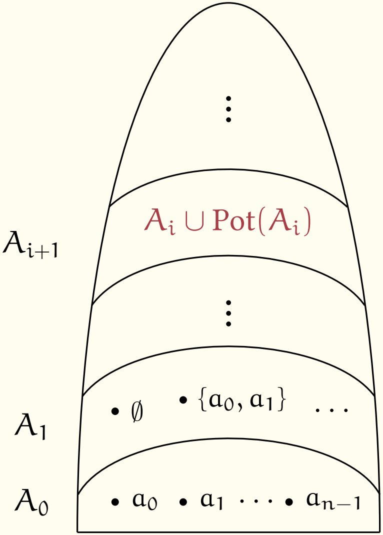

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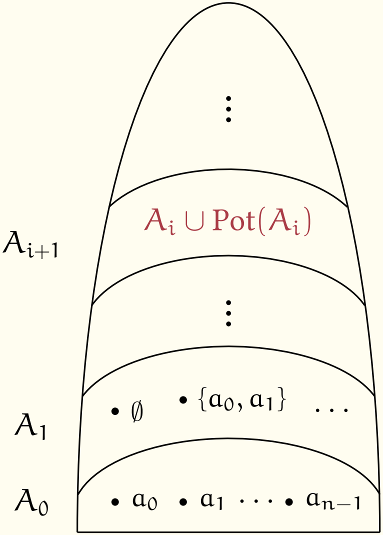

Choiceless Polynomial Time:

- set-theoretic operations to manipulate states
- iterated applications ⇝ run
- polynomial bounds on
  - length of runs
  - complexity of states

q-bounded structures with Abelian colours

CFI-query in CPT [Dawar et al., 2008]

- bounded colour size (CFI $\rightsquigarrow$ 2-bounded structures)

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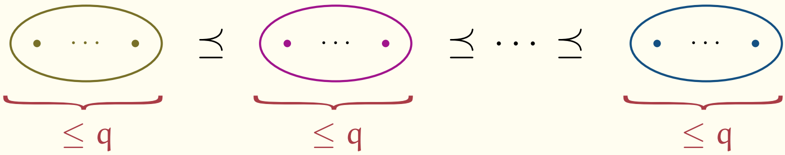

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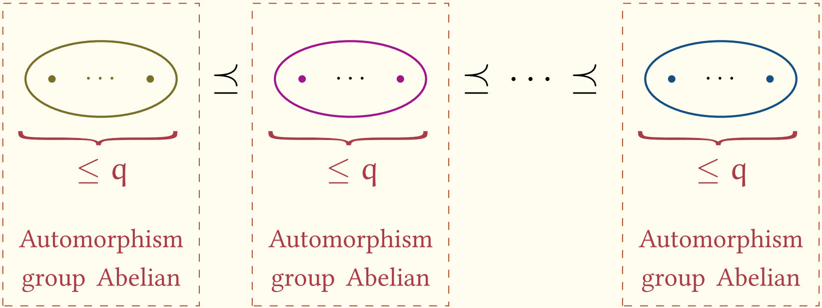

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Fact

There is a logic capturing PTIME on q-bounded structures.

Main question

Does CPT capture PTIME on q-bounded structures?

Our results: CPT on q-bounded structures

Theorem

CPT = PTIME on 2-bounded structures.

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- Solves open question of Blass, Gurevich, Shelah: isomorphism of multipedes is CPT-definable

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The method of logical canonisation

$$\mathfrak{A} \xrightarrow{\quad \cong \quad} \varphi^{\mathfrak{A}}$$

relational τ-structure — ordered $\tau \uplus \{<\}$-structure

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Remark

- $\mathbf{L} \in \{\text{FPC, FPR, CPT}, \dots\}$
- $\mathbf{K}$ class of τ-structures which allows $\mathbf{L}$-canonisation

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Theorem ([Grohe et al.])

FPC captures PTIME on trees, graphs of bounded treewidth, planar graphs, ..., graphs with excluded minor.

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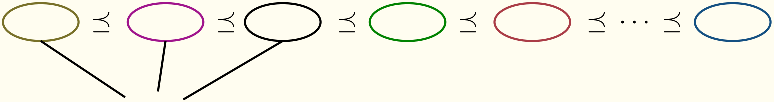

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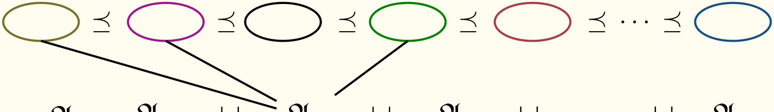

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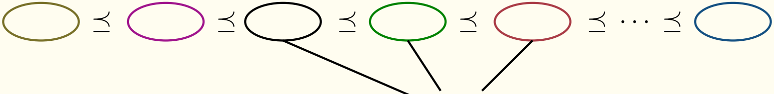

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Abelian colours $\rightsquigarrow$ linear algebraic structure

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(3) Linear equation systems in CPT

succinct representations of exponential-sized sets of equations

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- Graphs with bounded degree, bounded eigenvalue multiplicity, moderately growing tree-width

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- Graphs with bounded degree, bounded eigenvalue multiplicity, moderately growing tree-width
- Linear algebra in CPT