A Logic for Polynomial Time: Recent Results and Challenges

Wied Pakusa

RWTH Aachen University

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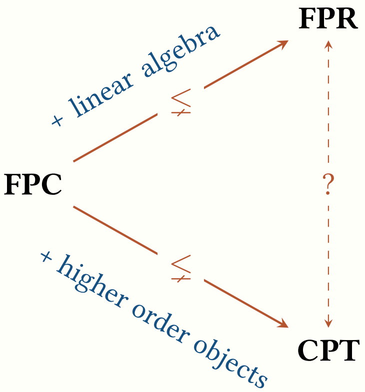
➜ **Linear equation systems** and the Search for a Logical Characterisation of Polynomial Time ➜

Two candidates, two results



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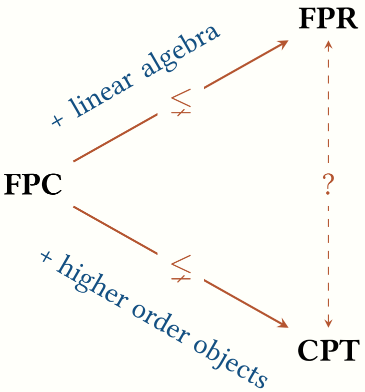

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Rank logic (original version) fails to capture polynomial time

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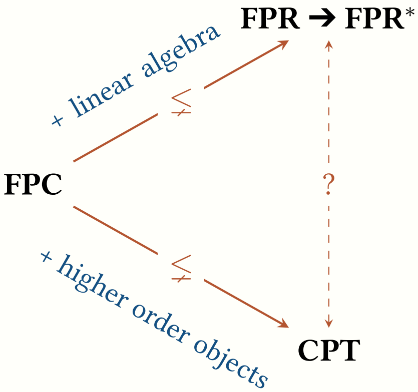

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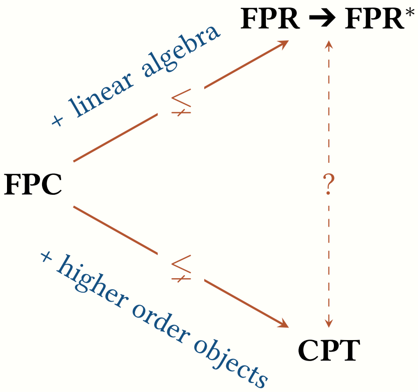

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- Rank operators over prime fields $\mathbb{F}_p$

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$\bar{b}$

$$M_\vartheta^{\mathfrak{A}} = \bar{a} \quad \vartheta(\bar{a}, \bar{b}) \qquad \text{mod } p$$

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Theorem (Grädel, P. '15)
For $\Omega, \Theta \subseteq \mathbb{P}$, $\Omega \neq \Theta$ we have

$$\mathbf{FPR}_\Omega \neq \mathbf{FPR}_\Theta$$

In particular

$$\mathbf{FPR} < \mathbf{FPR}^*$$

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Main ideas:

- Cai, Fürer, Immerman graphs over $\mathbb{F}_q$
- (Non-empty) solution spaces of definable linear equation systems over $\mathbb{F}_p$ always contain symmetric solutions
- Symmetric solutions can be defined in **FPC**

Structures with Abelian colours

CPT = P?

Total order $\overset{0}{\bullet} < \overset{1}{\bullet} < \cdots < \overset{n-1}{\bullet}$

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Total order	0 •	$<$	1 •	$<$	$\cdots$	$<$	n-1 •	✓

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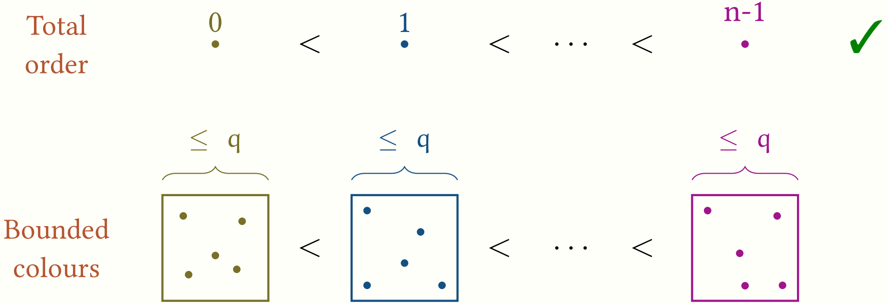

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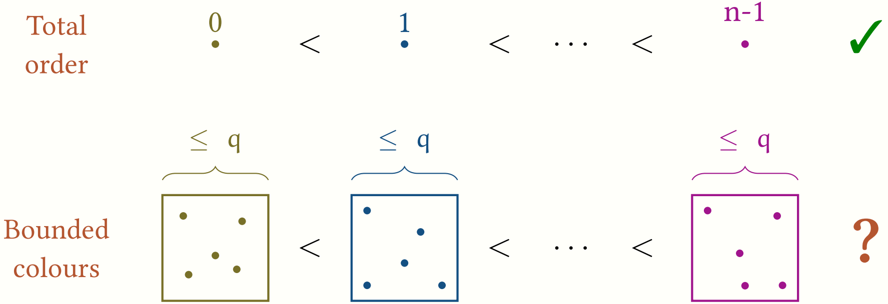

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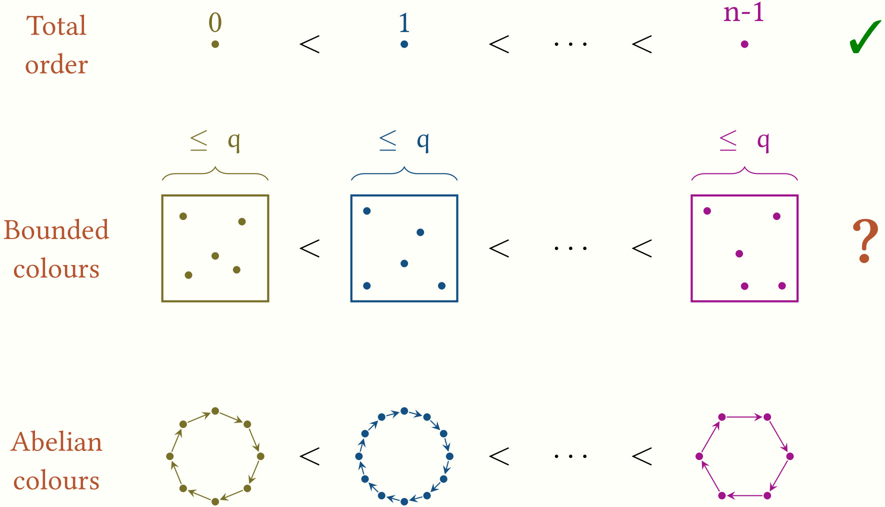

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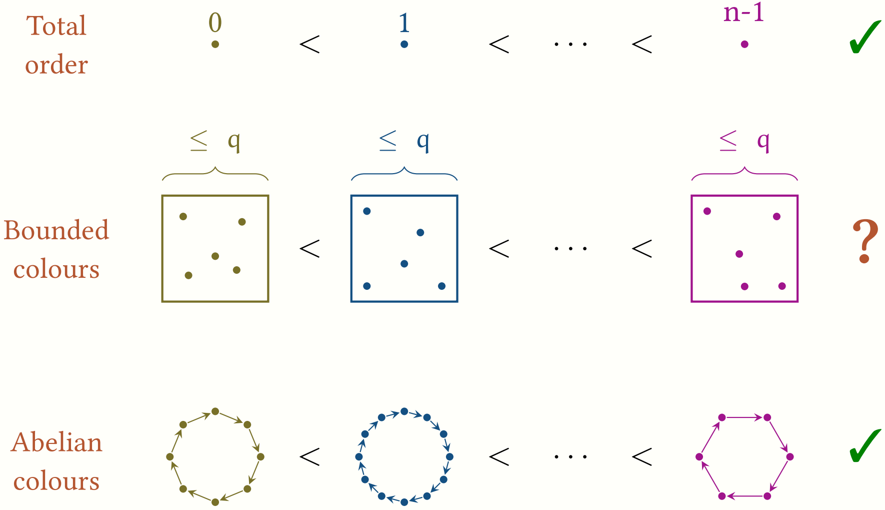

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Theorem (P. '15)

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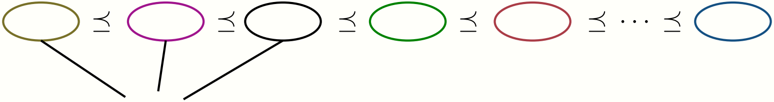

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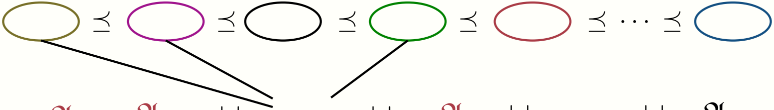

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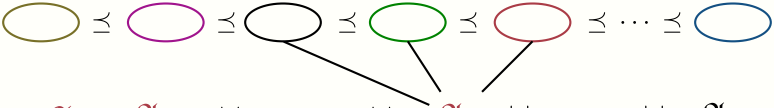

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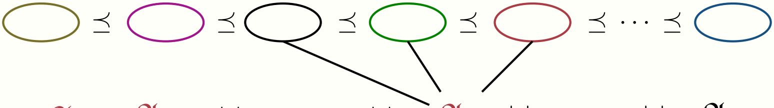

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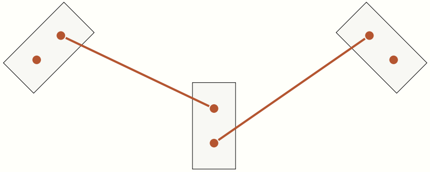

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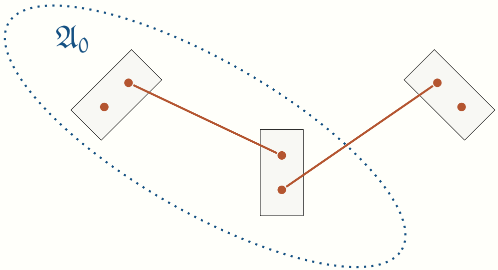

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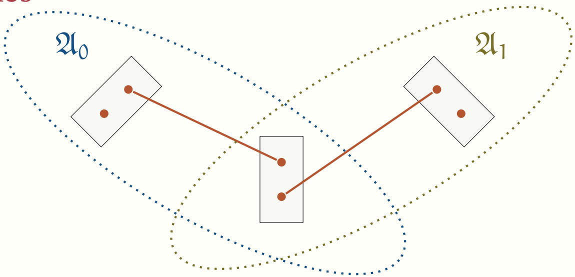

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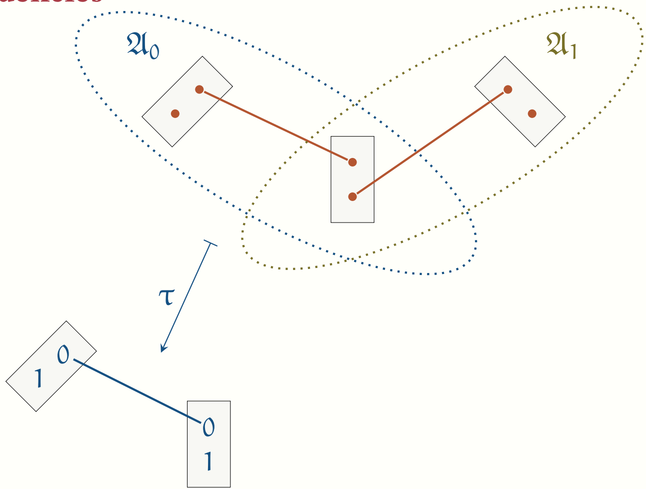

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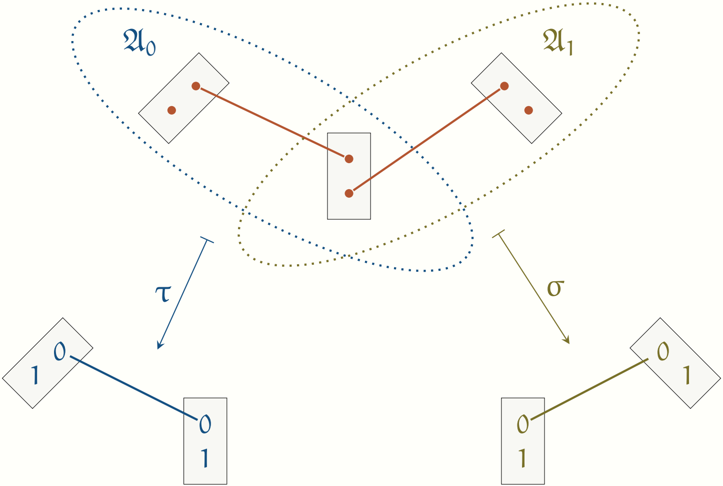

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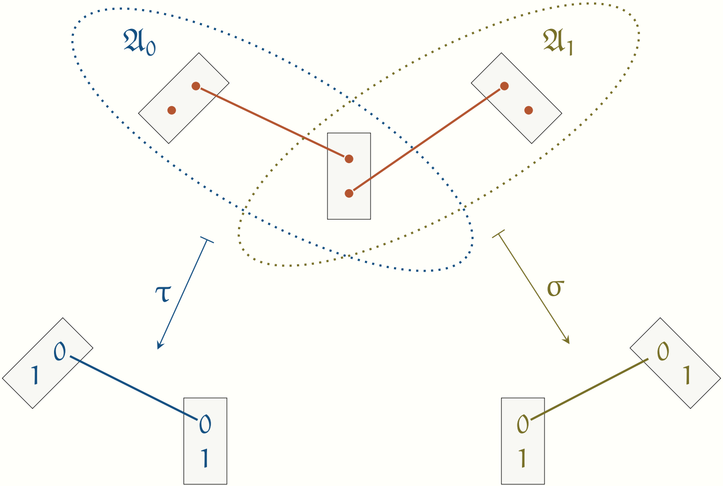

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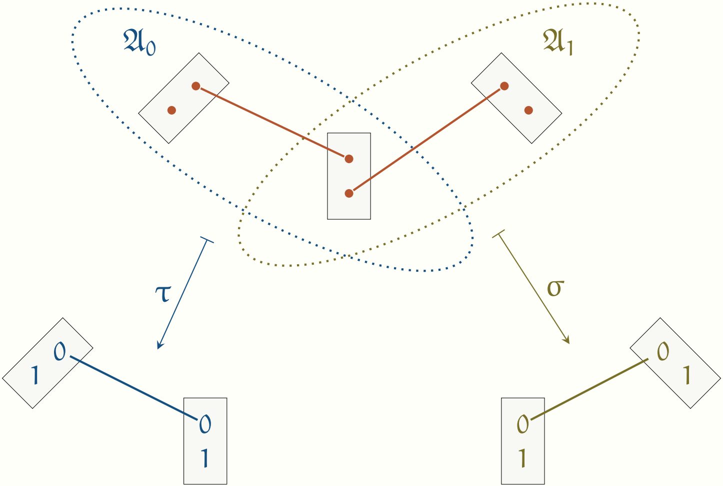

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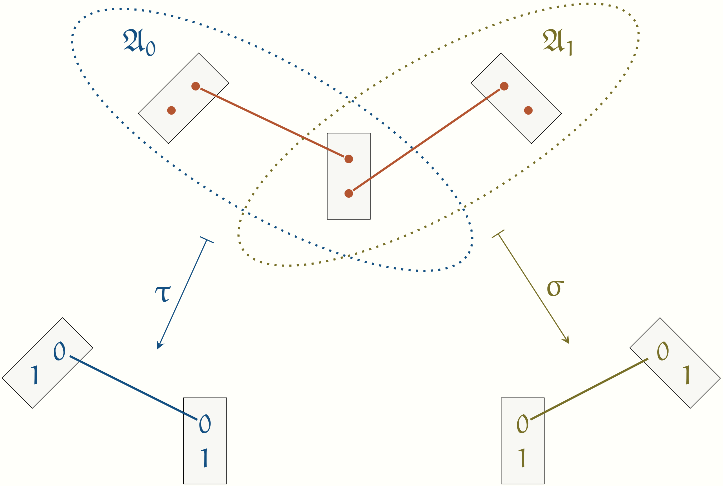

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✐ Representation via cyclic linear equation systems

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Compose isomorphism with projections onto summands $\mathbb{Z}_2$.

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Corollary

Asum is definable in (a certain variant of) rank logic.

Assumptions: $k \geq 3$ and $(G, +) \cong (\mathbb{Z}_2^n, +)$

Let $A, B, C \subseteq G$ be $\equiv^k_{\infty\omega}$-equivalence classes

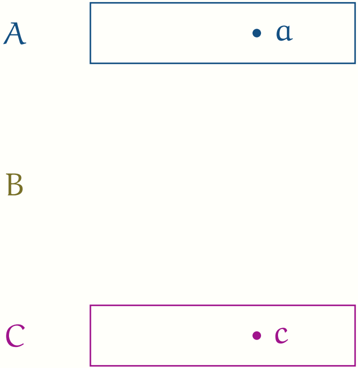
Assumptions: $k \geq 3$ and $(G, +) \cong (\mathbb{Z}_2^n, +)$

Let $A, B, C \subseteq G$ be $\equiv^k_{\infty\omega}$-equivalence classes , that is for all $a, a' \in A$ (likewise for B, C) and all $\varphi(x) \in C^k_{\infty\omega}$ we have

$$(G, +, M) \models \varphi(a) \iff (G, +, M) \models \varphi(a').$$

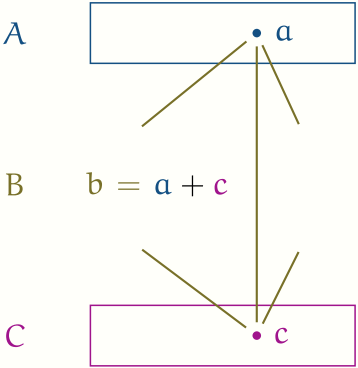
Assumptions: $k \geq 3$ and $(G, +) \cong (\mathbb{Z}_2^n, +)$

Let $A, B, C \subseteq G$ be $\equiv_{\infty\omega}^{k}$-equivalence classes



Assumptions: $k \geq 3$ and $(G, +) \cong (\mathbb{Z}_2^n, +)$

Let A, B, $C \subseteq G$ be $\equiv^k_{\infty\omega}$-equivalence classes



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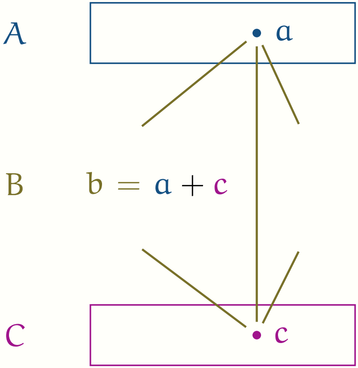

➜ Degree invariants:

$$d_A \cdot |A| = d_B \cdot |B| = d_C \cdot |C|$$

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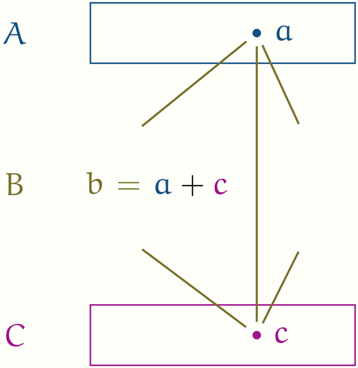

➜ Degree invariants:

$$d_A \cdot |A| = d_B \cdot |B| = d_C \cdot |C|$$

$$d_A \cdot \sum A + d_B \cdot \sum B = d_C \cdot \sum C$$

Application of the colour refinement technique

Lemma
There is an FPC-reduction (from Asum to Asum)

$$(G, +, M) \quad \longmapsto \quad (G, +, M')$$

such that M' is an odd $\equiv^k_{\infty\omega}$-equivalence class.

Question: Can we say something about sums over odd $\equiv^k_{\infty\omega}$-equivalence classes $M \subseteq G$?

For certain cases we can, e.g., if M is closed under addition.

Rank Logic

Choiceless Polynomial Time

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$\mathbf{FPR} < \mathbf{FPR}^* \leq \textsc{Ptime}$

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Choiceless Polynomial Time

CPT $=$ PTIME on structures with Abelian colours.

Rank Logic

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Choiceless Polynomial Time

$\mathbf{CPT} = \textsc{Ptime}$ on structures with Abelian colours.

Open questions

- Is $\mathbf{FPR}^* = \textsc{Ptime}$?

Rank Logic

FPR $<$ **FPR*** $\leq$ PTIME

Choiceless Polynomial Time

CPT $=$ PTIME on structures with Abelian colours.

Open questions

- Is **FPR*** $=$ PTIME?

- On solvable colours?
- On bounded colours?

Rank Logic

FPR $<$ **FPR***$\leq$ PTIME

Choiceless Polynomial Time

CPT $=$ PTIME on structures with Abelian colours.

Open questions

- Is **FPR*** $=$ PTIME?

- On solvable colours?
- On bounded colours?

Abelian summation problem (ASUM)

- Is ASUM definable in **CPT**?
- Is ASUM definable in **FPC**?