Logics with Operators from Linear Algebra

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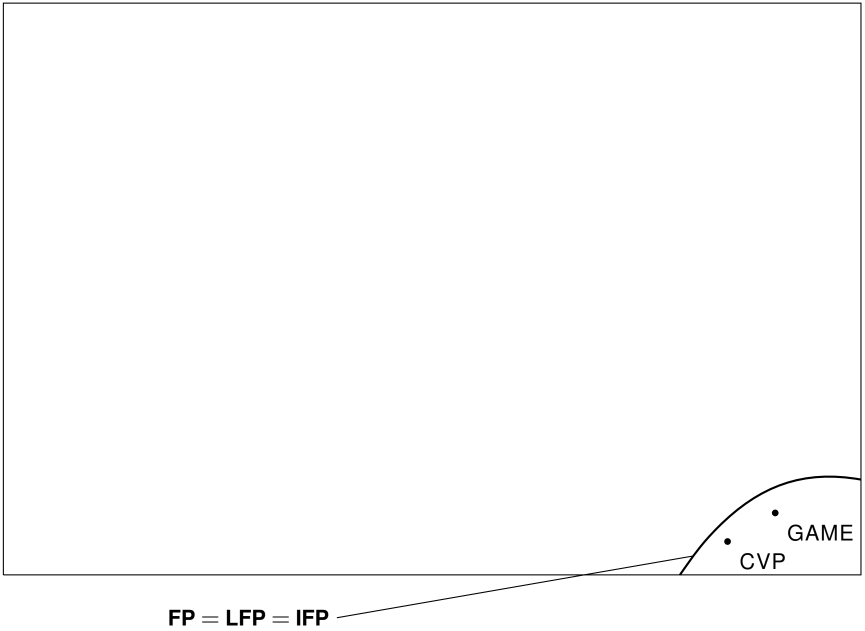
A logic for polynomial time



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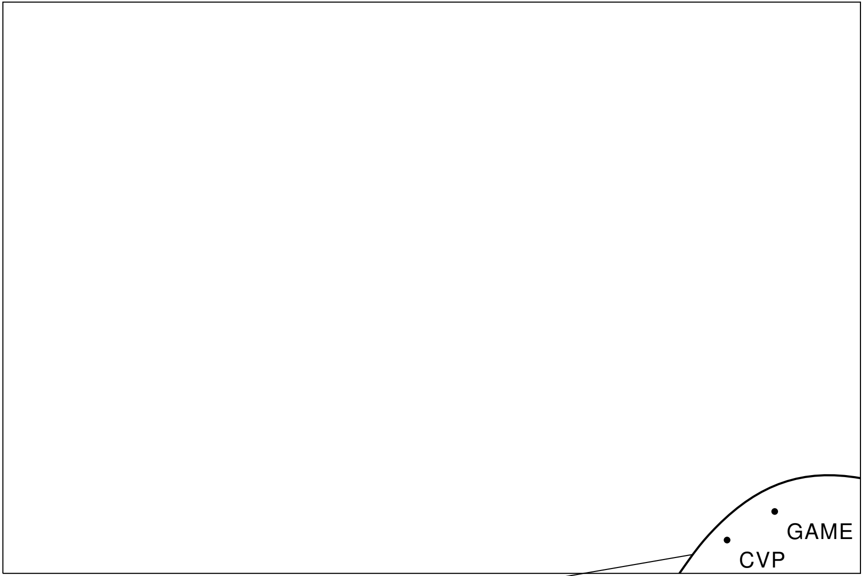

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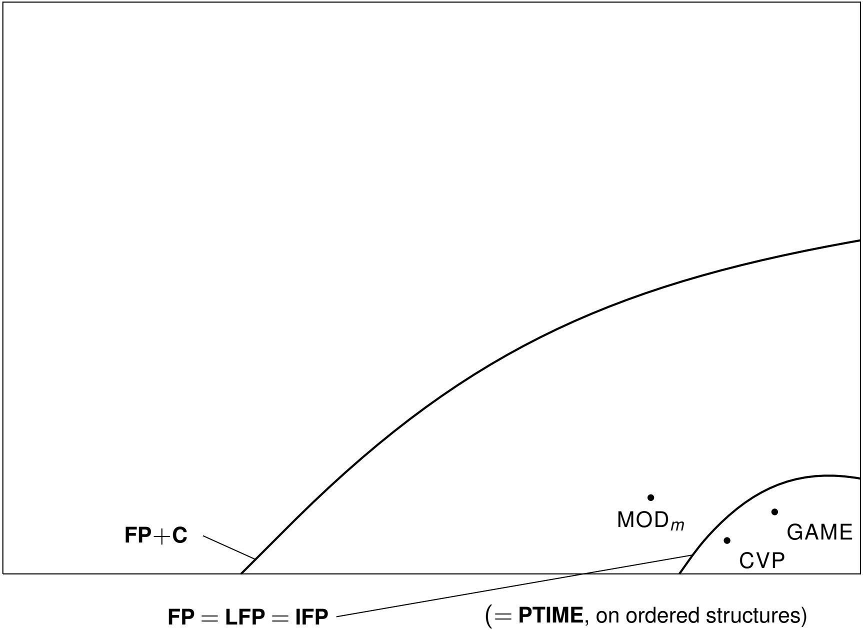

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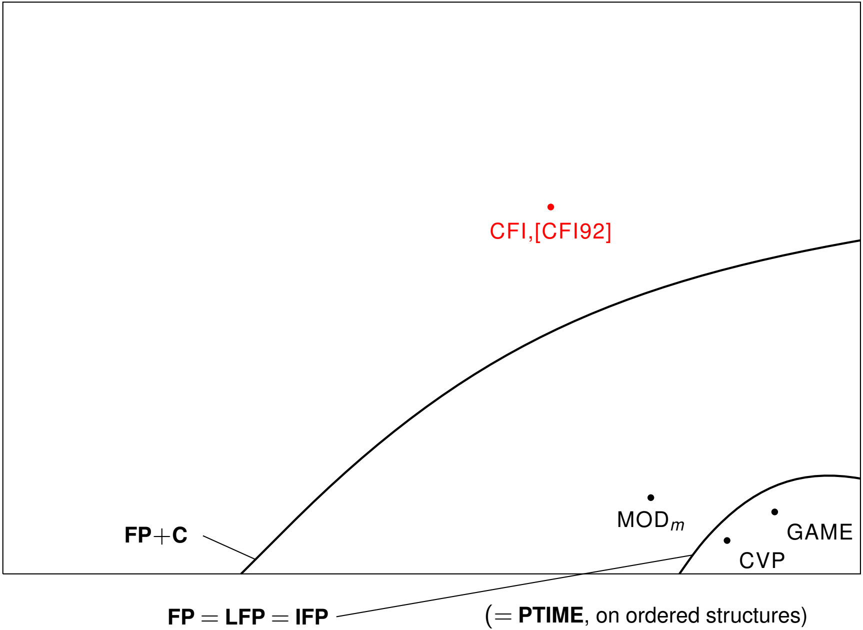

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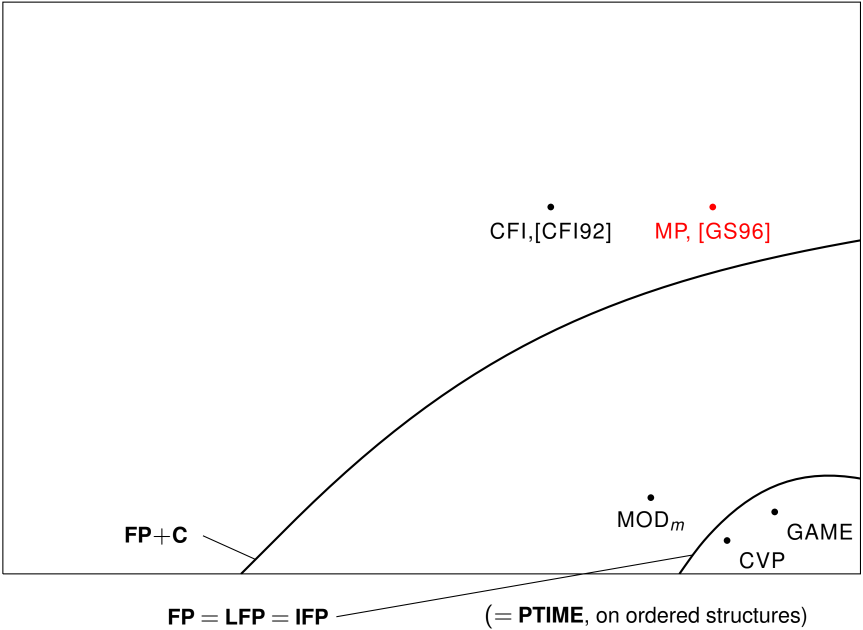

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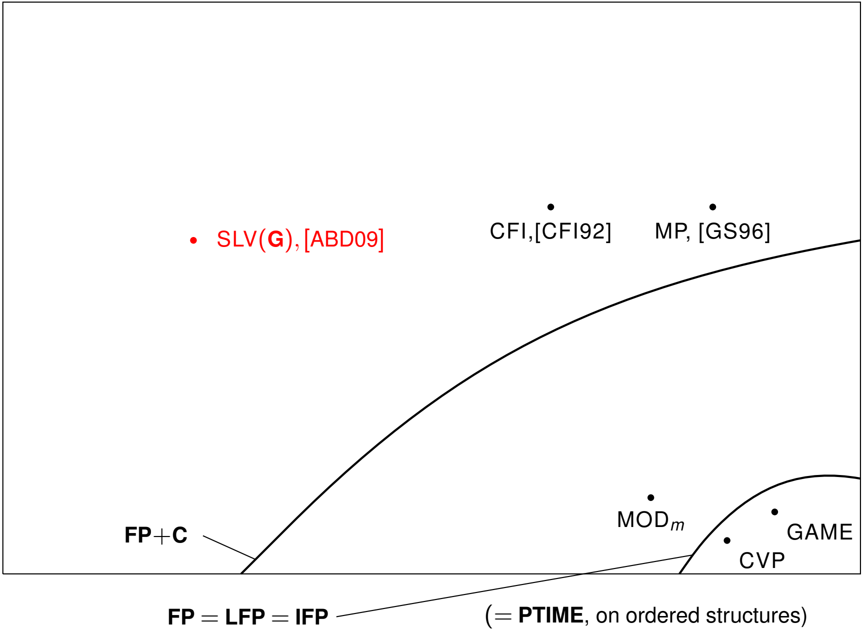

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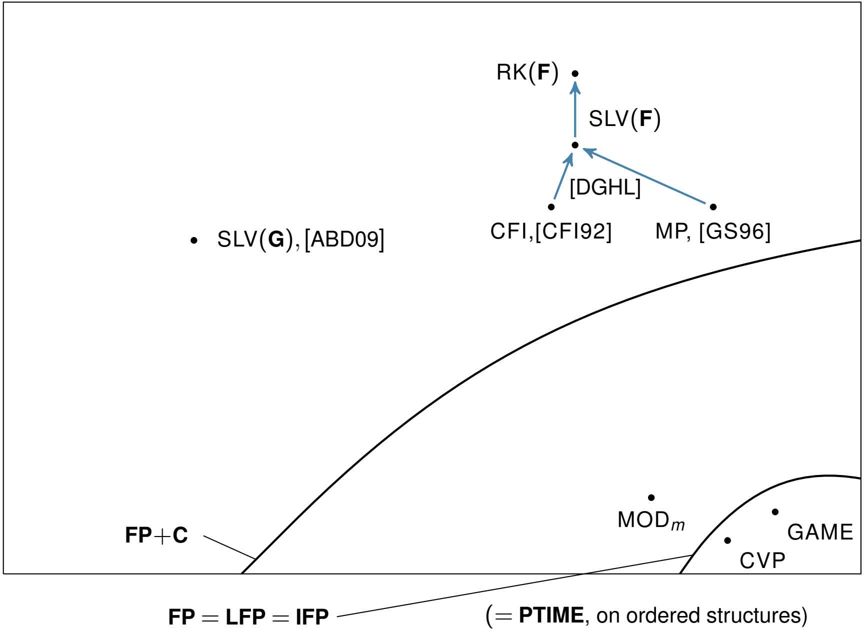

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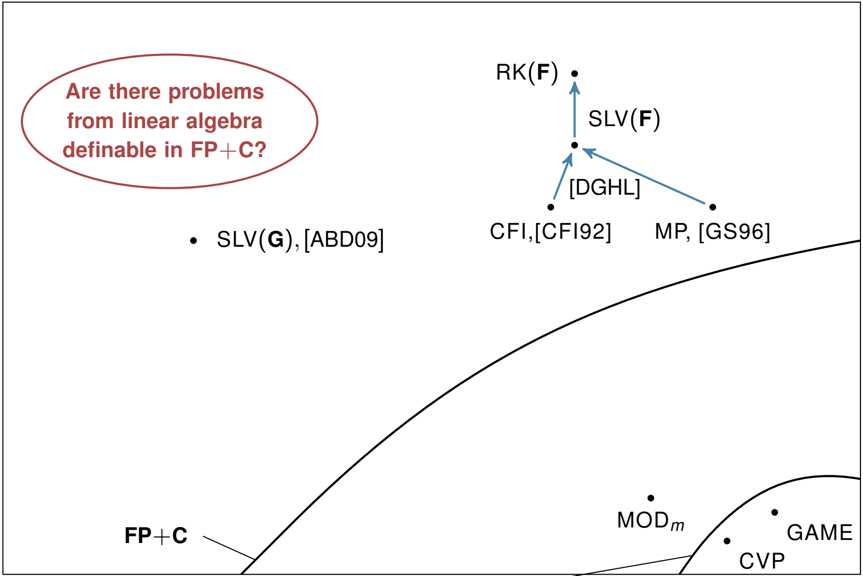

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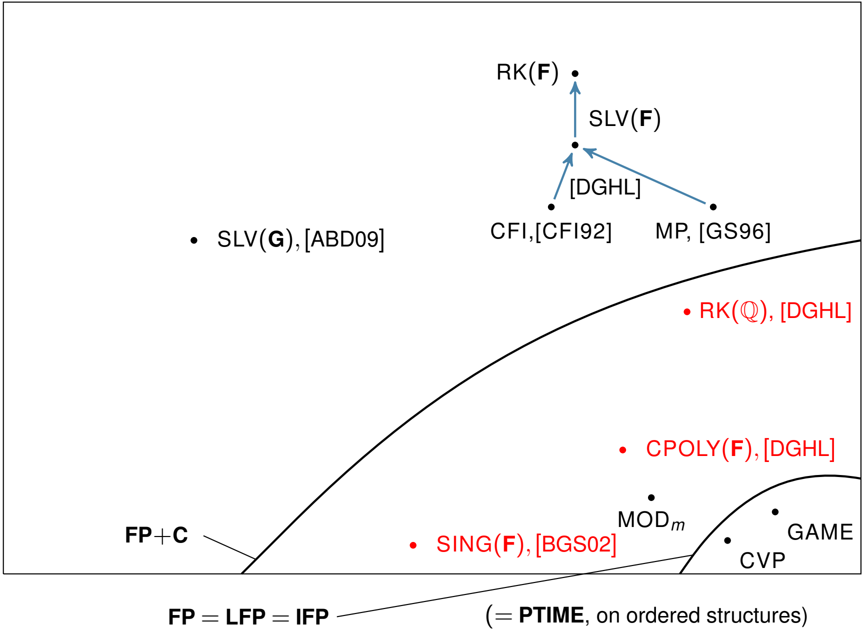

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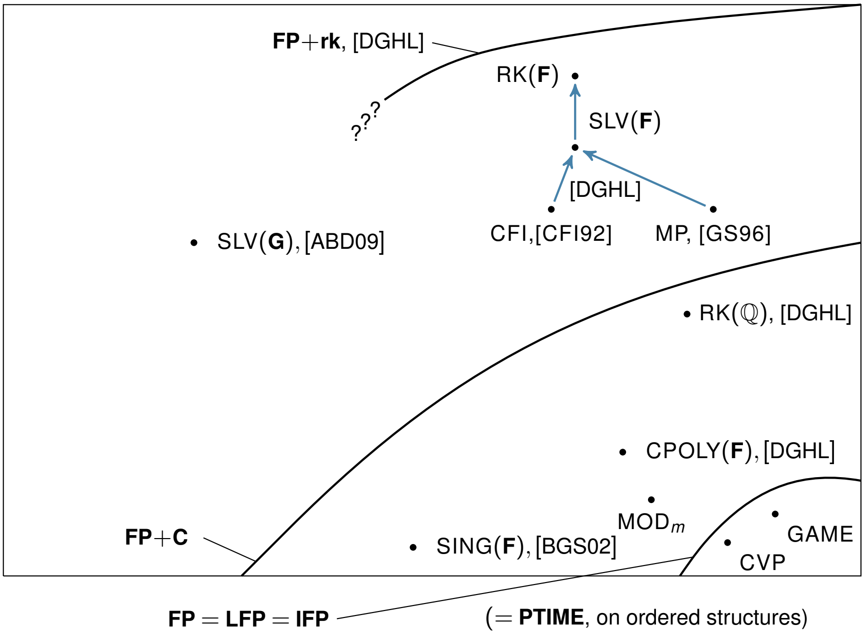

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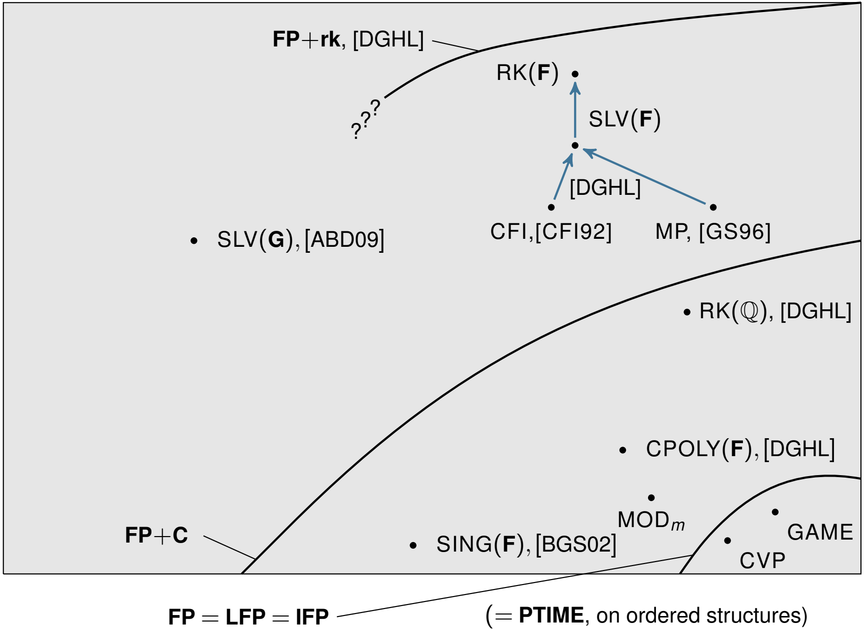

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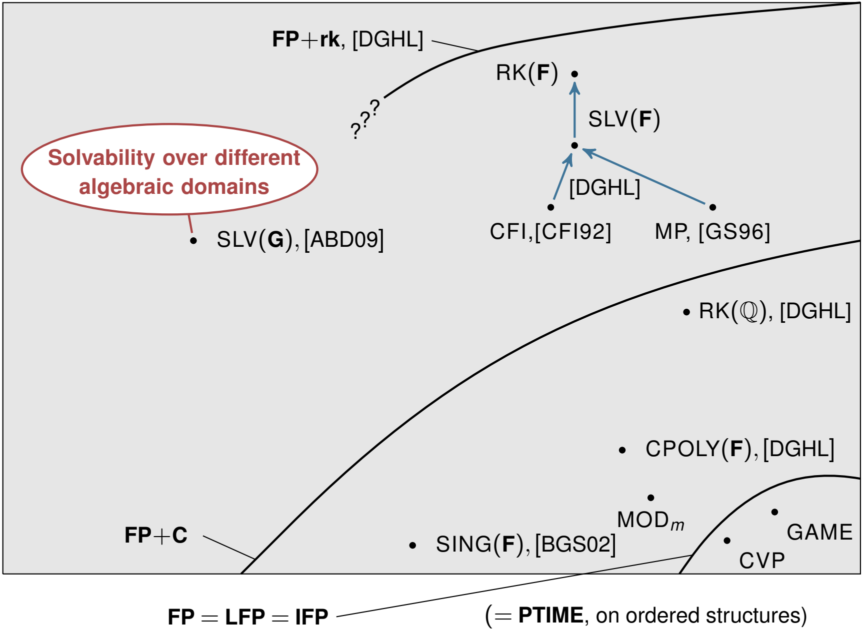

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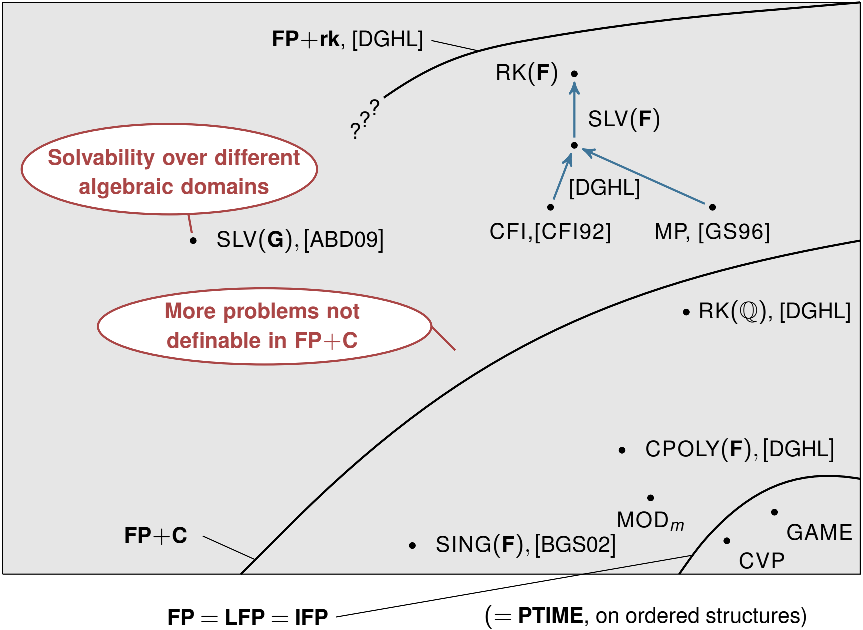

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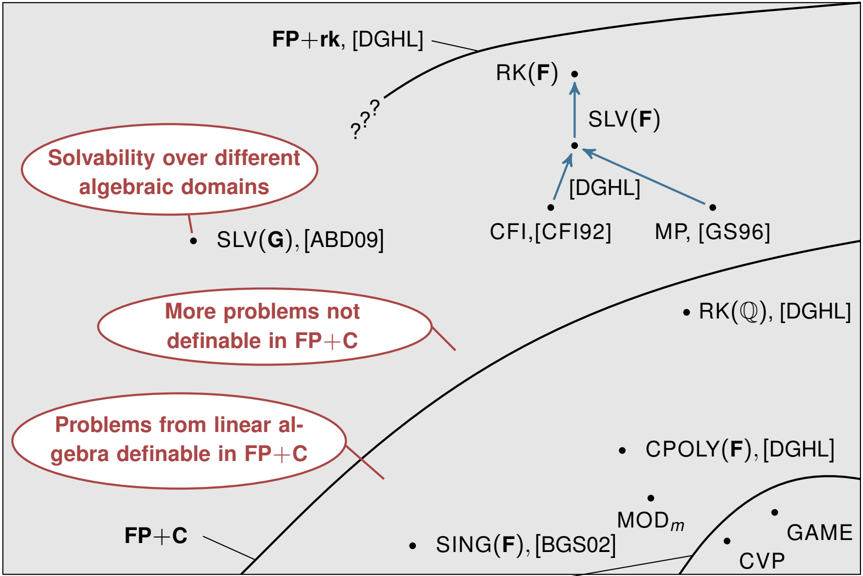

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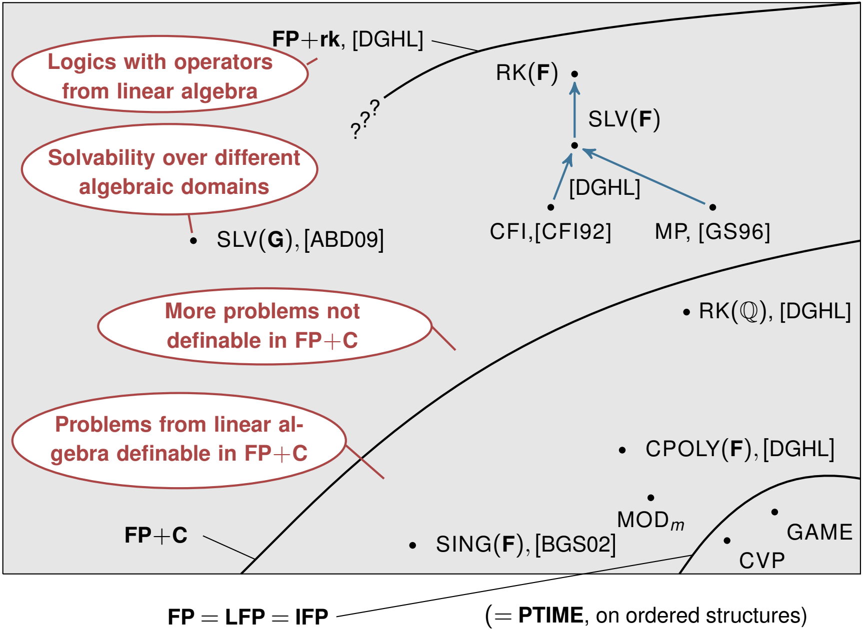

Linear algebra in relational structures

Relational structure	Logical interpretation	Unordered matrix over finite commutative ring
	φ_M	
A		
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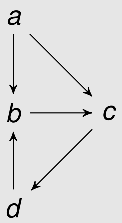

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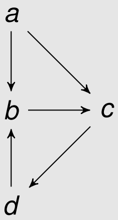

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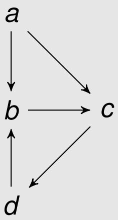

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$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & 1 & 1 & 0 & 1 \\ b & 1 & 1 & 0 & 0 \\ c & 0 & 0 & 1 & 0 \\ d & 1 & 0 & 0 & 1 \end{array}$$

Linear algebra in relational structures

Reachability in undirected graphs

- $\mathbf{G} = (V, E)$ undirected graph, $\varphi_R = \mathrm{GF}(2)$
- $\varphi_M(x, yz) = Eyz \wedge (x = y \vee x = z)$, the incidence matrix of $\mathbf{G}$

$$M_\varphi^{\mathbf{G}} = \begin{array}{c} \\ v \\ \\ w \end{array} \begin{pmatrix} vw \in E \\ 1 \\ \\ 1 \end{pmatrix} \implies \begin{array}{l} \text{If } \vec{x} \cdot M_\varphi^{\mathbf{G}} = 0 \\ \text{then } \vec{x}(v) = \vec{x}(w) \end{array}$$

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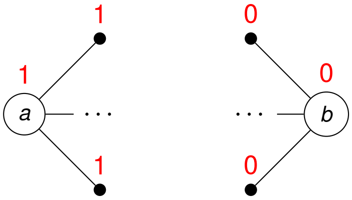
As a linear equation system:

for all $(v, w) \in E$: $v = w$

$a = 1$

$b = 0$



First-order logic with solvability quantifiers

Solvability quantifier:

$$slv(\bar{x},\bar{y},\bar{r}_i).\big[\underbrace{\varphi_M(\bar{x},\bar{y},\bar{r})}_{\text{Coefficient matrix}},\underbrace{\varphi_b(\bar{x},\bar{r})}_{\text{Solution vector}},\underbrace{\varphi_R(\bar{r}_1,\bar{r}_2),\varphi_+(\bar{r}_1,\bar{r}_2,\bar{r}_3),\varphi_\cdot(\bar{r}_1,\bar{r}_2,\bar{r}_3)}_{\text{Finite ring}}\big]$$

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Coefficient matrix — Solution vector — Finite ring

$$\Downarrow$$

FO+slv: First-order logic closed under solvability quantifier

FO+slv$_{\mathbf{PI}}$: Solvability quantifier over principal ideal rings

FO+slv$_{\mathbf{F}}$: Solvability quantifier over a fixed finite field F

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Theorem

Every $\mathbf{FO{+}slv_{PI}}$-formula is equivalent to an $\mathbf{FO{+}slv_{PI}}$-formula in negation normal form.

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Proof illustration over fields

$$\text{Non-solvability} \equiv \neg\exists\vec{x} : \mathbf{M}\vec{x} = \mathbf{b} \stackrel{?}{\equiv} \exists\vec{y} : \mathbf{M}'\vec{y} = \mathbf{b}' \equiv \text{Solvability}$$

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Over fields the method of *Gaussian elimination* implies:

$$\neg\exists\vec{x} : \mathbf{M}\vec{x} = \mathbf{b} \equiv \exists\vec{y} : \vec{y}(\mathbf{M}|\mathbf{b}) = (0, \ldots, 0|1).$$

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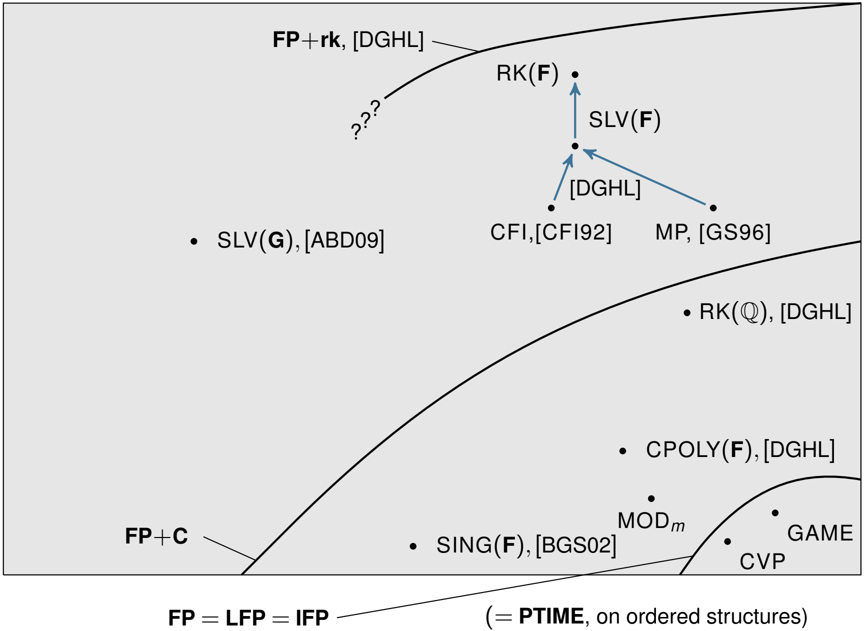
For a fixed finite field F, we can say even more:

Theorem

Every $\mathbf{FO{+}slv_{F}}$-formula is equivalent to an $\mathbf{FO{+}slv_{F}}$-formula

$$slv(\bar{x}, \bar{y}, \bar{z}).\left[\varphi_M, \varphi_b\right], \text{ with } \varphi_M, \varphi_b \text{ quantifier-free.}$$

Some answer to the questions



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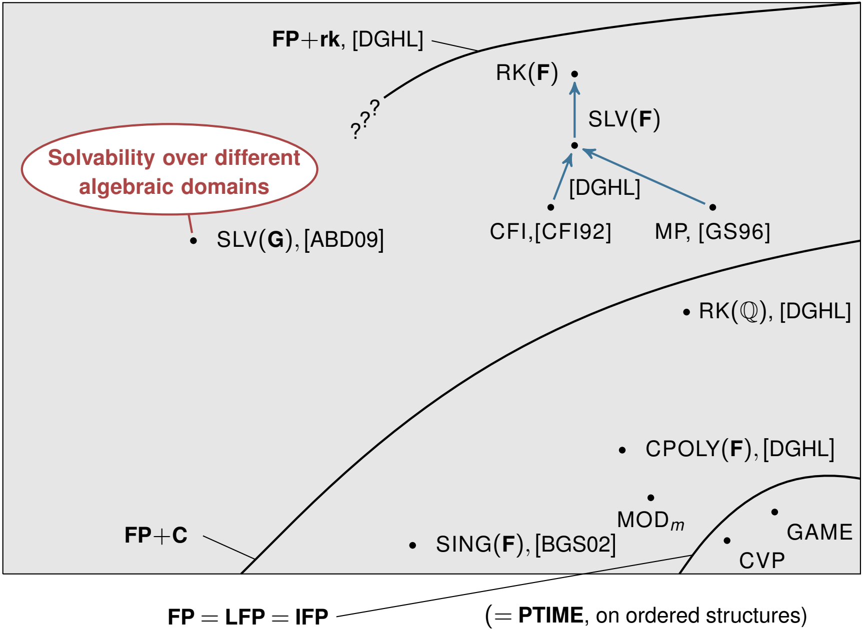

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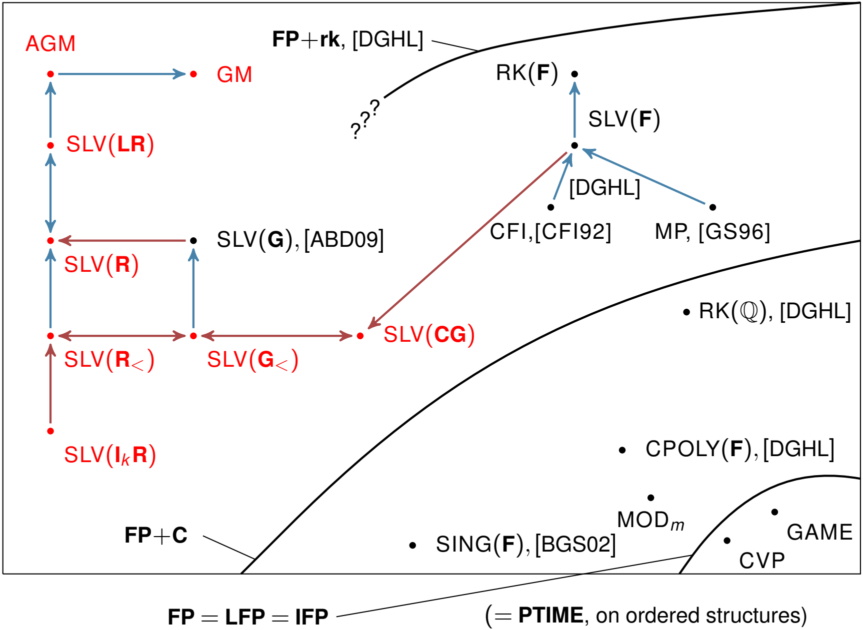

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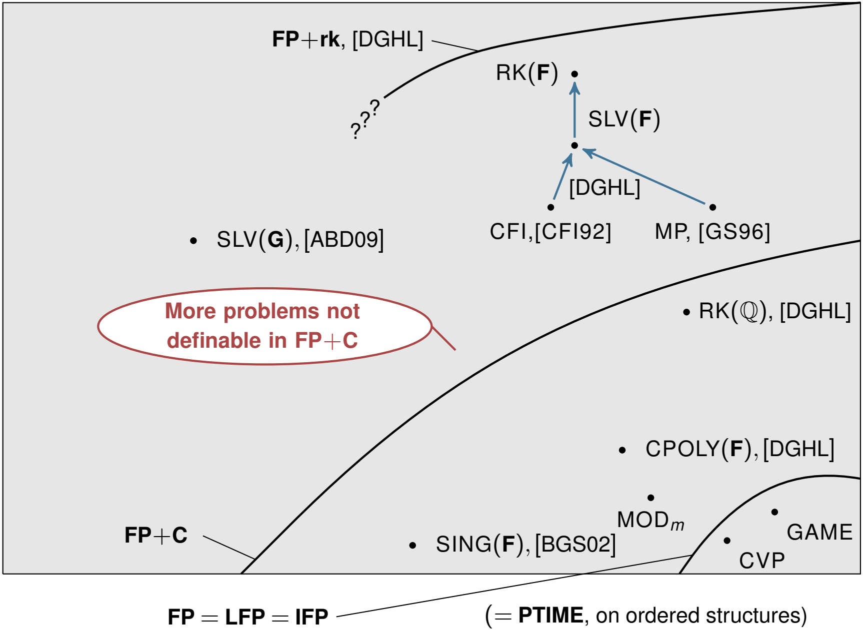

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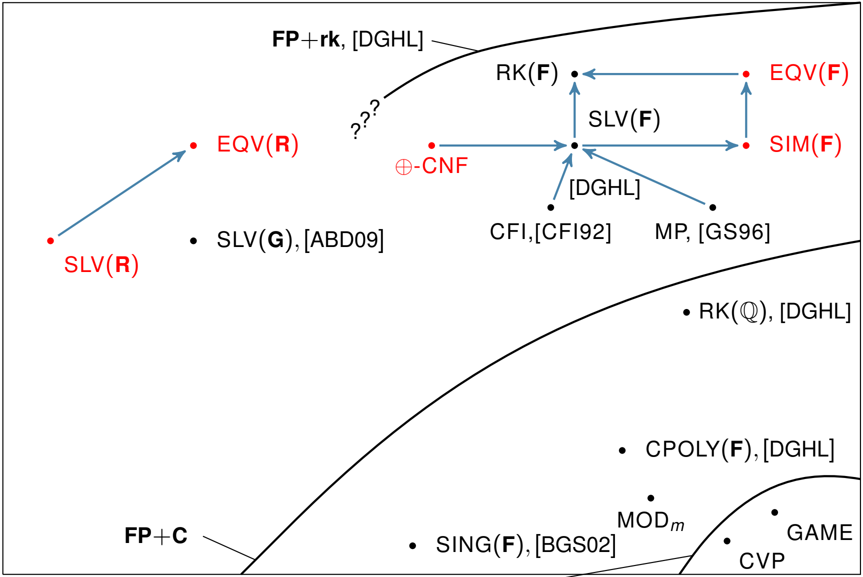

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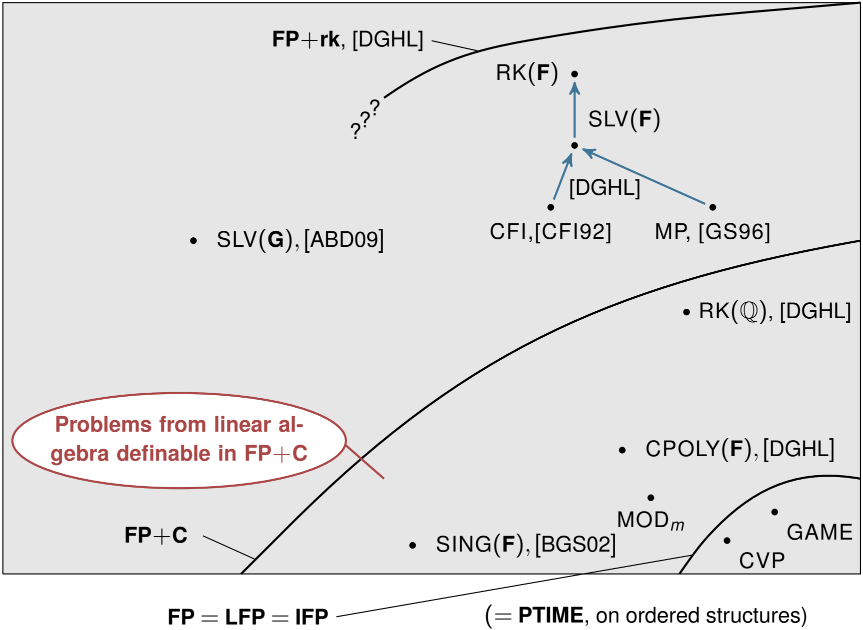

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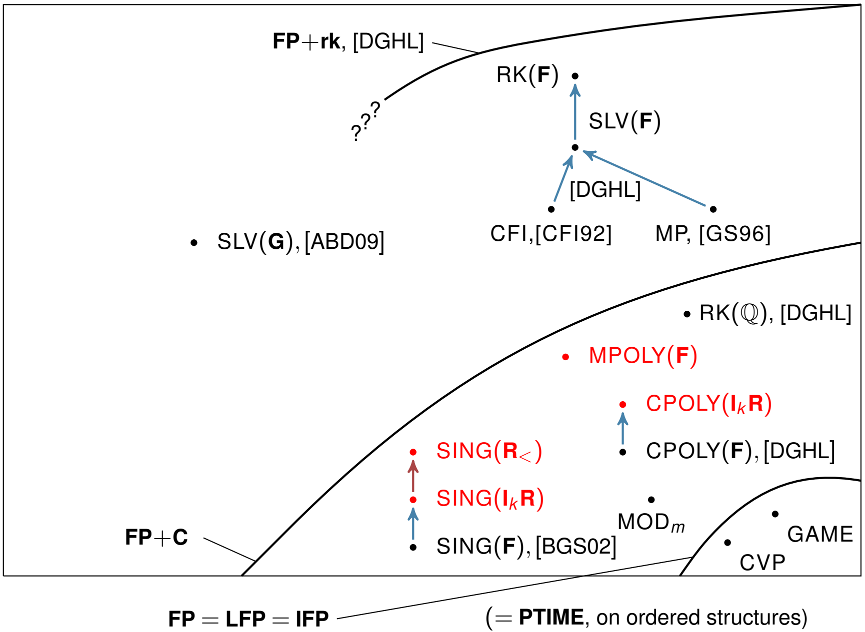
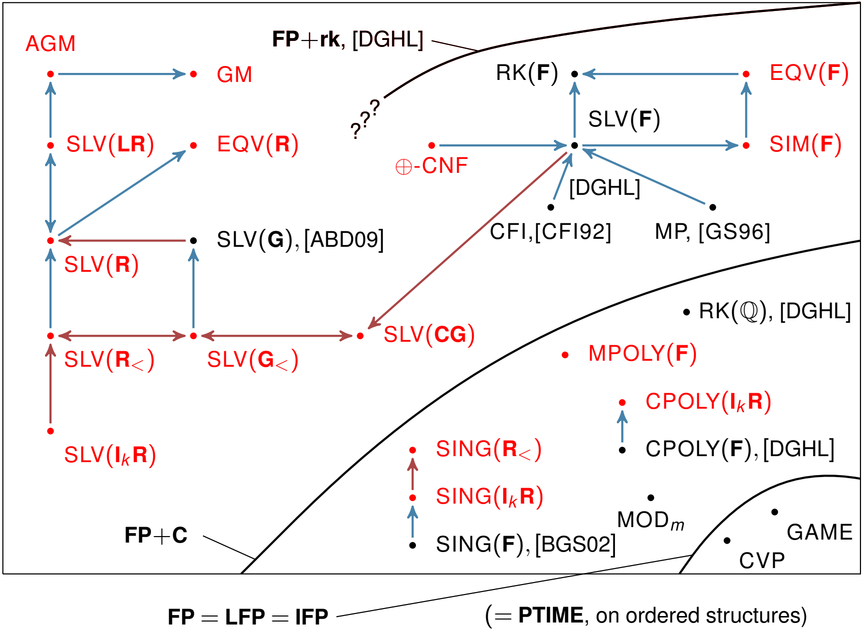


Thank you for your attention!



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