Definability of linear equation systems over groups and rings

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AlMoTh, Ilmenau, 17 February 2012

A logic for polynomial time

Atserias, Bulatov, Dawar $\mathsf{Slv}(\mathbf{G}) \notin \mathsf{FP{+}C}$

Dawar, Grohe, Holm, Laubner $\mathsf{FP{+}C} \lneq \mathsf{FP{+}rk} \leq \mathsf{PTIME}$

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Matrix rank and linear equation systems

Fields $A \cdot x = b$ solvable iff $\mathrm{rk}(A) = \mathrm{rk}(A|b)$:

If $r = \mathrm{rk}(A)$, then $a_1 \cdot \mathbf{c}_1 + \cdots + a_r \cdot \mathbf{c}_r + a \cdot \mathbf{b} = \mathbf{0}$

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Rings Many notions (linear dependence, McCoy, inner rank, ...), unknown complexity, above characterisation fails

Groups Undefined

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Groups Undefined

Question: Is $\mathsf{Slv}(\mathbf{G}) \in \mathsf{FP{+}rk}$?

A systematic study of solvability

Inter-definability: $\rightsquigarrow$ natural domain for Slv

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Theorem

k-ideal rings $\overset{\text{FP-red.}}{\Longrightarrow}$ cyclic groups of prime power order.

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Theorem

Normal form for $\mathsf{FO}+\mathsf{slv}_{\mathsf{F}}$.

Inter-definability: a natural class for solvability

$\mathsf{Slv}(\mathbf{CG})$: Cyclic groups ($\mathbb{Z}_{p^e}$)
$\mathsf{Slv}(\mathbf{I}_k\mathbf{R})$: k-gen. ideal rings ($I \trianglelefteq R \Rightarrow I = \pi_1 R + \cdots + \pi_k R$)

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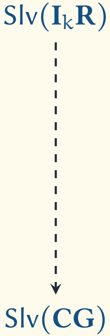

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$\mathsf{Slv}(\mathbf{I}_k\mathbf{R})$ $\qquad$ $\mathsf{Slv}(\textbf{local-}\mathbf{I}_k\mathbf{R})$

$\downarrow$

$\mathsf{Slv}(\mathbf{CG})$ $\qquad$ $\mathsf{Slv}(\mathbf{R}_<)$

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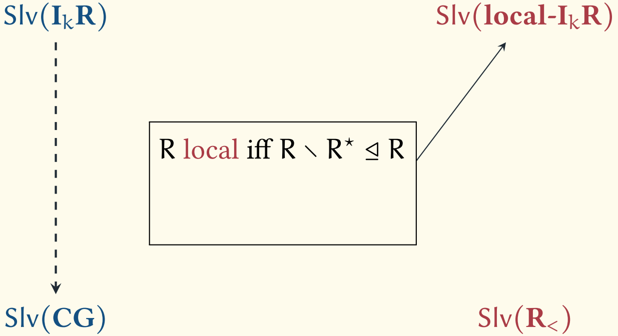

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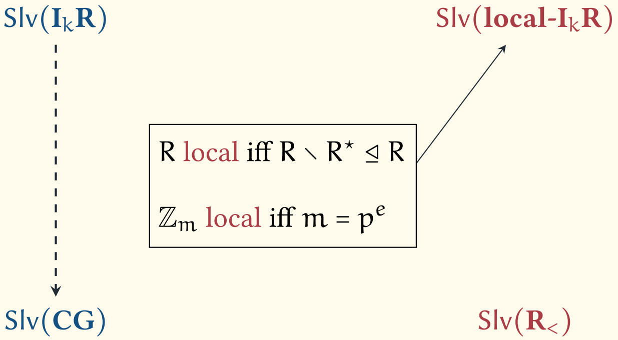

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$$\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \xrightarrow{R = \bigoplus_{e \in \varphi} eR,\ eR \text{ local}} \mathsf{Slv}(\textbf{local-}\mathbf{I}_k\mathbf{R})$$

$$\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \dashrightarrow \mathsf{Slv}(\mathbf{CG})$$

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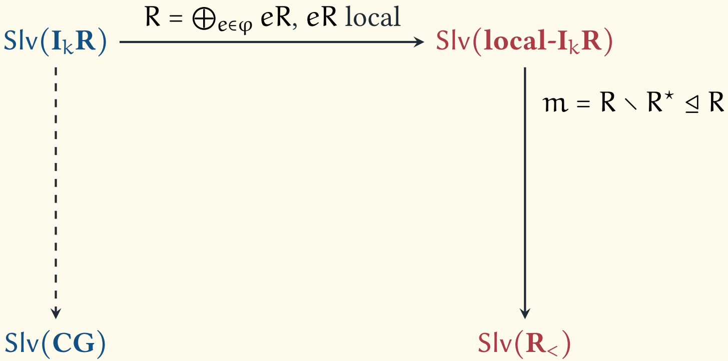

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$\mathsf{Slv}(\textbf{local-}\mathbf{I}_k\mathbf{R}) \longrightarrow \mathsf{Slv}(\mathbf{R}_<)$, with

$\mathfrak{m} = R \setminus R^\star \trianglelefteq R$

$\Gamma(R) = \{a : a^{|R/\mathfrak{m}|} = a\}$

$\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \dashrightarrow \mathsf{Slv}(\mathbf{CG})$

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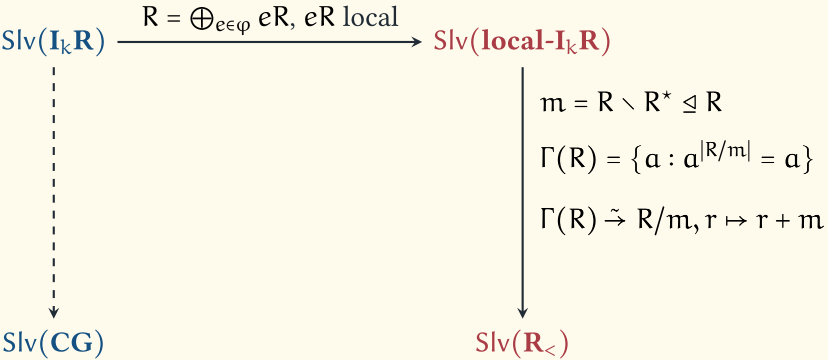

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$\mathsf{Slv}(\mathbf{I}_k\mathbf{R})$ → $\mathsf{Slv}(\mathbf{local}\text{-}\mathbf{I}_k\mathbf{R})$: $R = \bigoplus_{e \in \varphi} eR$, eR local

$\mathsf{Slv}(\mathbf{local}\text{-}\mathbf{I}_k\mathbf{R})$ → $\mathsf{Slv}(\mathbf{R}_<)$:

$\mathfrak{m} = R \setminus R^\star \trianglelefteq R$

$\Gamma(R) = \{a : a^{|R/\mathfrak{m}|} = a\}$

$\Gamma(R) \xrightarrow{\sim} R/\mathfrak{m}, r \mapsto r + \mathfrak{m}$

$r \mapsto \sum \underbrace{a_{i_1 \dots i_k}}_{\in \Gamma(R)} \pi_1^{i_1} \cdots \pi_k^{i_k}$

$\mathsf{Slv}(\mathbf{I}_k\mathbf{R})$ ⇢ $\mathsf{Slv}(\mathbf{CG})$

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$\mathsf{Slv}(\mathbf{R}_<) \longrightarrow \mathsf{Slv}(\mathbf{CG})$:

$(R, +) = \oplus_{g \in \psi} \langle g \rangle$

$(A, \mathbf{b}) \mapsto (A_1, \mathbf{b_1}), \ldots, (A_k, \mathbf{b_k})$

$\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \dashrightarrow \mathsf{Slv}(\mathbf{CG})$

Inter-definability: a natural class for solvability

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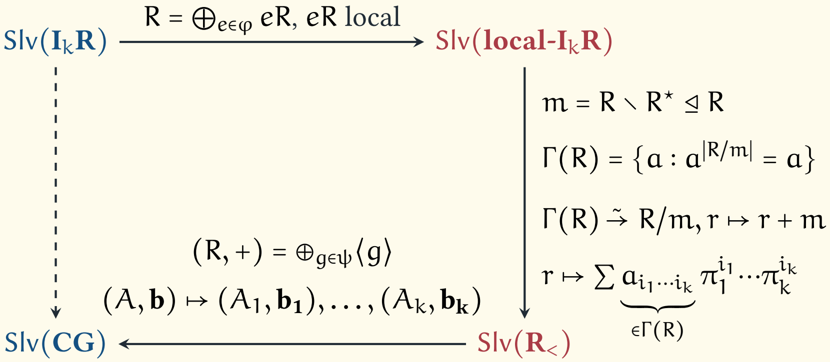

Theorem $\mathsf{Slv}(\mathbf{I}_k\mathbf{R}) \leq^T_{FP} \mathsf{Slv}(\mathbf{CG})$

Intra-definability: solvability as a logical operator

$$\mathsf{slv}(\bar{x}, \bar{y}, \bar{r}_i).\Big[\varphi_M(\bar{x}, \bar{y}, \bar{r}), \varphi_b(\bar{x}, \bar{r}), \underbrace{(\varphi_R, \varphi_+, \varphi_\cdot)(\bar{r}_1, \bar{r}_2, \bar{r}_3)}\Big]$$

coefficient matrix — solution vector — finite ring

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$$\mathsf{slv}(\bar{x}, \bar{y}, \bar{r}_i).\Big[\underbrace{\varphi_M(\bar{x}, \bar{y}, \bar{r})}_{\text{coefficient matrix}}, \underbrace{\varphi_b(\bar{x}, \bar{r})}_{\text{solution vector}}, \underbrace{(\varphi_R, \varphi_+, \varphi_\cdot)(\bar{r}_1, \bar{r}_2, \bar{r}_3)}_{\text{finite ring}}\Big]$$

$$\Downarrow$$

$\mathsf{FO{+}slv}$: First-order logic closed under solvability quantifier

$\mathsf{FO{+}slv}_\mathsf{F}$: Solvability quantifier over a fixed finite field F

Intra-definability: solvability as a logical operator

Theorem

Every $\mathsf{FO}+\mathsf{slv}_\mathsf{F}$-formula equivalent to an $\mathsf{FO}+\mathsf{slv}_\mathsf{F}$-formula

$$\mathsf{slv}(\bar{x},\bar{y}).\big[\varphi_M(\bar{x},\bar{y}),\mathbf{1}\big],\ \text{ with } \varphi_M \text{ quantifier-free.}$$

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Proof illustration: (negation)

$$\neg\mathsf{slv}(\bar{x}, \bar{y}).\big[\varphi, \mathbf{1}\big]$$

$$\text{Non-solvability} \equiv \neg\exists \mathbf{x} : \mathbf{M}\mathbf{x} = \mathbf{b} \overset{?}{\equiv} \exists \mathbf{y} : \mathbf{M}'\mathbf{y} = \mathbf{b}' \equiv \text{Solvability}$$

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Gaussian elimination implies:

$$\neg\exists\mathbf{x}:\mathbf{Mx}=\mathbf{b} \equiv \exists\mathbf{y}:\mathbf{y}(\mathbf{M}|\mathbf{b})=(0,\ldots,0|1).$$

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Proof illustration: (conjunction)

$$\mathsf{slv}(\bar{x},\bar{y}).\big[\varphi,\mathbf{1}\big] \wedge \mathsf{slv}(\bar{x},\bar{y}).\big[\psi,\mathbf{1}\big]$$

$$\boxed{\ \varphi\ } \cdot \mathbf{v_y} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

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Intra-definability: solvability as a logical operator

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Proof illustration: (universal quantification)

$$\forall z\,(\mathsf{slv}(\bar{x},\bar{y}).\big[\varphi(\bar{x},\bar{y},z),\mathbf{1}\big])$$

$$\boxed{\varphi(z_1)}\cdot\mathbf{v_y} = \boxed{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}$$

$$\vdots$$

$$\boxed{\varphi(z_n)}\cdot\mathbf{v_y} = \boxed{\begin{matrix}1\\ \vdots\\ 1\end{matrix}}$$

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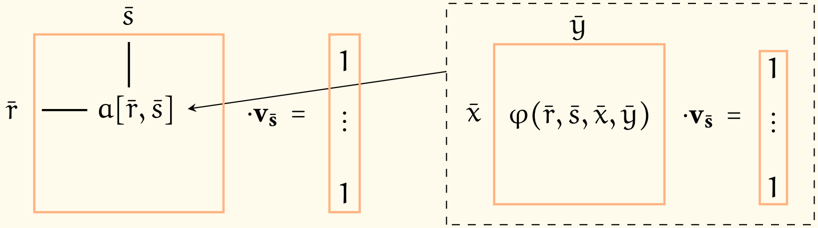
Proof illustration: (nesting of solvability)

$$\mathsf{slv}(\bar{r}, \bar{s}).\big[\mathsf{slv}(\bar{x}, \bar{y}).\big[\varphi(\bar{r}, \bar{s}, \bar{x}, \bar{y}), \mathbf{1}\big], \mathbf{1}\big]$$



Outer system: $\mathbf{S}$ Inner system: $\mathbf{I}[\bar{r}, \bar{s}]$

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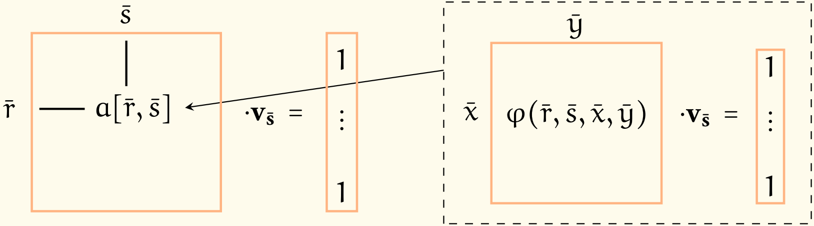

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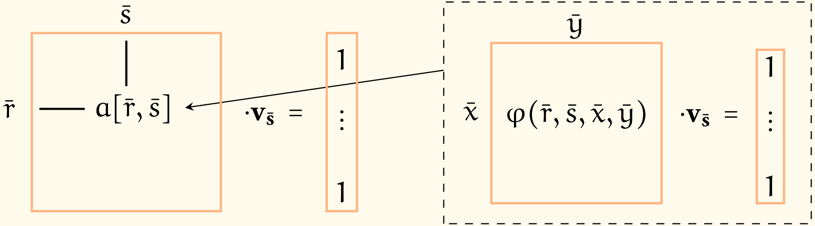

For $\bar{r}$: $\sum_{\bar{s}} \underbrace{a[\bar{r}, \bar{s}] \cdot v_{\bar{s}}} = 1$

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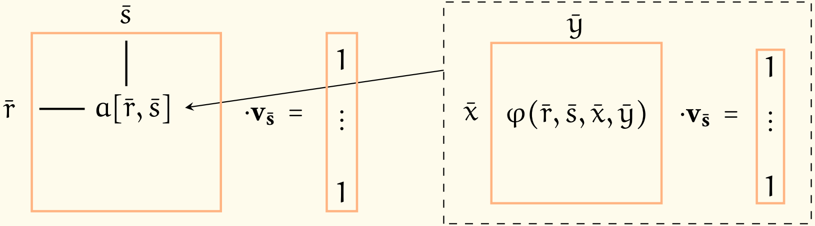

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$\rightsquigarrow$

For $\bar{r}$: $\sum_{\bar{s}} 1\cdot v[\bar{r},\bar{s}] = 1$

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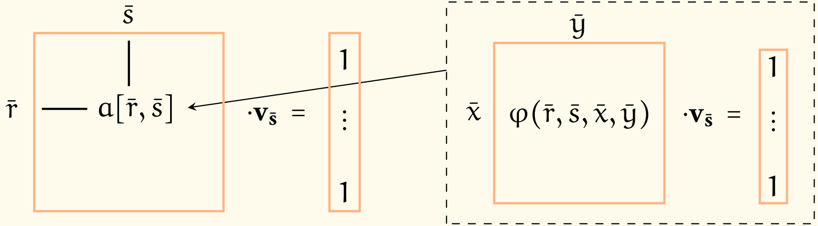

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Consistency conditions:

$v[\bar{r},\bar{s}] = 1 \Rightarrow a[\bar{r},\bar{s}] = 1$

$v[\bar{r},\bar{s}] \neq v[\bar{r}',\bar{s}] \Rightarrow a[\bar{r},\bar{s}] \neq a[\bar{r}',\bar{s}]$

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For $\bar{r}$: $\sum_{\bar{s}} \underbrace{a[\bar{r}, \bar{s}] \cdot v_{\bar{s}}} = 1$ $\rightsquigarrow$

$\downarrow$

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How to formalise: "If $v = 1$ then $A \cdot x = 1$ solvable"

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How to formalise: "If $v = 1$ then $A \cdot x = 1$ solvable"

$$\begin{bmatrix} A & \begin{matrix} -v+1 \\ \vdots \\ -v+1 \end{matrix} \end{bmatrix} \cdot \mathbf{x} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Conclusion and outlook

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Every $\mathsf{FO}+\mathsf{slv}_\mathsf{F}$-formula is equivalent to an $\mathsf{FO}+\mathsf{slv}_\mathsf{F}$-formula

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Outlook: Permutation group membership (GM)

Given: Permutations $\pi_1, \ldots, \pi_k$ and π on a set A

Question: Is $\pi \in \langle \pi_1, \ldots, \pi_l \rangle \leq S_A$?

Conclusion and outlook

Theorem

Every FO+slv_F-formula is equivalent to an FO+slv_F-formula

$$\mathsf{slv}(\bar{x}, \bar{y}).\left[\varphi_M, \mathbf{1}\right], \text{ with } \varphi_M \text{ quantifier-free.}$$

Theorem

k-ideal rings $\overset{\text{FP-red.}}{\Longrightarrow}$ cyclic groups of prime power order.

Outlook: Permutation group membership (GM)

Given: Permutations $\pi_1, \ldots, \pi_k$ and π on a set A

Question: Is $\pi \in \langle \pi_1, \ldots, \pi_l \rangle \leq S_A$?

$$\mathsf{Slv}(\mathbf{D}) \overset{\text{FO-reduction}}{\mapsto} \mathsf{GM} \qquad \text{(Cayley's theorem)}$$