Banach-Mazur Games with Simple Winning Strategies

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Hiermit versichere ich, dass ich die Arbeit selbständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt sowie Zitate kenntlich gemacht habe.

(Simon Robert Leßenich)

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Introduction

Mathematical games have been examined for more than a century by now, and have proven to be of great use in many different areas. On the one hand, the field referred to as classical game theory - with e.g. its concept of games in strategic form - has had huge influence on disciplines that range from biology to economy and social sciences. Although several people had worked on similar topics before, what brought game theory to a broader attention and started its successful rise was the the foundational book of von Neumann and Morgenstern [24]. On the other hand, there is the field of games which have their applications in a more theoretical area of mathematics and computer science. Examples from this field include games used in topology (e.g. the classical Banach-Mazur game and its different variants and modifications [21]) as well as classical infinite games on graphs (which can be viewed as games derived from automata theory on infinite words, e.g. Büchi games and Muller games) and games that can be formulated as games of this form (e.g. Gale-Stewart games [8]).

In this work, games from the latter field are considered, with the main focus being laid upon a certain kind of infinite two-player games on graphs. While games from classical game theory are mostly finite and many problems discussed in this area nowadays are based on the introduction of additional players or imperfect information, in topology, logic and many areas of computer science, games are used to describe infinite processes or model infinite behavior. When focusing on modeling infinite behavior, two major classes of games come to mind. In many situations, one considers infinite games played on a finite arena, i.e. on a finite graph. As an example, Büchi and Muller games in their traditional form are of this kind, and so are model checking games of fixed point logics on finite structures. As in the field of Model Checking it is often the task to check finite systems against a logical specification, many problems occurring there can also be described by such games. However, sometimes the information required to model a situation cannot be encoded in a finite graph, which leads to infinite games on infinite graphs. In our context, this difference in graph size is seldom explicitly used, as in many cases it is of low importance. Instead, the main difference between the games described above and the ones discussed in this thesis is that in contrast to representing moves by edges in the graph, the players are allowed to choose finite paths. These path games are a natural extension of classical infinite games on graphs, and have also already been used in different contexts in computer science. For example, in situations one encounters when discussing planning in nondeterministic domains [19], there is no singular execution path of a system anymore, but, because of the nondeterministic outcomes of an action, an execution tree. When requiring a system to satisfy a given specification, one differentiates between several variants of satisfaction (e.g. weak, strong or cyclic planning), which can all be modeled via path games with different move-alternation patterns between a player responsible for "good" outcomes of actions and one responsible for "bad" ones. Note that while in classical infinite games on graphs, the graph is partitioned into vertices of the first player and vertices where the second player moves, this is not a useful convention for path games, since instead of moving two times in a row, a player could directly choose the longer path. It turns out that there are only several different variants of "who-moves-when" (ranging from all moves by a single player over two alternations to strictly alternating) that provide different outcomes with respect to different kinds of planning. However, in our scenario we only consider path games with strictly alternating moves, and where it is always the first player that begins.

As such games coincide with topological Banach-Mazur games (played on a certain topology), they are usually referred to by this name, and because they are simply a special case of their topological variant, results from topology can be adopted. For example, the Banach-Mazur Theorem [17], which relates the topological size or complexity of the goal, i.e. of the winning condition, to the existence of winning strategies for the respective players, can directly be applied to Banach-Mazur games on graphs. This provides a practical means to examine whether a subset of the infinite paths in a graph is meager or co-meager in the space of all infinite paths in that graph, and has already been put to use in this manner. For example, one sensible notion of fairness is that the respective fair paths form a co-meager set [4, 23], i.e. only a topologically small fragment of the possible behaviors of a system is unfair. Whether a property is thus a fairness property can be checked via finding a winning strategy in a corresponding Banach-Mazur game. Furthermore, using results about positional determinacy for certain Banach-Mazur games, it can be shown that for ω-regular properties, a system is topologically fair if and only if it is probabilistically fair [22].

In a similar way, topological semantics of timed automata were defined [1, 2]. In this context, an LTL formula is said to be satisfied by an automaton if the set of runs satisfying the formula is topologically large, i.e. intuitively there is only a "small number" of runs violating the specification. Again, this test for co-meagerness can be done by checking whether winning strategies for a corresponding Banach-Mazur game exist. In addition, Banach-Mazur games have been used in the above papers to prove that probabilistic semantics for timed automata (i.e. an automaton satisfies a formula if the probability that a randomly chosen run satisfies it equals 1) and topological semantics coincide, i.e. an automaton largely satisfies an LTL formula (via the topological semantics) if and only if an automaton almost surely satisfies it (via the probabilistic semantics).

For this thesis, we use the variant of Banach-Mazur games on graphs as formalized in [9] - which is also already mentioned in [5, 19] - and continue the exploration of simple winning strategies for such games and which classes of winning conditions they guarantee determinacy for. In our context, simple winning strategies are strategies that are not simply functions mapping the history to a next move, but functions that only have restricted access of some kind to the history. Examples of such strategies include positional strategies and strategies using various (simple) kinds of finite or infinite memory, e.g. the Finite Appearance Records as introduced in [10]. We examine the strength of these strategies - as well as that of other kinds of strategies we will introduce later on - especially with respect to different classes of Muller winning conditions and a generalization of Muller winning conditions to finite words of colors.

Outline

We start this work with a definition of Banach-Mazur games on graphs. As Banach-Mazur games on graphs are a special case of topological Banach-Mazur games, we introduce this special topology in Section 1.2. We then illustrate various properties of this topology, which culminate in the Banach-Mazur Theorem. In Section 1.3, we begin with the discussion of the main question handled in this thesis, namely determinacy and determinacy via classes of strategies. We describe - following [9] - how the Baire property for sets and the Borel hierarchy can be used to conclude that a game is determined or that a winning condition guarantees determinacy. It will also be explained why decomposition invariant strategies are equally powerful as the most general form of strategies, which allows us to restrict general strategies to decomposition invariant ones as a basic concept. We then present a connection between determinacy via a class of strategies in classical games on graphs and determinacy via that class in Banach-Mazur games.

To further motivate the search for simple winning strategies, we recite the result from Varacca and Völzer [22] about the equivalence of probabilistic and topological largeness for certain kinds of fairness properties (or winning conditions).

In the third chapter, two different variants of prefix independent winning conditions are discussed. We begin with Muller winning conditions, where those over a finite set of colors and those over a countably infinite one are treated separately.

For Muller winning conditions over a finite set of colors, we solely present a result from [5, 9] that such Muller winning conditions essentially reduce to reachability conditions (i.e. Pl. 0 wins if and only if a certain kind of stable vertex can be reached). Hence we directly move to Muller winning conditions over a countably infinite set of colors.

For those, the discussion (cf. Chapter 3.2.2) is once more divided. We begin with Muller winning conditions where one of the sets $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton, and then increase the cardinality of this set to finite and infinite ones. We will see that for these three classes where one of the $\mathcal{F}$-sets is still countable, a connection between the existence of a winning strategy and the reachability of certain colors can be obtained. In other words, some good colors can be enforced, while others can simultaneously be prevented from being seen infinitely often. We also provide an example to demonstrate that Muller winning conditions with infinitely many colors where the sets of winning sets of both players are uncountably infinite do not generally guarantee determinacy.

In Section 3.3, we introduce a generalization of Muller winning conditions where the goal is not anymore to see exactly a specific set of colors infinitely often. Instead, the goal for Pl. 0 is to see certain words (i.e. finite sequences) of colors infinitely often, while other words may be seen only finitely many times. It turns out that in many aspects, these winning conditions are similar to Muller winning conditions, and we thus present results connecting properties of these winning conditions to the existence of winning strategies, i.e. we derive a connection similar to the one in the previous section. Again, we mainly focus on such sequential winning conditions where the set of colors is countable, so that the winning conditions still guarantee determinacy.

Chapter 4 in turn deals with different classes of winning strategies. We begin this discussion with the most basic/restrictive kind of a simple strategy, namely positional ones. We cite results from [5, 9] that these strategies suffice for Muller winning conditions over a finite set of colors, and also for ω-regular winning conditions. For classical games on graphs, an important class of simple winning strategies is the one of strategies using finite memory. As it has already been shown in [9] that in the setting of Banach-Mazur games, strategies using finite memory reduce to positional strategies, we mention such strategies only briefly and continue with strategies using infinite memory.

The first class of simple strategies using infinite memory considered here is the one of counting strategies, i.e. strategies having access to an auto-increasing non-decreasing counter. We discuss two different variants, namely strategies that count the number of moves, and such that count the length of the prefix played so far. We show that the former is weaker than the latter, but that the former already suffices for many interesting classes of Muller and sequential winning conditions over a countable set of colors.

The next class of infinite memory strategies examined with regard to Banach-Mazur games in this work are strategies using FAR-memory, as introduced in [10]. Instead of adapting the proofs for Muller games presented there, we provide new simpler ones for Banach-Mazur games showing that many Muller winning conditions guarantee determinacy via strategies using FAR-memory. We then discuss the relation between such strategies and counting strategies, and provide examples demonstrating that the classes are in a sense incomparable, i.e. there exist games determined via one, but not via the other, and vice versa.

In the last chapter, we introduce the notion of a bounded strategy, i.e. a strategy where the (maximal) length a move is permitted to have is restricted in some way. We analyze two different concepts of such restrictions and examine their effects. It turns out that in both cases, restricting the players to such bounded strategies may change the outcome of a play. Furthermore, the usual notion of a winning strategy fails to be of much use, as certain games can still be thought of as determined, despite not allowing for winning strategies. In addition, we will see that letting the players choose bounding functions introduces - via imperfect information - some kind of nondeterminism.

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Chapter 1

Preliminaries

We begin this chapter with introducing Banach-Mazur games on graphs and then discuss basic properties of such games. Starting with the definition of a game, its components and how it is played, we introduce a topology on the possible plays and present the Banach-Mazur Theorem, which characterizes the winning conditions for which one of the players has a winning strategy. We then explain what determinacy is, and present some characterizations of winning conditions which guarantee determinacy. Finally, we argue that decomposition invariant strategies suffice as a general notion of strategies.

1.1 The Banach-Mazur game

In its original form (cf. [17], Problem 43), a Banach-Mazur game is played on the real line. The winning condition consists of a set of real numbers W and both players take turns choosing refined intervals $d_i \subseteq \mathbb{R}$ such that at any time $d_{i+1} \subsetneq d_i$. Pl. 0 wins a play if and only if the countable intersection of all intervals contains an element of W.

Banach-Mazur games have also been studied in descriptive set theory [15, Chapter 8.H] and topology [21]. In the topological setting, a winning condition is a subset $W \subseteq X$ of some topological space X, and instead of intervals of reals, a sequence $V_0 \supsetneq V_1 \supsetneq V_2 \supsetneq \cdots$ of non-empty open sets is chosen in an alternating fashion by the two players. Pl. 0 wins if the intersection of this sequence is a subset of W.

The notion of Banach-Mazur games on graphs was introduced in [5, 9, 19] as a special case of topological Banach-Mazur games. Such games are essentially infinite games on graphs, with the major difference in comparison to classical games on graphs being that the players choose finite paths instead of edges. Throughout this work we use the following definition of such games.

Definition 1.1.1. A *Banach-Mazur game* $\mathcal{G} = (G, v_0, \mathsf{Win})$ is a triplet consisting of

- a directed graph $G = (V, E)$ called the *arena*, where every node has a successor,
- an initial vertex $v_0 \in V$, and
- a subset Win of the infinite paths in G that start in v_0.

A game is played by two players, Pl. 0 and Pl. 1, which alternate in choosing finite prolongations of a path. Pl. 0 begins with choosing a finite path π_0 which starts in v_0. Pl. 1 then prolongs π_0 with another finite path π_1 and so on. After an infinite number of moves an infinite path α - called a *play* of $\mathcal{G}$ - has been chosen. Pl. 0 wins the play α if and only if $\alpha \in \mathsf{Win}$.

Sometimes, Win is implicitly defined by means of a function $\Omega\colon V \to C$ and a language $\mathcal{W} \subseteq C^\omega$. Pl. 0 wins a play α in this variant if for the word

$$\mathsf{w}(\alpha) = \Omega(\alpha[0]) \cdot \Omega(\alpha[1]) \cdot \Omega(\alpha[2]) \cdots$$

it holds that $\mathsf{w}(\alpha) \in \mathcal{W}$, i.e. we have

$$\mathsf{Win} = \{\alpha : \mathsf{w}(\alpha) \in \mathcal{W}\}.$$

Notice that the two variants are in a sense equivalent, since any Win can be replaced by $\Omega\colon V \to V, v \mapsto v$ and $\mathcal{W} = \mathsf{Win}$.

The major concept discussed in this work is that of a *strategy*. In its most general form for Banach-Mazur games, a strategy for Player σ is a function $f\colon \mathsf{FinPaths}(G)^* \to \mathsf{FinPaths}(G)$ that satisfies the following constraint: if $(\pi_1, \cdots, \pi_n)$ is a tuple of finite paths such that $\pi_1 \cdots \pi_n \in \mathsf{FinPaths}(G, v_0)$, then $f(\pi_1, \cdots, \pi_n) \in \mathsf{FinPaths}(G, \mathsf{last}(\pi_n))$. Such a function essentially assigns a finite path, i.e. a next move, to a possible position in the game, i.e. after the moves $\pi_1, \cdots, \pi_n$ have been made. Since the game is played in a strictly alternating form, it is always known who chose which π_n. We say that a play α is *consistent* with a strategy f if α can be decomposed in such a way that all moves of the respective player are moves according to f. Furthermore, we say that a strategy for Player σ is *winning* if every consistent game is won by that Player σ.

We will see in Section 1.3.3 that the type of strategy introduced above can always be replaced by a simpler variant (the decomposition of the path played so far into the players' moves is unnecessary), and we examine winning conditions for which even simpler kinds of strategies suffice. Before doing so, we start by defining a topology which allows us to adapt classical results on Banach-Mazur games for our setting. We then establish the notion of determinacy (via a class of strategies) and present the Borel hierarchy.

Notation Before we talk about topology and properties of Banach-Mazur games on graphs in more detail, we briefly introduce some notation. For a graph $G = (V, E)$, the set of vertices reachable from a vertex v will be called

$$vE^* := \{v' \in V : \text{ there ex. a finite path } \pi \text{ from } v \text{ to } v'\}.$$

Furthermore, we write $\mathsf{last}(\pi)$ to reference the vertex in which a finite path π ends. We use $\mathsf{FinPaths}(G)$ as a name for the set of finite paths in G, and $\mathsf{Paths}(G)$ for the set of infinite ones. If, in addition to G, a vertex is given as a second argument, we mean the set of finite/infinite paths in G that start in this vertex. As already used above, with $\mathsf{w}(\alpha)$ we denote the word over the alphabet C induced by α.

1.2 Topology and the Banach-Mazur Theorem

As mentioned above, Banach-Mazur games have been extensively studied in topology. In this section we introduce a topology on the infinite paths in a graph showing that the Banach-Mazur games on graphs studied here do not essentially differ from the usual notion in set theory or topology, as for example in [15]. For this topology we then provide a proof of the Banach-Mazur Theorem. This section is based on [9].

Definition 1.2.1. A *topological space* $(X, \mathcal{O})$ consists of a set X and a set $\mathcal{O}$ of *open* subsets of X, for which the following hold:

- $\emptyset, X \in \mathcal{O}$
- For any $x_1, x_2 \in \mathcal{O}$: $x_1 \cap x_2 \in \mathcal{O}$
- Any union of sets from $\mathcal{O}$ is again in $\mathcal{O}$.

The set $\mathcal{O}$ is then called a *topology* on X.

Definition 1.2.2. A *base* for a topology $\mathcal{O}$ on a set X is a subset of $\mathcal{O}$ whose elements are called *basic open sets* such that every open set is a union of basic open sets.

We now introduce a topology on graphs (or on infinite paths in a graph, to be precise) by providing a suitable collection of basic open sets. For this collection, we then show that the resulting set of open sets is indeed a topology.

As said above, the topology will be one on the space of infinite paths in a graph. So let $G = (V, E)$ be a directed graph without terminal nodes and let $v \in V$ be some vertex. In the following, we use $X = \mathsf{Paths}(G, v)$. The collection of basic open sets is

$$\{\mathcal{BO}(\pi) : \pi \in \mathsf{FinPaths}(G, v)\},$$

where $\mathcal{BO}(\pi) := \pi \cdot \mathsf{Paths}(G, \mathsf{last}(\pi))$ is the set of all infinite paths that start with the prefix π. As open sets are unions of basic open sets, an open set Γ can be characterized as

$$\Gamma = \bigcup_{\pi \in P} \mathcal{BO}(\pi) = P \cdot V^\omega \cap \mathsf{Paths}(G, v)$$

for some $P \subseteq \mathsf{FinPaths}(G, v)$. It remains to show that the class $\mathcal{O}$ of all such Γ is a topology on $\mathsf{Paths}(G, v)$:

- $\emptyset$ can be written as the empty union of basic open sets, while $\mathsf{Paths}(G, v) = \mathcal{BO}(v)$, hence both are in $\mathcal{O}$.

- For two open sets $\Gamma_0 = P_0 \cdot V^\omega \cap \mathsf{Paths}(G, v)$ and $\Gamma_1 = P_1 \cdot V^\omega \cap \mathsf{Paths}(G, v)$, let $P := \{\pi \in \mathsf{FinPaths}(G, v) : \pi \text{ minimal such that } \mathcal{BO}(\pi) \subseteq \Gamma_0 \cap \Gamma_1\}$. The intersection can be written as $\Gamma_0 \cap \Gamma_1 = P \cdot V^\omega \cap \mathsf{Paths}(G, v)$, hence $\Gamma_0 \cap \Gamma_1 \in \mathcal{O}$. (Essentially, for any $\pi \in P_i$, in order to belong to P, there has to be a prefix π' of π such that $\pi' \in P_{1-i}$.)

- Let $Y = \{\Gamma_i = P_i \cdot V^\omega \cap \mathsf{Paths}(G, v) : i \in I\} \subseteq \mathcal{O}$ be a set of open sets. We have to show that the union of Y is again an open set:

  $$\bigcup Y = \bigcup_{i \in I} \Gamma_i = \bigcup_{\pi \in Z} \mathcal{BO}(\pi) \in \mathcal{O},$$

  where $Z = \bigcup_{i \in I} P_i$.

As we have shown that $\mathcal{O}$ as characterized above is a topology on $\mathsf{Paths}(G, v)$, it is easy to see that Banach-Mazur games on graphs are a special case of topological Banach-Mazur games on the topology $\mathcal{O}$, since with each move a player selects a refined basic open set.

We continue by introducing characterizations for closed, dense, nowhere dense and meager sets valid in the graph topology, since these simplify the formulation and the proof of the Banach-Mazur Theorem.

Recall that a *closed* set is the complement of an open set. Let now Λ be a closed set, i.e. $\Lambda = \Gamma^{\mathcal{C}}$ for some $\Gamma = P \cdot V^\omega \cap \mathsf{Paths}(G, v) \in \mathcal{O}$. This means that

$$\Lambda = \mathsf{Paths}(G, v) \setminus (P \cdot V^\omega) = \bigcap_{\pi \in P} \mathcal{BO}(\pi)^{\mathcal{C}}.$$

When looking at the infinite tree $\mathsf{FinPaths}(G, v)$, the elements from $\mathsf{Paths}(G, v)$ are precisely the infinite paths in this tree with root v. The above characterization of closed sets states that a closed set is in fact the set of infinite paths of some tree, namely the tree on $\mathsf{FinPaths}(G, v)$ that remains after removing all subtrees with roots in P. It can easily be seen that the converse is also true, i.e. for each tree on $\mathsf{FinPaths}(G, v)$ with root v, the set of infinite paths in this tree is a closed set.

A set Γ is *dense*, if its intersection with every basic open set is non-empty. Formally, we have

$$\Gamma \text{ dense} \iff \Gamma \cap \mathcal{BO}(\pi) \neq \emptyset \text{ for every } \pi \in \mathsf{FinPaths}(G, v).$$

In other words, a set Γ is dense, if every finite path π that starts in v can be prolonged to an infinite path in Γ. If a set Γ is both open and dense, i.e. $\Gamma = P \cdot V^\omega \cap \mathsf{Paths}(G, v)$ is dense, this means that every finite path can be prolonged to a finite path that has a prefix in P (otherwise no infinitely prolonged path would be in Γ). Hence every finite path π can be extended to a finite π' such that $\mathcal{BO}(\pi') \subseteq \Gamma$.

Having said what a dense set is, we are ready to define *nowhere dense* sets. A set Γ is nowhere dense if it is not dense in any open set, i.e. for all open $P \cdot V^\omega \cap \mathsf{Paths}(G, v)$ it holds that $\Gamma \cap \mathcal{BO}(\pi) = \emptyset$ for at least one π that has a prefix in P. Nowhere dense sets can also be characterized by the fact that a set Γ is nowhere dense if the complement $\Gamma^{\mathcal{C}}$ contains a dense open set. Intuitively this states that every finite path π can be prolonged to a finite π' such that $\mathcal{BO}(\pi') \subseteq \Gamma^{\mathcal{C}}$, or equivalently $\mathcal{BO}(\pi') \cap \Gamma = \emptyset$.

A *meager* set Γ is a countable union of nowhere dense sets, i.e.

$$\Gamma = \bigcup_{n \in \omega} \Gamma_n, \text{ where all } \Gamma_n \text{ are nowhere dense.}$$

With the above characterization of nowhere dense sets, this means that for any $n \in \omega$ and every $\pi \in \mathsf{FinPaths}(G, v)$ there exists a prolongation π' of π with $\mathcal{BO}(\pi\pi') \cap \Gamma_n = \emptyset$. A set whose complement is meager is called a *co-meager* set.

We proceed with introducing the notion of a Baire space, then prove that the topological space established above is a space of such kind and show a property of Baire spaces we will use in the proof of the Banach-Mazur Theorem.

Definition 1.2.3. A topological space $(X, \mathcal{O})$ is a *Baire space* if and only if every non-empty set is not both open and meager. The latter condition is equivalent to the condition that every countable intersection $y = \bigcap_{n\in\omega} y_n$ of dense open y_n is again dense.

Proposition 1.2.1. $(\mathsf{Paths}(G, v), \mathcal{O})$ *is a Baire space.*

Proof. Let $\Gamma = \bigcap_{n\in\omega} \Gamma_n$ be a countable intersection of dense open sets Γ_n. It needs to be shown that for any $\pi \in \mathsf{FinPaths}(G, v)$ it holds that $\Gamma \cap \mathcal{BO}(\pi) \neq \emptyset$. Let therefore π be an arbitrary finite path starting in v. We iteratively create an infinite path α that starts with π as a prefix such that $\alpha \in \Gamma_n$ for every $n \in \omega$. The main idea is to use the fact that, for dense open sets, each finite path can be finitely prolonged such that the basic open set defined by this prolongation is contained in the dense open set. We hence start with $\pi_0 = \pi \cdot \rho_0$, where ρ_0 is a finite prolongation of π such that $\mathcal{BO}(\pi_0) \subseteq \Gamma_0$. In the same way, π_n is π_{n-1} prolonged with ρ_n, again so that $\mathcal{BO}(\pi_n) \subseteq \Gamma_n$. Since all Γ_i are dense and open, such a prolongation is always possible. The construction results in an infinite α for which it holds that $\alpha \in \Gamma_n$ for every $n \in \omega$. But then $\alpha \in \Gamma$. □

Proposition 1.2.2. *Let $(X, \mathcal{O})$ be a Baire space. For any set y it holds that y is co-meager if and only if y contains a dense set y' which is a countable intersection of open sets.*

Proof. For the direction from left to right, let y be a co-meager set. Then $y^{\mathcal{C}}$ is meager, so $y^{\mathcal{C}} = \bigcup_{n\in\omega} y_n$, where all y_n are nowhere dense. Hence $y = \bigcap_{n\in\omega} y_n^{\mathcal{C}}$, and each $y_n^{\mathcal{C}}$ contains a dense open set y_n'. Thus let $y' := \bigcap_{n\in\omega} y_n' \subseteq y$. Because the space is a Baire space, y' is dense, which was to prove.

For the remaining direction, let y be a set that contains a dense set $y' = \bigcap_{n\in\omega} y_n$ which is a countable intersection of open sets y_n. It follows immediately from the fact that y' is dense that all y_n are dense open sets. Since $y' \subseteq y$, we have $y = \bigcap_{n\in\omega}(y_n \cup y)$, and each $(y_n \cup y)$ contains a dense open set, namely y_n. From the characterization of nowhere dense sets it follows that every $(y_n \cup y)^{\mathcal{C}}$ is nowhere dense. But then $y^{\mathcal{C}} = \bigcup_{n\in\omega}(y_n \cup y)^{\mathcal{C}}$ is meager, or equivalently, y is co-meager. □

With the above, we are ready to formulate the Banach-Mazur Theorem, in the version in which it was presented in [9]. The theorem relates the fact that a player has a winning strategy in a game to topological properties of the set of paths that are winning for Pl. 0. For a Banach-Mazur game $\mathcal{G} = (G, v_0, \mathsf{Win})$ we use the topology on $\mathsf{Paths}(G, v_0)$ introduced above.

Theorem 1.2.1 (Banach-Mazur [17, 9])**.** *Let $\mathcal{G} = (G, v_0, \mathsf{Win})$ be a Banach-Mazur game on a graph G, for some $\mathsf{Win} \subseteq \mathsf{Paths}(G, v_0)$.*

(i) Pl. 1 has a winning strategy for $\mathcal{G}$ $\iff$ Win is meager.

(ii) Pl. 0 has a winning strategy for $\mathcal{G}$ $\iff$ there exists a finite path π starting in v_0 such that Win is co-meager in $\mathcal{BO}(\pi)$.

Proof. We start the proof with showing (i):
First, assume that Pl. 1 has a winning strategy f for the game $\mathcal{G}$. We show that the set $\mathsf{Plays}(f)$ of all plays consistent with the strategy f is co-meager.

Because of Proposition 1.2.2, it suffices to show that $\mathsf{Plays}(f)$ is a countable intersection of dense open sets (since such sets are always dense in a Baire space). It is easy to see that $\mathsf{Plays}(f) = \bigcap_{n\in\omega} \mathsf{Plays}_n(f)$, where $\mathsf{Plays}_n(f)$ is the set of all infinite paths or plays in which Pl. 1 plays according to f in his first n moves. It remains to show that each of these sets is both dense and open. To see that it is open, we rewrite the set as $\mathsf{Plays}_n(f) = F_n \cdot V^\omega \cap \mathsf{Paths}(G, v_0)$, where F_n is the set of minimal finite prefixes of paths in $\mathsf{Plays}_n(f)$ such that Pl. 1 has already moved n times. Each of these sets is also dense, as Pl. 0 can start with an arbitrary finite path π and Pl. 1 can always prolong this path to an infinite one that is consistent with f.

Since f is a winning strategy it follows that $\mathsf{Plays}(f) \subseteq \mathsf{Win}^{\mathcal{C}}$. Since $\mathsf{Plays}(f)$ is co-meager, $\mathsf{Win}^{\mathcal{C}}$ is also co-meager, which entails that Win is meager.

For the other direction, let Win be meager. In this case, we construct a winning strategy for Pl. 1. Since Win is meager, it can be written as $\mathsf{Win} = \bigcup_{n\in\omega} \Gamma_n$, where each Γ_n is nowhere dense. In his n-th move, Pl. 1 now prolongs the current path π to a π' in such a way that $\mathcal{BO}(\pi') \cap \Gamma_n = \emptyset$ (this is always possible since Γ_n is nowhere dense). For such a strategy f it is easy to see that $\mathsf{Plays}(f) \cap \mathsf{Win} = \emptyset$, from which it immediately follows that f is winning.

The proof of (ii) is basically analogous with the difference being that instead of Win, $\mathsf{Win}^{\mathcal{C}}$ is considered, and that this set is not meager, but meager in some basic open set. This is due to the fact that Pl. 0 starts each play and the players switch roles after this opening move, in a sense. Thus the finite path π from the theorem is essentially $f(v_0)$

for a winning strategy f. Correspondingly, it can easily be shown that $\mathsf{Plays}(f)$ for a strategy of Pl. 0 is a countable intersection of dense open subsets of $\mathcal{BO}(f(v_0))$. Using this, the proof of (i) can be adapted to a proof for (ii). □

1.3 Determinacy, strategies and the Borel hierarchy

One of the major questions of interest in the field of infinite games is that of *determinacy*. In its basic form, one asks whether a given game $\mathcal{G}$ with a winning condition Win is determined, i.e. whether one of the players has a winning strategy. Using the Banach-Mazur Theorem, the following corollary can be shown which states that if Win satisfies certain conditions, then $\mathcal{G}$ is determined. However, the opposite direction of the corollary is not true since winning conditions of determined games can be arbitrarily complex, e.g. by combining a simple determined game with a complex undetermined one. In this case the winning condition is complex, but the game might still be trivially determined.

To formulate the corollary, the following definition is required.

Definition 1.3.1. Let $(X, \mathcal{O})$ be a topological space. A set $x \subseteq X$ has the *Baire property* if there exists an open set $y \in \mathcal{O}$ such that the symmetric difference $(x \cup y) \setminus (x \cap y)$ is meager.

Corollary 1.3.1 ([9])**.** *Let $\mathcal{G} = (G, v_0, \mathsf{Win})$ be a Banach-Mazur game. If $\mathsf{Win} \subseteq \mathsf{Paths}(G, v_0)$ has the Baire property, then $\mathcal{G}$ is determined.*

Proof. To prove the corollary, we distinguish between two cases. If Win is meager, then Pl. 1 has a winning strategy by the Banach-Mazur Theorem. So assume that Win is not meager. Since Win has the Baire property, there exists an open set $\Gamma = P \cdot V^\omega \cap \mathsf{Paths}(G, v)$ such that $(\mathsf{Win} \cup \Gamma) \setminus (\mathsf{Win} \cap \Gamma)$ is meager. Since any subset of a meager set is also meager, it follows that $\mathsf{Win} \setminus \Gamma$ is meager, and so is $\Gamma \setminus \mathsf{Win}$. If $\Gamma \setminus \mathsf{Win} \neq \emptyset$, there exists a $\pi \in P$ such that $\mathcal{BO}(\pi) \setminus \mathsf{Win}$ is again meager. In other words, $\mathsf{Win}^{\mathcal{C}}$ is meager in $\mathcal{BO}(\pi)$, and equivalently Win is co-meager in $\mathcal{BO}(\pi)$. By the Banach-Mazur Theorem it follows that Pl. 0 has a winning strategy. If $\Gamma \setminus \mathsf{Win} = \emptyset$, then $\Gamma \subseteq \mathsf{Win}$. In this case, Win is obviously co-meager in Γ (notice that $\mathsf{Win} \neq \emptyset$, as Win is not meager, and also $\Gamma \neq \emptyset$, as this would entail that Win is meager). Again Pl. 0 has a winning strategy.

In any case, one of the players has a winning strategy, hence $\mathcal{G}$ is determined. □

1.3.1 The Borel hierarchy

An important class of sets when discussing infinite games is the class of *Borel sets*. This class consists of all sets that are on some level in the *Borel hierarchy*, which is obtained by taking the closure of the open sets under union, countable intersection and complementation.

Definition 1.3.2 (Borel hierarchy)**.** For a topology $\mathcal{O}$ on a set X, the Borel hierarchy consists of levels Σ^0_n, Π^0_n which are defined as follows:

- The first level Σ^0_1 is the set of open sets:

$$\Sigma^0_1 = \mathcal{O}.$$

- For any n, $\Pi^0_n := \{x^{\mathcal{C}} : x \in \Sigma^0_n\}$. (Hence Π^0_1 is the set of all closed sets.)
- For any n, Σ^0_{n+1} is the set of all countable unions of Π^0_n sets, and accordingly Π^0_{n+1} is the set of all countable intersections of Σ^0_n sets.
- For any limit ordinal λ, Σ^0_λ is the union of all Σ^0_n for $n < \lambda$.

We say that a set x is on level Σ^0_n, if n is minimal such that $x \in \Sigma^0_n$ and $x \notin \Sigma^0_i \cup \Pi^0_i$ for any $i < n$ (for Π^0_n analogously). We say that a set x is Borel, if it is on some level of the Borel hierarchy.

The reason why Borel sets are of particular interest in game theory can be found in Martin's Theorem, which provides a practical means to show determinacy for certain winning conditions.

Theorem 1.3.1 (Martin [16])**.** *Every Borel game is determined.*

This theorem implies that every Banach-Mazur game with a winning condition that lies on some level of the Borel hierarchy is determined. Instead of adapting the proof from Martin, we prove the theorem by showing that every Borel set has the Baire property, from which Martin's Theorem in the context of Banach-Mazur games directly follows.

Proposition 1.3.1. *Let $(X, \mathcal{O})$ be a topological space, and let $x \subseteq X$ be a Borel set. Then x has the Baire property.*

Proof. The proof of the proposition consists of three steps. First, we show that every open set (i.e. every set on level Σ_1^0) has the Baire property. We then show that the complement of a set with the Baire property also has it and end in proving that the countable union of sets with the Baire property has the property as well. Using this, it directly follows from the definition of the Borel hierarchy that every Borel set has the Baire property. For the latter two parts we use ideas from the proof of Proposition 3.5.1 in [20], that shows that the sets that have the Baire property form a σ-algebra, and [14, page 505].

Let at first x be an open set. Recall that we have to show that there exists an open set such that the symmetric difference of x with this set is meager. Consider the symmetric difference of x with itself:

$$(x \cup x) \setminus (x \cap x) = \emptyset.$$

Obviously $\emptyset$ is meager, since it is nowhere dense.

Let now x be a set that has the Baire property. We want to show that $x^{\mathcal{C}}$ also has this property. Let therefore y be an open set such that the symmetric difference $(x \setminus y) \cup (y \setminus x)$ is meager (y exists because x has the Baire property). Consider the unique smallest closed set $\bar{y}$ such that $y \subseteq \bar{y}$ (called the *closure* of y). Then $\bar{y} \setminus y$ is nowhere dense. (Assume it is not. Then there exists a basic open set $\mathcal{BO}(\pi)$ such that $\bar{y} \setminus y$ is dense in $\mathcal{BO}(\pi)$. Since $\bar{y} \setminus y$ is closed and dense in $\mathcal{BO}(\pi)$, it follows that $\mathcal{BO}(\pi) \cap (\bar{y} \setminus y) = \mathcal{BO}(\pi)$, which entails $\mathcal{BO}(\pi) \subseteq (\bar{y} \setminus y)$. Thus, $\mathcal{BO}(\pi) \cap y = \emptyset$, and since $\bar{y}$ is minimal such that $y \subseteq \bar{y}$, it follows that $\mathcal{BO}(\pi) \not\subseteq \bar{y}$, which is a contradiction.) Since $\bar{y}$ is closed, $\bar{y}^{\mathcal{C}}$ is open. For the symmetric difference of $\bar{y}^{\mathcal{C}}$ and $x^{\mathcal{C}}$ it holds that

$$(x^{\mathcal{C}} \setminus \bar{y}^{\mathcal{C}}) \cup (\bar{y}^{\mathcal{C}} \setminus x^{\mathcal{C}}) = (\bar{y} \setminus x) \cup (x \setminus \bar{y}).$$

Since the symmetric difference of x and y is meager, and $\bar{y} \setminus y$ is nowhere dense, the symmetric difference of x and $\bar{y}$ is also meager, hence $x^{\mathcal{C}}$ has the Baire property.

To show that countable unions preserve the Baire property, let $x_0, x_1, \cdots$ be a countable sequence of sets such that each x_i has the Baire property. Thus, for each x_i there exists an open set y_i such that the symmetric difference of x_i and y_i is meager. Let $y_0, y_1, \cdots$ be a sequence of such sets y_i. We need to show that for $z := \bigcup_{n\in\omega} x_i$, there exists an open set z' such that the symmetric difference is again meager. The idea is to use the union of all sets y_i as z'. (Note that open sets are closed under union.)

$$\begin{aligned}
&\left(\bigcup_{n\in\omega} x_n \setminus \bigcup_{n\in\omega} y_n\right) \cup \left(\bigcup_{n\in\omega} y_n \setminus \bigcup_{n\in\omega} x_n\right)\\
&= \bigcup_{n\in\omega} \underbrace{\left(x_n \setminus \bigcup_{i\in\omega} y_i\right)}_{\subseteq (x_n \setminus y_n)} \cup \bigcup_{n\in\omega} \underbrace{\left(y_n \setminus \bigcup_{i\in\omega} x_i\right)}_{\subseteq (y_n \setminus x_n)}\\
&\subseteq \bigcup_{n\in\omega} (x_n \setminus y_n) \cup \bigcup_{n\in\omega} (y_n \setminus x_n)\\
&= \bigcup_{n\in\omega} ((x_n \setminus y_n) \cup (y_n \setminus x_n))
\end{aligned}$$

Since meager sets are closed under countable union, and subsets of meager sets are also meager, a countable union of sets with the Baire property has the Baire property. □

1.3.2 Determinacy for classes of winning conditions

In the previous section we introduced a sufficient condition for a game to be determined. However, in general the question of whether a winning condition or a class of winning conditions guarantees that games are determined is more relevant. As classes of winning conditions are usually formalized by sets of words over the set of colors, we use the variant of Banach-Mazur games where we have a coloring function Ω and a language $\mathcal{W}$ rather than a set of winning paths Win when discussing this question.

Definition 1.3.3. We say that a winning condition $\mathcal{W} \subseteq C^\omega$ for some C *guarantees determinacy*, if for all graphs $G = (V, E)$, all $v_0 \in V$ and all coloring function $\Omega\colon V \to C$, the Banach-Mazur game $\mathcal{G} = (G, v_0, \Omega, \mathcal{W})$ is determined.

For games with a winning condition Win that has the Baire property, it has already been shown that they are determined. This result can be formulated in a similar way for classes of winning conditions.

Theorem 1.3.2 ([9])**.** *Let C be an arbitrary set of colors, and let $\mathcal{C} \subseteq \mathcal{P}(C^\omega)$ be a class of winning conditions. Then all $\mathcal{W} \in \mathcal{C}$ guarantee determinacy if and only if all $\mathcal{W} \in \mathcal{C}$ have the Baire property.*

Proof. For all coloring functions it holds that if $\mathcal{W}$ has the Baire property, the set $\Omega^{-1}(\mathcal{W})$ of all paths α such that $\mathsf{w}(\alpha) \in \mathcal{W}$ also has the Baire property, hence the direction from left to right follows from Corollary 1.3.1.

To prove the remaining direction, we construct a nondetermined game starting with a set $\mathcal{W}$ that does not have the Baire property. Instead of using $\mathcal{W}$ as a winning condition, we use the symmetric difference Z of $\mathcal{W}$ with the open set

$$Y := \bigcup\{\mathcal{BO}(\pi) : \mathcal{BO}(\pi) \setminus \mathcal{W} \text{ is meager}\}.$$

Let $\mathcal{G} = ((C, C \times C), c_0, \Omega = \mathsf{id}, Z)$ be a Banach-Mazur game. If Pl. 1 had a winning strategy, then Z would be meager. But if Z was meager, the symmetric difference of $\mathcal{W}$ with some open set were, and $\mathcal{W}$ would have the Baire property.

Thus assume that Pl. 0 has a winning strategy. In this case, it follows from the Banach-Mazur Theorem that Z is co-meager in some basic open set $\mathcal{BO}(\pi)$. If $\mathcal{BO}(\pi) \subseteq Y$, this cannot hold since in this case $\mathcal{BO}(\pi) \cap Z = \mathcal{BO}(\pi) \setminus \mathcal{W}$ is meager. It follows that $\mathcal{BO}(\pi) \not\subseteq Y$. Assume that there is a ρ such that $\mathcal{BO}(\rho) \subseteq Y$ and $\mathcal{BO}(\pi) \cap \mathcal{BO}(\rho) \neq \emptyset$. Since in this case, $\mathcal{BO}(\rho) \cap Z = \mathcal{BO}(\rho) \setminus \mathcal{W}$ is meager, Z would not be co-meager in $\mathcal{BO}(\pi)$, so we have $\mathcal{BO}(\pi) \cap Y = \emptyset$.

It follows that $\mathcal{BO}(\pi) \cap Z = \mathcal{BO}(\pi) \cap \mathcal{W}$, and that this set is co-meager (in $\mathcal{BO}(\pi)$). However, since in this case $\mathcal{BO}(\pi) \setminus \mathcal{W}$ is meager, this entails $\mathcal{BO}(\pi) \subseteq Y$, which is a contradiction. Thus Pl. 0 cannot have a winning strategy. □

1.3.3 Determinacy via classes of winning strategies

While in the basic notion of determinacy only the existence of a general winning strategy was required, one sometimes wants to confine winning strategies to certain classes of strategies. We thus say that a game is *determined via a class $\mathcal{K}$ of strategies*, if the player who has a winning strategy also has one from $\mathcal{K}$. We now show two results regarding this notion, namely that we can restrict the notion of a strategy to decomposition invariant strategies, and that if all classical graph games with a winning condition X are determined via some class of strategies, then so are all Banach-Mazur games with the same winning condition.

In the case of Banach-Mazur games, it turns out that a strategy does not need information about which player contributed what part to the prefix π played so far. We will see that *decomposition invariant strategies*, i.e. strategies that instead of getting the decomposition of π into moves as an input get π itself, are strong enough in the sense that determined Banach-Mazur games are also determined via decomposition invariant strategies. Formally, such a decomposition invariant strategy is a function of the form $f\colon \mathsf{FinPaths}(G,v) \to \mathsf{FinPaths}(G)$, satisfying the constraint that for any π, $f(\pi) \in \mathsf{FinPaths}(G, \mathsf{last}(\pi))$.

Theorem 1.3.3 ([9]). *Let $\mathcal{G}$ be a determined Banach-Mazur game. Then $\mathcal{G}$ is also determined via decomposition invariant strategies.*

Proof. We show that if Pl. 1 has a winning strategy, he also has a decomposition invariant one. It is easy to see that if this is true, the same holds in the case where Pl. 0 has a winning strategy, only that the first move is important, i.e. Pl. 0 has to make sure that the remaining game is played in a certain basic open set (cf. Theorem 1.2.1).

Let thus Pl. 1 have a winning strategy f. By the Banach-Mazur Theorem we know that the set Win of winning paths of Pl. 0 is meager, i.e. $\mathsf{Win} = \bigcup_{n\in\omega} \Gamma_n$ for some nowhere dense Γ_n. Since every Γ_n is nowhere dense, the complement contains a dense open set Λ_n. One characterization of dense open sets was that any finite path π can be extended to π' in such a way that $\mathcal{BO}(\pi') \subseteq \Lambda_n$. Let now, for every $n \in \omega$, g_n be a function that assigns to any finite path a finite path that prolongs it in the above way. We define a decomposition invariant strategy f' as follows: for any finite path π, take the minimal n such that $\mathcal{BO}(\pi) \nsubseteq \Lambda_n$, and set $f'(\pi) := g_n(\pi)$.

We claim that the f' defined above is a winning strategy for Pl. 1. Therefore assume that it is not. Then there exists a consistent play α that is won by Pl. 0. This means however, that $\alpha \in \Gamma_n$ for some n. But because of the first n moves of Pl. 1 we know by the above definition of f' that $\alpha \in \Lambda_n \subseteq \Gamma_n^{\mathcal{C}}$, which already is a contradiction to f' not being a winning strategy. □

What can also be shown is that if classical games on graphs are - for a specific class of winning conditions - determined via some kind of strategies, Banach-Mazur games with such a winning condition are as well. For example, parity games with countably many colors are determined via a very simple class of strategies (they are *positionally determined* [12]), and from the following theorem we know that Banach-Mazur games with a parity winning conditions over countably many colors are as well.

Theorem 1.3.4. *Let C be a set of colors, and let $\mathcal{W} \subseteq C^\omega$ be a language of infinite words over C. If $\mathcal{W}$ guarantees determinacy via a class $\mathcal{K}$ of strategies for classical graph games, then $\mathcal{W}$ also guarantees determinacy via strategies from $\mathcal{K}$ for Banach-Mazur games.*

Proof. Let $\mathcal{W}$ be a winning condition that guarantees determinacy via strategies from some class $\mathcal{K}$ for classical graph games, and let $\mathcal{G} = (G, v_0, \Omega\colon V \to C, \mathcal{W})$ be a Banach-Mazur game. We prove the theorem by constructing a classical graph game $(G' = (V_0, V_1, E'), v_0', \Omega', \mathcal{W})$ and show that any winning strategy for this game can be mapped to a winning strategy of the same type for $\mathcal{G}$. The idea is to add new edges and vertices for every finite path, such that every move in the new game determines a move in $\mathcal{G}$ and vice versa.

The formal definition of the new game is the following:

$$V_0 := (V \times \{0\}) \cup \{v_2^{\pi,0}, \dots, v_{n-1}^{\pi,0} : \pi = v_1 \cdots v_n \in \mathsf{FinPaths}(G)\}$$

$$V_1 := (V \times \{1\}) \cup \{v_2^{\pi,1}, \dots, v_{n-1}^{\pi,1} : \pi = v_1 \cdots v_n \in \mathsf{FinPaths}(G)\}$$

$$\begin{aligned} E' :=& \{((v,\sigma),(v',1-\sigma)) : (v,v') \in E, \sigma \in \{0,1\}\} \\ &\cup \{(v_i^{\pi,\sigma}, v_{i+1}^{\pi,\sigma}) : i < |\pi| - 1, \pi \in \mathsf{FinPaths}(G), \sigma \in \{0,1\}\} \\ &\cup \{((v,\sigma), v_2^{\pi,\sigma}) : \pi \in \mathsf{FinPaths}(G,v), \sigma \in \{0,1\}\} \\ &\cup \{(v_{n-1}^{\pi,\sigma}, (v, 1-\sigma)) : v = \mathsf{last}(\pi), \pi \in \mathsf{FinPaths}(G), \sigma \in \{0,1\} \end{aligned}$$

$$v_0' := (v_0, 0)$$

$$\Omega'(v) := \begin{cases} \Omega(u) & , v = (u,\sigma), \sigma \in \{0,1\} \\ \Omega(v_i) & , v = v_i^{\pi,\sigma}, \pi \in \mathsf{FinPaths}(G), \sigma \in \{0,1\}. \end{cases}$$

As $\mathcal{W}$ guarantees determinacy via $\mathcal{K}$-strategies, there exists a strategy $f \in \mathcal{K}$ that is winning for one of the players. We claim that this strategy - after adapting it to work on $\mathcal{G}$ - is also winning for this player in $\mathcal{G}$. Since the only vertices of Pl. σ that have more than one successor are those of the form (v, σ) for some $v \in V$, any strategy is fully characterized by describing what successor to choose at such a vertex after a path ρ has already been seen. The next move in the strategy will either be a move to a $(v', 1-\sigma)$ (which can be copied in $\mathcal{G}$), or to a $v_2^{\pi,\sigma}$. In this case the next move in $\mathcal{G}$ would be π. Hence any strategy for the new game can in essence also be applied to $\mathcal{G}$, and is winning if and only if it is in $\mathcal{G}$. (Given a play as a sequence of moves, there is a one to one correspondence between the two games.) □

Chapter 2

Applications: Fair Model Checking

In this chapter we present an application of Banach-Mazur games which exemplifies why simple winning strategies are of importance. As the Banach-Mazur Theorem (Theorem 1.2.1) relates the topological size of a set - i.e. a set being topologically small (meager) or topologically large (co-meager) - to the existence of winning strategies for the game with this winning condition, Banach-Mazur games are a useful means to determine whether a set is of either size.

In the following, we consider a characterization of *fairness* [23] that uses Banach-Mazur games in the way just described, and then recite a result (using the existence of simple winning strategies) that two different notions of fairness coincide for certain classes of properties, or winning conditions [22].

2.1 Topological and probabilistic concepts of fairness

In Model Checking - i.e. when asking whether a system matches a specification, or satisfies certain properties - the usual notion of *correctness* is that a system is correct if all possible runs are. In other words, all runs of the system, regardless of the extent to which they are realistic or plausible, have to satisfy a previously specified property. As in many situations, unrealistic or otherwise artificial runs can be neglected, the notion of *fairness* was introduced. The intuition of this notion is that a system is *fairly correct*, if all fair runs satisfy the property, meaning all but some artificial or irrelevant runs are correct. There have been several approaches towards finding a precise definition of fairness, of which we consider two important ones.

A natural definition of fairness can be formulated in a probabilistic manner. As all but some negligible runs should be correct, one of the first notions of fairness that comes to mind is hence requiring a system to *almost surely* match its specification:

Definition 2.1.1. A system is *fairly correct* (with respect to probabilistic fairness), if the probability that a random run is correct equals 1. This can equivalently be characterized in the way that for a certain probability measure on the set of all runs, the set of all correct runs is of measure 1.

Since appropriate probability measures are hard to find, this definition may not be the one of choice in many applications, thus other variants of fairness need to be considered. In [4, 23], a topological characterization of fairness was proposed. In this approach, the idea is to use a topologically large set instead of a probabilistically large one, using co-meagerness as a precise notion of topological largeness.

Definition 2.1.2. A system is *fairly correct* (with respect to topological fairness), if the set of correct runs is co-meager in the space of all runs.

This notion of fairness turns out to cover many existing concepts of fairness, and can be considered as a general definition. We thus follow this concept in order to explicate a result by Varacca and Völzer [22] that for certain properties, topological and probabilistic fairness coincide.

2.2 Systems, properties and probability measures

When it comes to Model Checking, systems are usually represented by *transition systems*, consisting of a set of states, an initial state, a transition relation and a labeling of the states with subsets of some fixed set of atomic propositions. In order to simplify the later proofs, we assume without loss of generality that every state is labeled with only one atomic proposition (this can be done by using the powerset of the old one as a new set of atomic propositions), so the labeling can be viewed as a simple coloring of the states. To keep notation consistent with the previous chapter, a transition system is a tuple $(G, v_0, \Omega\colon V \to C)$. A property can be given in two different ways, either as the set of infinite paths in G that satisfy it, or as a language of infinite words over C. Following the above definition of topological fairness, a system is fairly correct with respect to a property, if the property is topologically large, or the set of runs inducing words in the language is, respectively. By the Banach-Mazur Theorem, this amounts to checking which player has a winning strategy in the corresponding Banach-Mazur game, using the property as the winning condition. To further simplify things, for the rest of this chapter we consider a slightly modified version of the Banach-Mazur game, we namely let Pl. 1 move first. Thus, a property Y is co-meager (or induces a co-meager set) if and only if Pl. 0 has a winning strategy in the game $\mathcal{G} = (G, v_0, \Omega, Y)$.

In literature (see e.g. [3]), a common characterization of properties for Model Checking uses the notions of *liveness* and *safety* properties. On the one hand, a property X is said to be a liveness one, if after any finite prefix, a run can always be continued in such a way that it satisfies the property. In the previous chapter it was shown that this is precisely the characterization of a dense set, thus a property is *live* if and only if it corresponds to a dense set of infinite paths. On the other hand, a safety property is a property where any run not satisfying it has a finite prefix such that no run with this prefix satisfies the property. Using the terminology of the previous chapter, we see that closed sets correspond to safety properties. As fairness is usually formulated with respect to a safety property, this also fits the setting explained above, as the set of all paths is a closed set.

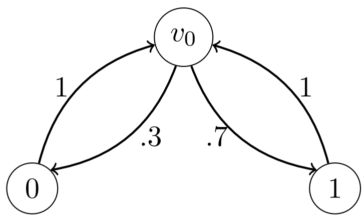
In order to show the equivalence between topological and probabilistic fairness, we need to make precise what kind of *measure* we mean. If a probability measure μ on the infinite paths in a graph is defined over the Borel sets, we call it a *Borel measure*. Such a measure is *positive*, if it is positive for any basic open set, i.e. $\mu(\mathcal{BO}(\pi)) > 0$ for all π. Furthermore, a measure is *bounded*, if there exists a constant $c > 0$ such that for any finite path π and every $v \in \mathsf{last}(\pi)E$,

$$\mu(\mathcal{BO}(\pi v) \mid \mathcal{BO}(\pi)) > c.$$

A set X is *probabilistically large* (with respect to μ), if $\mu(X) = 1$.

To see that probabilistic and topological largeness do not coincide in general, we use an example from [22]:

Example 2.2.1. Consider the following graph, where each edge is chosen with the indicated probability.



Let X be the set of paths where infinitely many times the same amount of 0s and 1s has been seen, i.e.

$$X := \{\alpha : \text{ for infinitely many prefixes } \pi,\ |\pi|_0 = |\pi|_1\}$$

Since when being at vertex v_0, the probability of seeing vertex 0 is less than the probability of seeing 1, the probability of the set X is equal to 0, hence X is not probabilistically large. However, Pl. 0 can easily win the corresponding Banach-Mazur game, from which it follows that X is topologically large.

Before we explain under which conditions the two notions do coincide, we recall a lemma regarding probabilities of limits of infinite sequences, namely the second Borel-Cantelli lemma, as can be found in [6, page 41].

Lemma 2.2.1 (Borel-Cantelli). *Let $\{A_i : i \geq 0\}$ be a sequence of independent events. Let $\mathbb{A}$ be the set of those elements that are contained in infinitely many A_i,*

$$\mathbb{A} := \{A_i \ i.o.\} = \lim_{m\to\infty} \bigcup_{n>m} A_n.$$

Then the following holds:

$$\text{If } \sum_i \mu(A_i) = \infty, \text{ then } \mu(\mathbb{A}) = 1.$$

2.3 Coincidence of topological and probabilistic largeness

In the following, we will see that for certain properties and for finite systems (in our setting: finite graphs), the properties are topologically large if and only if they are probabilistically large. This means that both concepts of fairness coincide for certain fairness properties. The result, which is due to [22], heavily relies on the existence of simple winning strategies for ω-regular properties, or winning conditions. It will later be seen (cf. Section 4.1.1) that ω-regular winning conditions guarantee positional determinacy, i.e. one player has a winning strategy of the form $f : V \to \mathsf{FinPaths}(G)$.

Theorem 2.3.1 ([22]). *Let $\mathcal{G}$ be a finite Banach-Mazur game with an ω-regular winning condition X, and let μ be a bounded Borel measure on $\mathsf{Paths}(G, v_0)$. Then*

$$X \text{ is topologically large} \iff X \text{ is probabilistically large (wrt. } \mu).$$

Proof. We begin with the direction from left to right. Let thus X be topologically large. From the Banach-Mazur Theorem, it follows that Pl. 0 has a winning strategy. Since X is ω-regular, by Theorem 4.1.3 it follows that he also has a positional winning strategy f. Since there are only finitely many vertices, f is bounded, i.e. there exists a $k \in \omega$ such that for all $v \in V$, $|f(v)| < k$. Furthermore, since μ is bounded, playing any successor after some finite path has a probability of at least c, for some $c > 0$. This means that the probability that a random path is - after any prefix π - continued with a move according to f can be bounded from below by

$$\mu(\mathcal{BO}(\pi \cdot f(\mathsf{last}(\pi))) \,|\, \mathcal{BO}(\pi)) > c^k > 0.$$

Let X_i be the event that a random path continues according to a move of f at position i. By the above, each event X_i has a positive probability of at least c^k. As a path is consistent with f if and only if there are infinitely many positions where it continues according to f, we can write the set $\mathsf{Plays}(f)$ as

$$\mathsf{Plays}(f) = \lim_m \bigcup_{n>m} X_n.$$

Since all X_i are independent, we can apply the Borel-Cantelli Lemma, and since the sum over all measures of the X_i diverges,

$$\sum_i \mu(X_i) > \sum_i c^k = \infty,$$

it follows that $\mu(\mathsf{Plays}(f)) = 1$. As f is a winning strategy, X is a superset of all plays consistent with f, which entails that X is probabilistically large (with respect to μ).

For the direction from right to left, assume that X is not topologically large. Since Pl. 0 does not have a winning strategy in this case, but the game is still determined (all ω-regular winning conditions guarantee determinacy), Pl. 1 has a winning strategy. Recall that we adapted the game so that Pl. 1 starts, thus let π_0 be his first move. Obviously, any winning strategy of Pl. 1 in the game is also a winning strategy for Pl. 0 in the game with the winning condition $X^{\mathcal{C}}$ that starts in $\mathsf{last}(\pi_0)$. Again, since ω-regular languages are closed under complementation, $X^{\mathcal{C}}$ is of measure 1 in this game ($\mu(X^{\mathcal{C}} \,|\, \mathcal{BO}(\pi_0)) = 1$). Since $\mu(\mathcal{BO}(\pi_0)) > 0$, it follows that $X^{\mathcal{C}}$ has a positive measure in the original game, hence $\mu(X) < 1$, and X cannot be probabilistically large (with respect to μ). □

We stress that the finiteness of the graph and the ω-regularity of the winning condition are crucial to proof, because otherwise the existence of bounded positional strategies cannot be guaranteed. Whether similar results can also be shown for other winning conditions is still an open problem, but it motivates the search for other kinds of simple winning strategies.

Chapter 3

Classes of winning conditions

Before we investigate various kinds of simple winning strategies, we introduce different classes of winning conditions and closely examine these in order to discover and obtain properties useful for finding simple winning strategies these classes guarantee determinacy for. We focus on prefix independent winning conditions, as many important winning conditions are in this class.

3.1 Prefix independent winning conditions

A winning condition is prefix independent, if the presence of any finite prefix of some play has no effect on whether or not this play belongs to the winning condition. From this, we derive the following definition:

Definition 3.1.1. Let $\mathcal{W} \subseteq C^\omega$ for some set C of colors. $\mathcal{W}$ is *prefix independent*, if for all $\alpha \in C^\omega$ and all $x \in C^*$,

$$\alpha \in \mathcal{W} \iff x\alpha \in \mathcal{W}.$$

From the definition it can be seen that a major class of prefix independent winning conditions is the class of those that only distinguish between finite infixes (e.g. single colors or sequences of colors) that have to occur infinitely often and finite infixes that may occur only finitely many times. A very famous class of winning conditions of this kind is the class of Muller winning conditions, which we examine first. We then extend these winning conditions to ones where finite sequences of colors have to be seen infinitely often, and will see that these are similar when it comes to properties concerning determinacy.

Many other properties, e.g. a being seen at least/most n times, are not in this class, and mostly not prefix independent at all ($a^n b^\omega$ is winning, but $a^{n-1}b^\omega$ / $aa^n b^\omega$ is not). However, ω-regular properties like the above are not treated in this chapter, but will be mentioned in the chapter on strategies.

3.2 Muller winning conditions

One of the most important classes of winning conditions is the class of Muller winning conditions, as they allow to differentiate between vertices (or states in a system) that should appear infinitely often, and those that are only allowed to occur finitely many times. Formally, a Muller winning condition defining a language $\mathcal{W}$ is of the form $(\mathcal{F}_0, \mathcal{F}_1)$ for some $\mathcal{F}_0 \subseteq \mathcal{P}(C)$, $\mathcal{F}_1 = \mathcal{P}(C) \setminus \mathcal{F}_0$. For any $\alpha \in C^\omega$, we have $\alpha \in \mathcal{W}$ if and only if the set $\mathsf{Inf}(\alpha)$ of colors appearing infinitely often in α is in $\mathcal{F}_0$. Accordingly, a Banach-Mazur game with a Muller winning condition is a tuple $\mathcal{G} = (G, v_0, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$, where $(\mathcal{F}_0, \mathcal{F}_1)$ represents the language $\mathcal{W}$ as described above.

For Muller winning conditions, different cardinalities of sets of colors lead to different results. We hence divide the class of Muller winning conditions into three disjoint subclasses, namely the one where C is finite, the one where it is countably infinite, and the one where C is uncountable. Only the first two classes are treated in this thesis.

Notation As already mentioned above, for any $\alpha \in C^\omega$ we write $\mathsf{Inf}(\alpha)$ for the set of colors that appear infinitely often in α, i.e.

$$\mathsf{Inf}(\alpha) := \{c \in C : \text{ there are infinitely many } i \text{ such that } \Omega(\alpha[i]) = c\}.$$

For finite paths π, we use $\Omega(\pi)$ as an abbreviation for the set of colors that appear in π:

$$\Omega(\pi) := \{c \in C : \Omega(\pi[i]) = c \text{ for some } i\}.$$

For any vertex $v \in V$, we use C^∞ to refer to the sets of colors that can at least be seen infinitely often on some infinite path starting in v:

$$C^\infty(v) := \{A \subseteq C : \text{ there ex. an } \alpha \in \mathsf{Paths}(G, v) \text{ such that } A \subseteq \mathsf{Inf}(\alpha)\}.$$

With $P_\sigma(v) := C^\infty(v) \cap \mathcal{F}_\sigma$ we mean the set of all winning sets of Pl. σ that are still possible to be seen infinitely often from v.

At last, for any set $A = \{a_1 < a_2 < \cdots\} \subseteq \omega$ we call the subset of the initial n elements $\mathsf{first}(A, n)$:

$$\mathsf{first}(A, n) := \begin{cases} \{a_1, \cdots, a_n\} & \text{, if } n < |A| \\ A & \text{, otherwise.} \end{cases}$$

3.2.1 Muller winning conditions over a finite set of colors

The simplest and best known class of Muller winning conditions is the one where the set C of colors is finite. In contrast to standard Muller games, Banach-Mazur games with a Muller winning condition over only finitely many colors are simple in the way that they are already positionally determined. This is due to a property of such games which has been established in [5]. The idea is that Pl. 0 can only win such a game if he reaches a *stable* vertex, which intuitively is a vertex v such that after v has been reached, the set of colors that can still be seen does not change anymore.

Definition 3.2.1. For any $v \in V$, let

$$C(v) := \{c \in C : \text{ there ex. a } w \in vE^* \text{ with } \Omega(w) = c\}.$$

A vertex v is *stable*, if $C(v) = C(u)$ for all $u \in vE^*$.

It is easy to see that in games with a Muller winning condition over a finite set of colors, stable vertices are always reachable, as only finitely many colors can be made impossible to be seen again.

Lemma 3.2.1 ([5])**.** *Let $\mathcal{G}$ be a Banach-Mazur game with a Muller winning condition over a finite set C of colors. Pl. 0 has a winning strategy for $\mathcal{G}$ if and only if a stable vertex v with $C(v) \in \mathcal{F}_0$ can be reached from v_0.*

Proof. There are two directions to the proof.

If a stable vertex v with $C(v) \in \mathcal{F}_0$ is reachable, Pl. 0 can move there in his first move. Since in every later move, the current vertex is stable, Pl. 0 can play a finite path on which $C(v)$ is seen completely. With playing as just described, Pl. 0 ensures that $\mathsf{Inf}(\alpha) \in \mathcal{F}_0$ for a consistent play α, hence he has a winning strategy.

For the other direction, assume that no such v is reachable, hence - stable positions are always reachable in such a game due to the finiteness of C - for all stable v we have $C(v) \in \mathcal{F}_1$. It follows immediately from the above that Pl. 1 has a winning strategy. □

It can easily be seen that the strategies described in the proof are already positional, which was also claimed above (positional strategies are formally introduced in Section 4.1.1).

3.2.2 Muller winning conditions over a countably infinite set of colors

As we have dealt with Muller winning conditions over only a finite set of colors so far, the question arises what happens if infinite sets of colors are allowed. Since more subclasses of Muller winning conditions over such a set of colors guarantee determinacy, we focus on these infinite C that are still countable. For simplicity and without loss of generality, we assume throughout this section that $C = \omega$. Note further that we implicitly use Theorem 1.3.3 that allows us to restrict the discussion to decomposition invariant strategies.

The outline of this section is as follows. We begin by looking at such Muller winning conditions where one of the sets of winning sets $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton, and afterwards loosen the size restriction to finite and countably infinite ones.

Before we start with a specific class of winning conditions, we observe the following: let $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ be a decomposition invariant winning strategy of Pl. 0. Then all vertices reachable from $\mathsf{last}(f(v_0))$ are in the winning region of Pl. 0 (meaning Pl. 0 can win from there, regardless of whose move it is). The difference to other non-path games is that the winning region of Pl. 0 cannot be left anymore after the initial move of a winning strategy (by any of the two players), because otherwise Pl. 1 would do so in his first move. This is already implied in the Banach-Mazur Theorem (Theorem 1.2.1), namely in (*ii*).

Singleton winning conditions The simplest kind of Muller winning conditions over ω are those where one of the sets of winning sets (i.e. either $\mathcal{F}_0$ or $\mathcal{F}_1$) is a singleton, as the respective player does not have to choose which set to play. We separate the two different cases, i.e. those games where the player with the singleton $\mathcal{F}$-set has a winning strategy, and those where the opponent does, and begin with these games where Pl. 0 has a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ and $\mathcal{F}_0$ is a singleton (the corresponding case for Pl. 1 turns out to be analogous).

Lemma 3.2.2. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition over a countable set of colors such that $F_0 = \{A\}$, for some $A \subseteq \omega$. Let further Pl. 0 have a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$.*

Then for any $v \in \mathsf{last}(f(v_0))E^*$ *it holds that*

(i) $A \in C^\infty(v)$*, and*

(ii) for all $b \notin A$*, there exists a vertex* v' *reachable from* v *such that* $\{b\} \notin C^\infty(v')$*.*

Proof. Assume not. Then there would exist a v reachable from $\mathsf{last}(f(v_0))$ for which (i) or (ii) is false (remember that Pl. 0 can win from v no matter whose move it is). If (i) was false, Pl. 0 could not win from v, because A cannot be seen infinitely often. So (i) has to be true. Thus, there must exist a $b \notin A$ such that from all $v' \in vE^*$, a b-vertex is reachable. This means that Pl. 1 could make sure (even with a positional strategy) that $b \in \mathsf{Inf}(\alpha)$ for any play α that visits v, namely by moving to a b-vertex in every move. However, this is a contradiction to v being in the winning region of Pl. 0. □

Note that if $\mathcal{F}_1$ was a singleton and Pl. 1 had a winning strategy, the above lemma would also hold, but in fact both (i) and (ii) would now be true for him for any v reachable from v_0, as Pl. 0 starts every play.

For the other direction, i.e. the situation where the opponent of the player with the singleton set of winning sets has a winning strategy, we - for simplicity - look at the situation where $\mathcal{F}_1 = \{B\}$ for some $B \subseteq \omega$.

Lemma 3.2.3. *Let* $\mathcal{G}$ *be a game as above, i.e. such that Pl. 0 has a winning strategy, but with* $\mathcal{F}_1 = \{B\}$ *being a singleton. Let* $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ *be a winning strategy of Pl. 0. Then it holds for* v_0 *that*

(i) there exists a v' *reachable from* v_0 *such that* $B \notin C^\infty(v')$*, or*

(ii) there exists an $a \notin B$ *and a* v_a *reachable from* v_0 *such that* $\{a\} \in C^\infty(v')$ *for all* $v' \in v_aE^*$*.*

Proof. Assume that neither (i) nor (ii) hold. It would then be true that

1. for all reachable vertices v, $B \in C^\infty(v)$. Hence Pl. 1 could make sure with his moves that $B \subseteq \mathsf{Inf}(\alpha)$, and

2. for all $a \notin B$ and all reachable v, there always exists a reachable v' such that $\{a\} \notin C^\infty(v')$. Thus, Pl. 1 could make sure that $\mathsf{Inf}(\alpha) \subseteq B$.

Taking 1 and 2 together, we have that Pl. 1 can enforce $\mathsf{Inf}(\alpha) = B$ for a play consistent with f (by alternating moves according to 1 and 2), contradicting that f is a winning strategy of Pl. 0. □

For the symmetric case - i.e. $\mathcal{F}_0 = \{A\}$ and Pl. 1 has a winning strategy - the situation is similar. But we again have that (i) or (ii) hold for any vertex reachable in any play, instead of just v_0.

Combining the previous lemmas, we obtain the following corollary (the corollary can analogously be formulated for Pl. 1, in whose case such a v is always reachable):

Corollary 3.2.1. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition such that $\mathcal{F}_0 = \{A\}$ for some $A \subseteq \omega$. The following are equivalent:*

- *Pl. 0 has a winning strategy.*
- *There exists a reachable v such that for all $v' \in vE^*$:*
  1. *$A \in C^\infty(v')$, and*
  2. *for all $b \notin A$, there exists a w reachable from v' with $\{b\} \notin C^\infty(w)$.*

Finitely many winning sets In general, several different sets of colors may be equally good to appear infinitely often. We now consider those winning conditions where instead of one $\mathcal{F}$-set being a singleton, either $\mathcal{F}_0$ or $\mathcal{F}_1$ must be finite, which is the next natural step in complexity of the winning condition. As it turns out, in games with such a winning condition we can always find a vertex from which the game can - in a way - be reduced to one with a singleton $\mathcal{F}$-set. This is already a hint towards determinacy via the same classes of strategies for both cases.

The first situation we examine is again the situation where a game $\mathcal{G}$ with a finite $\mathcal{F}_0$ is given, such that Pl. 0 has a winning strategy. We argue that there exists a vertex to which Pl. 0 can move in his opening move such that the game reduces to a game $\mathcal{G}'$ with $\mathcal{F}'_0 = \{A_i\}$ for some $A_i \in \mathcal{F}_0$.

Lemma 3.2.4. *Let $\mathcal{G}$ be a game with a Muller winning condition where $\mathcal{F}_0 = \{A_1, \ldots, A_n\}$, such that Pl. 0 has a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$. Then there exists a vertex $\tilde{v}$ in the winning region of Pl. 0, such that $P_0(\tilde{v}) = \{A, A_{i_1}, \ldots, A_{i_s}\} = P_0(v')$ for all $v' \in \tilde{v}E^*$, and $A_{i_l} \subseteq A$ for all $1 \leq l \leq s$.*

Proof. We show the existence of $\tilde{v}$ by iteratively constructing a path that starts in v_0 and that eventually ends in $\tilde{v}$. This path π is created as follows:

- We start with $\pi := v_0 \cdot f(v_0)$.
- While there still exists some $A_i \in P_0(\mathsf{last}(\pi))$ and a vertex v reachable from $\mathsf{last}(\pi)$ with $A_i \notin P_0(v)$, we prolong π with a shortest path from $\mathsf{last}(\pi)$ to this v.

Since $\mathcal{F}_0$ is finite and P_0 can only decrease in size on any path through G, it follows immediately that the process will terminate. We claim that $\tilde{v} := \mathsf{last}(\pi)$ has the desired properties. Notice first that $P(\mathsf{last}(\pi))$ cannot be empty, since Pl. 0 has a winning strategy from $\mathsf{last}(\pi)$, as π starts with the opening move of f. It is also obvious that $P_0(\mathsf{last}(\pi)) = P_0(v')$ for all $v' \in \mathsf{last}(\pi)E^*$, since otherwise the process would not yet have terminated.

Let $A := \bigcup P_0(\mathsf{last}(\pi))$. Then clearly $A' \subseteq A$ for all $A' \in P_0(\mathsf{last}(\pi))$. It remains to show that $A \in P_0(\mathsf{last}(\pi))$. Therefore suppose not. Let thus α be a play in which Pl. 0 plays according to his winning strategy f, while Pl. 1 moves to $\mathsf{last}(\pi)$ in his first move, and sees an increasing subset of A in every later move (which is always possible since for any u reachable from $\mathsf{last}(\pi)$ it holds that $A \in C^\infty(u)$ as P_0 is already constant). Since ω is countable, he hence enforces $A \subseteq \mathsf{Inf}(\alpha)$. Since α is consistent with f, $\mathsf{Inf}(\alpha) \in \mathcal{F}_0$ must be true. As $A \notin \mathcal{F}_0$ by assumption, it must hold that $A \subsetneq \mathsf{Inf}(\alpha)$. But then there must exist an $A' \in \mathcal{F}_0$ such that $A \subsetneq A'$. Since A' was seen infinitely often in α, $A' \in P_0(\mathsf{last}(\pi))$, contradicting $A = \bigcup P(\mathsf{last}(\pi))$.

All in all, $\mathsf{last}(\pi)$ satisfies the properties from the lemma, so $\tilde{v} = \mathsf{last}(\pi)$ exists. □

What remains to show is that the game $\mathcal{G}' = (G, \tilde{v}, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$ is equivalent (Pl. 0 wins $\mathcal{G}'$ if and only if he wins $\mathcal{G}''$, and winning strategies can be adapted) to the game $\mathcal{G}'' = (G, \tilde{v}, \Omega, (\{A\}, \mathcal{P}(\omega) \setminus \{A\}))$, for the maximal A from $P_0(\tilde{v})$.

Lemma 3.2.5. *Let $\mathcal{G}$ be a game as above and let $\tilde{v}$ be a vertex as in the previous lemma. Let furthermore A be the maximal set in $P_0(\tilde{v})$. Then Pl. 0 has a winning strategy for $\mathcal{G}' = (G, \tilde{v}, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$ if and only if Pl. 0 has a winning strategy for $\mathcal{G}'' = (G, \tilde{v}, \Omega, (\{A\}, \mathcal{P}(\omega) \setminus \{A\}))$.*

Proof. Since $A \in \mathcal{F}_0$ and the arenas of the games coincide, a winning strategy for Pl. 0 in $\mathcal{G}''$ is also a winning strategy for Pl. 0 in $\mathcal{G}'$.

So let $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ be a winning strategy for Pl. 0 in $\mathcal{G}'$. What we need to show is that there then also exists a winning strategy $f'\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ for Pl. 0 in $\mathcal{G}''$. We first note that for any v reachable in any play, $A \in C^\infty(v)$, since - in $\mathcal{G}'$ - $P_0(v) = P_0(\tilde{v})$ for all reachable v. We define f' as follows: $f'(\pi) := f(\pi) \cdot \rho$ where ρ prolongs $f(\pi)$ in such a way that

$$\mathsf{first}(A, |\pi|) \subseteq \Omega(\rho).$$

(Such a ρ can always be found since $A \in C^\infty(v)$ for all reachable v.) Let now α be any play consistent with f'. Since the ρ-part of any move could also be an initial segment of the subsequent move of Pl. 1, and f is a winning strategy for Pl. 0, we know that $\mathsf{Inf}(\alpha) \in \mathcal{F}_0$. Because for any $A' \in P_0(\tilde{v})$ we know (by the previous lemma) that $A' \subseteq A$, we can infer $\mathsf{Inf}(\alpha) \subseteq A$. Furthermore, we know that there exists an infinite sequence $n_1 < n_2 < \ldots$ such that, for every i,

$$\mathsf{first}(A, n_i) \subseteq \Omega(\alpha[n_i] \cdots \alpha[n_{i+1} - 1]).$$

From that it follows that $A \subseteq \mathsf{Inf}(\alpha)$, which actually means $A = \mathsf{Inf}(\alpha)$. Thus f' is indeed a winning strategy for Pl. 0 in $\mathcal{G}''$. □

If Pl. 1 has only finitely many winning sets and a winning strategy, then such a $\tilde{v}$ is reachable from any v that does not already satisfy the property that $P_1(v) = P_1(v')$ for all $v' \in vE^*$. As soon as a vertex that satisfies this property is reached, we know by the above that the union of P_1 is in $\mathcal{F}_1$, and by the previous lemma the game is then equivalent to one with a singleton $\mathcal{F}_1$. Since all vertices satisfying the property for different unions of P_1 induce disjoint subgraphs (i.e. the subgraph induced by the vertex set vE^*), the strategies for the singleton games can easily be merged to an overall winning strategy, where the actual winning set is determined in the first move.

What we have shown so far is that any game with a winning condition where the player that has a winning strategy only has finitely many winning sets can basically be played as if there was only one winning set and will still be won by this particular player.

To complete our discussion of Muller winning conditions where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is finite, we show that a similar lemma holds for the situation where the opponent of the player with finitely many winning sets has a winning strategy. We prove that for any vertex v in the winning region of this player, there exists a simple set B and a vertex v_B, such that he can win by moving to v_B and focusing on this very set B. In other words, there exists a realizable B such that he will always win if he enforces B to be seen infinitely often.

Lemma 3.2.6. *Let $\mathcal{G}$ be a game as above with $\mathcal{F}_0 := \{A_1, \ldots, A_n\}$, such that Pl. 1 has a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$. Then for any reachable v there exists a $v_B \in vE^*$ and a finite B with $B \in C^\infty(v')$, for all $v' \in v_B E^*$, such that for any $B' \in C^\infty(v)$ with $B \subseteq B'$, we have $B' \in \mathcal{F}_1$.*

Proof. Let $A \in P_0(v)$ be one of the winning sets of Pl. 0 that are still possible. Without loss of generality, we assume that P_0 is already stable, hence there exists no reachable v' such that $A \notin C^\infty(v')$. Since Pl. 1 has a winning strategy from v, we know that there exists some $b \notin A$ which Pl. 1 can see infinitely often. This actually means that he can move to a v_b such that $\{b\} \in C^\infty(u)$ for all $u \in v_b E^*$. But since $P_0(v)$ is finite, we can simply extend this in the following way: Let $P_0(v) = \{A_1, \ldots, A_m\}$, and let $\pi_0 := v$. For $i \in \{1, \ldots, m\}$, choose some $b_i \notin A_i$ and a v_i reachable from $\mathsf{last}(\pi_{i-1})$ such that $b_i \in C^\infty(u)$ for all $u \in v_i E^*$, and set $\pi_i := \pi_{i-1} \cdot \rho$ where ρ is a shortest path prolonging π_{i-1} to v_i.

Now set $B := \{b_1, \ldots, b_m\}$ and $v_B := \mathsf{last}(\pi_m)$. By definition we have $B \in C^\infty(v')$ for all $v' \in v_B E^*$, and obviously for any path α that visits v_B and for which $B \subseteq \mathsf{Inf}(\alpha)$ holds, we have $\mathsf{Inf}(\alpha) \neq A$ for any $A \in P_0(v)$. By definition of $P_0(v)$ we infer $\mathsf{Inf}(\alpha) \in \mathcal{F}_1$. □

As before, the situation where $\mathcal{F}_1$ is finite and Pl. 0 has a winning strategy is similar, with the only difference being that the lemma may not hold for any v anymore, but will still hold for v_0 and any v reachable from $\mathsf{last}(f(v_0))$, if f is a winning strategy of Pl. 0.

Countably many winning sets After discussing games where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is finite, we proceed with allowing infinite sets of winning sets. In order to ensure that the games are still determined, we only consider winning conditions where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is still countable at this point. (These winning conditions are on level Σ^0_4 and Π^0_4, respectively, of the Borel hierarchy; thus the games are determined, as was shown in Section 1.3.1: as the open sets are exactly those sets of paths in G that are of the form $P \cdot V^\omega \cap \mathsf{Paths}(G, v)$ for some $P \subseteq \mathsf{FinPaths}(G, v)$, for any set $A = \{a_1, a_2, \ldots\} \subseteq \omega$ the set X_A of paths α with $\mathsf{Inf}(\alpha) = A$ can be written as

$$X_A = \bigcap_{a \in A} \overbrace{X_a}^{\in \Pi^0_2} \cap \bigcap_{b \notin A} \overbrace{X_b^{\mathcal{C}}}^{\in \Sigma^0_2} \in \Pi^0_3,$$

where the set of paths X_a on which a is seen infinitely often is $X_a := \bigcap_{n\in\omega} Y_n^a V^\omega \cap \mathsf{Paths}(G) \in \Pi_2^0$, with Y_n^a being the set of finite paths on which a is seen n times. All in all, the winning paths of Pl. 0 for a countable $\mathcal{F}_0$ are $\bigcup_{A\in\mathcal{F}_0} X_A \in \Sigma_4^0$. For countable $\mathcal{F}_1$, this set of paths is in Π_4^0.)

We now examine games where the opponent of the player with a winning strategy has countably infinitely many winning sets. What will be shown is that in each move, the winning player can play in such a way that without focusing on his winning sets, he can render all of his opponent's winning sets useless. This is due to the fact that for any of the opponent's winning sets, there are only two ways to prevent them from being seen infinitely often. As he has a winning strategy, one of these must be exploitable by a winning strategy.

Lemma 3.2.7. *Let $\mathcal{G} = (G, v_0, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning conditions with colors in ω, where $\mathcal{F}_0 = \{A_1, A_2, \dots\}$ is countably infinite.*

If Pl. 1 has a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$, then for any reachable vertex v and any $A \in \mathcal{F}_0$, one of the following must be true:

(i) There exists a $v' \in vE^*$ such that $A \notin C^\infty(v')$.

(ii) There exists some $b \notin A$ and a $v_b \in vE^*$ such that $\{b\} \in C^\infty(w)$ for all $w \in v_bE^*$.

Proof. Suppose not, i.e. Pl. 1 has a winning strategy, but there exists a reachable vertex v and some $A \in \mathcal{F}_0$, such that, at position v, neither (i) nor (ii) holds. It is obvious that in this case both do also not hold at any later vertex. In particular, since there does not exist a vertex reachable from v such that from there on A cannot be seen infinitely often anymore, Pl. 0 can make sure that $A \subseteq \mathsf{Inf}(\alpha)$. Furthermore, since no $b \notin A$ can be forced to be seen infinitely often (since (ii) is false), Pl. 0 can also successively prevent every $b \notin A$ from being seen infinitely often, e.g. by always moving to a vertex where the minimal such b that could still be seen infinitely often cannot be seen infinitely often anymore. Since (i) and (ii) are false for this A at every later vertex, the strategies induced by each can be combined to an overall strategy that is winning for Pl. 0, namely move to v in the first move, and later simultaneously see a larger initial subset of A while making the minimal unwanted b impossible. For any consistent play α we would then have $\mathsf{Inf}(\alpha) = A \in \mathcal{F}_0$. But since Pl. 1 has a winning strategy, this is a contradiction. □

What we show next is that the other direction of this lemma also holds, i.e. that, if the conditions are satisfied at every vertex, Pl. 1 has a winning strategy.

Lemma 3.2.8. *Let $\mathcal{G}$ be a game as above with a countable $\mathcal{F}_0$.*

If for every reachable vertex v and every $A \in \mathcal{F}_0$

(i) there exists a $v' \in vE^*$ such that $A \notin C^\infty(v')$, or

(ii) there exist $b \notin A$ and $v_b \in vE^*$ such that $\{b\} \in C^\infty(w)$ for all $w \in v_bE^*$,

then Pl. 1 has a winning strategy.

Proof. Let $\mathcal{G}$ be a game as in the lemma for which the precondition holds. We prove the lemma by constructing a winning strategy for Pl. 1. The idea is that in each move, Pl. 1 will try to prevent another $A \in \mathcal{F}_0$ from being the actual winning set of Pl. 0. There are two ways to achieve this. The first - and simpler - way is to simply make it impossible to be contained in the Inf-set. The second one is to force a color not in A to be seen infinitely often, which can obviously not be done in a single move. Therefore, it is necessary to ensure that certain colors are seen repeatedly. The strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ for Pl. 1 is defined as follows:

For some arbitrary $\pi = v_0 \ldots v$ where Pl. 1 gets to move, we consider the set $P_0(v) = C^\infty(v) \cap \mathcal{F}_0$. Let i be minimal such that $A_i \in P_0(v)$ and there exists no $b \notin A_i$ with $\{b\} \in C^\infty(u)$ for all $u \in vE^*$. In other words, A_i is a new, i.e. not previously considered, winning set of Pl. 0 that is still possible and for which no "bad" color can yet be forced to be seen infinitely often. However, we know by assumption that (i) or (ii) is true for v and A_i. If (i) holds, Pl. 1 chooses the closest such v' and moves there. Otherwise, Pl. 1 chooses the smallest b for which a v_b exists and moves there. If no such A_i exists, we set $n := |v_0 \cdots v|$. We still have to take care of winning sets that could not be made impossible in previous moves, i.e. for which we have to see a "bad" color infinitely often. So after moving to v' or v_b, respectively, for every $j \leq i$ ($j \leq n$, if no A_i existed) such that $A_j \in P_0(v)$, Pl. 1 chooses the smallest $b_j \notin A_j$ for which $\{b_j\} \in C^\infty(w)$ for all $w \in v'E^*$ or $w \in v_bE^*$, respectively, is true and moves to a closest reachable vertex v'' with $\Omega(v'') = b_j$.

It remains to prove that f is a winning strategy for Pl. 1. Therefore we assume that it is not, i.e. there exists a play α which is consistent with f, but which is won by Pl. 0. In other words, we have a consistent play α for which $\mathsf{Inf}(\alpha) = A_i \in \mathcal{F}_0$. What we want to show is that this cannot be, i.e. such a play cannot be consistent with f, or $\mathsf{Inf}(\alpha) = A_i$ cannot be true. If α is consistent with f, one of the following must hold. Either there exists a vertex v that is visited in α such that $A_i \notin C^\infty(v)$ (from which it would immediately follow that $\mathsf{Inf}(\alpha) \neq A_i$), or there exists a move $f(\pi)$ in α after which A_i is smaller than the smallest (with respect to the index) unhandled winning set of Pl. 0 (because $\mathcal{F}_0$ is countable). Since no vertex u with $A_i \notin C^\infty(u)$ can exist in α, a smallest $b \notin A_i$ is seen in every move according to f after $f(\pi)$. Since $<$ is a well-order on ω, there must exist a b^* which is eventually the smallest - hence visited - such color, and for which no smaller one is chosen later. Since Pl. 1 moves infinitely often, it follows that $b^* \in \mathsf{Inf}(\alpha)$, which is a contradiction to $\mathsf{Inf}(\alpha) = A_i$.

Hence, f is a winning strategy, which proves the lemma. □

Again, the situation is similar if Pl. 0 has a winning strategy for a game where $\mathcal{F}_1$ is countably infinite. The only difference is that both lemmas now only hold for the part of the winning region that cannot be left, i.e. for all vertices reachable after an opening move of any winning strategy. (For the above lemma, this means that Pl. 0 has a winning strategy if there exists a reachable vertex such that the condition stated in the lemma holds for every vertex reachable from there.)

Given both lemmas above, we can derive a similar one about the opposite case, that is games where the player with countably infinitely many winning sets has a winning strategy.

Lemma 3.2.9. *Let $\mathcal{G}$ be a game as above, such that $\mathcal{F}_0$ is countably infinite. If Pl. 0 has a winning strategy, then there exists a reachable v and some $A \in \mathcal{F}_0$ such that*

(i) for all $v' \in vE^*$, we have $A \in C^\infty(v')$, and

(ii) for all $b \notin A$ and $v' \in vE^*$, there exists some $w \in v'E^*$ with $\{b\} \notin C^\infty(w)$.

Proof. Since $\mathcal{G}$ is determined, Pl. 0 has a winning strategy if and only if Pl. 1 does not have one. This however implies that the negation of the precondition of the previous lemma must be true, which is exactly the conclusion that was to prove. □

It is once more evident that the other direction of the lemma holds as well. Furthermore, the situation is similar if Pl. 1 has a winning strategy while he has countably infinitely many winning sets. There is again a difference, namely that there then not only exists one reachable v, but instead such a v is reachable from any possible position.

Uncountably many winning sets In order to guarantee determinacy, so far we restricted the winning condition in such a way that one of the $\mathcal{F}$-sets had to be countable.

We now explain why this was necessary by giving an example of a game where both $\mathcal{F}_0$ and $\mathcal{F}_1$ are uncountable and which is not determined. We do so by adapting the proof technique from [9, 12], which was used to show that nondetermined Muller games with infinitely many colors exist.

The proof uses the Boolean Prime Ideal Theorem, from which it follows, amongst others, that a free ultrafilter exists. A *free ultrafilter* $\mathcal{U}$ is a set of sets of natural numbers such that the following hold:

- For any $x \subseteq \omega$, either $x \in \mathcal{U}$, or $\omega \setminus x \in \mathcal{U}$.
- If $x, y \in \mathcal{U}$, then $x \cap y \in \mathcal{U}$.
- If $x \in \mathcal{U}$, then $x \cup y \in \mathcal{U}$, for any $y \subseteq \omega$.
- If x is cofinite, then $x \in \mathcal{U}$.

Lemma 3.2.10. *There exists a nondetermined Banach-Mazur game with colors in ω and a Muller winning condition.*

Proof. Let $\mathcal{G} = (G, v_0, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$ be defined as follows:

- As a vertex set we use a subset of $\mathcal{P}(\omega) \times \omega$, namely the tuples of sets together with one of their members, plus another vertex which is used as a start vertex.

$$V := \{(x, y) : y \in x \subseteq \omega\} \cup \{v_0\}$$

- There are edges between vertices which have the same first component, namely a chain following the natural order (with respect to the second component). Furthermore, there is an edge from a vertex where the second component is the maximum of the first to all vertices where the first component is the origin's first component plus one larger number, and the second one is its minimum. To start the game, v_0 is connected to all vertices with a singleton first component.

$$\begin{aligned} E := &\{((\bar{x}, x_i), (\bar{x}, x_{i+1})) : \bar{x} = \{x_1 < \dots < x_n\}, i < n\} \\ &\cup \{((\bar{x}, x_n), (\bar{x} \cup \{y\}, x_1)) : \bar{x} = \{x_1 < \dots < x_n\}, y > x_n\} \\ &\cup \{(v_0, (\{x\}, x)) : x \in \omega\} \end{aligned}$$

- The priority function Ω assigns to every vertex the value of its second component, and an arbitrary value to the start vertex.

$$\Omega\colon V \to \omega, v \mapsto \begin{cases} y & , v = (x, y) \\ 0 & , v = v_0 \end{cases}$$

- As winning condition, we set $\mathcal{F}_0 = \mathcal{U}$.

A play can be viewed as alternatingly creating (by choosing finitely many numbers in each turn) a sequence of increasing numbers, such that the colors seen infinitely often are precisely the chosen ones. Without loss of generality, each player selects - in every move - a new set $x \subseteq \omega$ of numbers larger than the already selected ones. What remains to show is that neither of the players can have a winning strategy for $\mathcal{G}$.

Claim 1: Pl. 0 cannot have a winning strategy. To show this let us assume he has a winning strategy f. The idea is that Pl. 1 simultaneously plays two games which both are consistent with f, but which lead into a contradiction, because it cannot be the case that both sets of colors seen infinitely often are in $\mathcal{U}$. For simplicity, instead of writing the actual path in the graph, we write the increasing sequence of selected numbers (which allows for a unique mapping to paths). The games are played as follows (where a_{n+1} is an arbitrary number larger than a_n):

$$\alpha_1 = \overbrace{a_1 < \cdots < a_n}^{f(v_0)} < \overbrace{a_{n+1}}^{Pl.\ 1} < \overbrace{a_{n+2} < \cdots < a_{n_2}}^{f(\ldots a_n)} < \overbrace{a_{n_4} + 1}^{Pl.\ 1} < \overbrace{a_{n_5} < \cdots < a_{n_6}}^{f(\ldots a_{n_4}+1)} < \ldots$$

$$\alpha_2 = \underbrace{a_1 < \cdots < a_n}_{f(v_0)} < \underbrace{a_{n_2} + 1}_{Pl.\ 1} < \underbrace{a_{n_3} < \cdots < a_{n_4}}_{f(\ldots a_{n_2}+1)} < \underbrace{a_{n_6} + 1}_{Pl.\ 1} < \underbrace{a_{n_7} < \cdots < a_{n_8}}_{f(\ldots a_{n_6}+1)} < \ldots$$

Since both games are consistent with f, it holds that $\mathsf{Inf}(\alpha_1) \in \mathcal{U}$, and $\mathsf{Inf}(\alpha_2) \in \mathcal{U}$. But then $\mathsf{Inf}(\alpha_1) \cap \mathsf{Inf}(\alpha_2) = \{a_1, \ldots, a_n\} \in \mathcal{U}$, which is a contradiction to $\mathcal{U}$ being a free ultrafilter. Hence a winning strategy f for Pl. 0 cannot exist.

Claim 2: Pl. 1 cannot have a winning strategy. To prove this we assume that Pl. 1 has a winning strategy f. The idea is again to simultaneously play two games, but Pl. 0 now plays in such a way that the sets of colors seen infinitely often cannot both be not included in $\mathcal{U}$. The plays are as follows:

$$\beta_1 = \overbrace{0}^{Pl.\ 0} < \overbrace{a_1 < \cdots < a_n}^{f(\ldots 0)} < \overbrace{b_{m_1} < \cdots < b_{m_2}}^{\{b_m+1,\ldots,a_{n_2}+1\}\setminus\{a_i : n_1 \le i \le n_2\}} < \overbrace{a_{n_3} < \cdots < a_{n_4}}^{f(\ldots b_{m_2})} < \ldots$$

$$\beta_2 = \underbrace{b_1 < \cdots < b_m}_{\{1,2,\ldots,a_n+1\}\setminus\{a_i : i \leq n\}} < \underbrace{a_{n_1} < \cdots < a_{n_2}}_{f(\ldots b_m)} < \underbrace{b_{m_3} < \cdots < b_{m_4}}_{\{b_{m_2}+1,\ldots,a_{n_4}+1\}\setminus\{a_i : n_3 \leq i \leq n_4\}} < \ldots$$

Since both games are consistent with f and f is a winning strategy for Pl. 1, we know that $\mathsf{Inf}(\beta_1) \notin \mathcal{U}$ and $\mathsf{Inf}(\beta_2) \notin \mathcal{U}$. But since $\mathsf{Inf}(\beta_2) = \omega \setminus \mathsf{Inf}(\beta_1)$ and $\mathcal{U}$ is an ultrafilter, one of the two must be included in $\mathcal{U}$, which is a contradiction. Hence a winning strategy f for Pl. 1 cannot exist either.

We thus have shown that $\mathcal{G}$ is indeed a nondetermined Banach-Mazur game with a Muller winning condition and colors in ω. □

For games with such a winning condition that are determined, we do not yet know if properties similar to the ones above can be proven. It is easy to see that, in case there exists a vertex v reachable from v_0 such that for some $A \in \mathcal{F}_0$ it holds that (i) $A \in C^\infty(w)$ for all $w \in vE^*$, and (ii) for all $w \in vE^*$ and all $b \notin A$ there exists a $w' \in wE^*$ with $\{b\} \notin C^\infty(w')$, then Pl. 0 has a winning strategy (a corresponding implication can be formulated for Pl. 1). The question whether the other direction also holds remains open.

3.3 Sequential winning conditions

Using Muller winning conditions, one can only express that certain colors have to be seen infinitely often while others may be seen at most finitely many times. In some applications, this is not enough, and one would rather specify certain words of colors that should appear infinitely often. Sequential winning conditions are such a generalization of Muller winning conditions, as in contrast to these, they require sequences of colors to be seen infinitely often, while simultaneously other sequences must not be seen more than finitely many times. Since the set of sequences that have to be seen infinitely often and those that are allowed to be seen only finitely many times do not have to be complemental, the formal definition differs from the definition of Muller winning conditions.

Definition 3.3.1. A *sequential winning condition* is a set $\mathcal{S} = \{(A_1, B_1), (A_2, B_2), \cdots\}$ of pairs (A_i, B_i), $A_i, B_i \subseteq C^*$, such that always $A_i \cap B_i = \emptyset$, defining a $\mathcal{W} \subseteq C^\omega$ in the following way: a word $\alpha \in C^\omega$ is in the language $\mathcal{W}$ if and only if there exists some i such that all $x \in A_i$ appear infinitely often as infixes in α, while all $y \in B_i$ appear only finitely many times, if at all.

Notation To simplify speaking about sequences, we introduce the following notation: for every finite path π we write $\mathsf{seq}(\pi)$ for the word $\Omega(\pi[0])\Omega(\pi[1])\cdots\Omega(\pi[|\pi|-1])$. Notice that an appropriate function for infinite paths α has already been defined, namely $\mathsf{w}(\alpha)$. Furthermore, for every vertex $v \in V$ we introduce two sets.

$$C_{\mathsf{seq}}(v) := \{\mathsf{seq}(\pi) : \pi = vv_1 \cdots v_n \in \mathsf{FinPaths}(G)\}$$

is the set of all sequences that start in vertex v, while

$$C^*_{\mathsf{seq}}(v) := \bigcup_{u \in vE^*} C_{\mathsf{seq}}(u)$$

is the set of all sequences that can be reached and seen from v.

As said above, sequential winning conditions are a generalization of Muller winning conditions, i.e. every Muller winning condition can be written as a sequential one: let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition defining a language $\mathcal{W}$, with $\mathcal{F}_0 = \{A_i : i \in I\}$. We define a sequential winning condition $\mathcal{S}$ with an associated language $\mathcal{W}'$ by

$$\mathcal{S} := \{(A_i, B_i) : i \in I\},$$

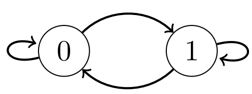
with $B_i := \{c \in C : c \notin A_i\}$. For any infinite path α in any game it now holds that

$$\mathsf{w}(\alpha) \in \mathcal{W} \iff \mathsf{w}(\alpha) \in \mathcal{W}'.$$

($\mathsf{w}(\alpha) \in \mathcal{W}$ is true if and only if there exists some $A_i \in \mathcal{F}_0$ such that $\mathsf{Inf}(\alpha) = A_i$. This means that all $a \in A_i$ are seen infinitely often, while every $b \in B_i = C \setminus A_i$ is seen at most finitely times. This in turn is equivalent to $\mathsf{w}(\alpha) \in \mathcal{W}'$.)

However, sequential winning conditions are a strict generalization of Muller winning conditions. There are many - even very simple - winning conditions that cannot be written as a Muller winning condition.

Example 3.3.1. Let $\mathcal{G} := (G, v_0, \Omega\colon v \mapsto v, \mathcal{S} = \{(1^+, \emptyset)\})$ be a Banach-Mazur game with a sequential winning condition, where G is the following graph:



Pl. 0 wins all those plays where 1^n is seen infinitely many times, for every $n \in \omega$. He obviously has a winning strategy for $\mathcal{G}$ (e.g. see larger and larger 1^n in each move), but clearly no positional one. Since there are only finitely many colors and Banach-Mazur games with a Muller winning condition over finitely many colors are positionally determined, the above sequential winning condition cannot be written as a Muller one.

Before we discuss different classes of sequential winning conditions in more detail, we observe a property of such winning conditions that will be useful later on, because certain special cases can be ignored. Let $\mathcal{S} := \{(A_i, B_i) : i \in I\}$ be a sequential winning condition. Whenever, for some $i \in I$, there exists some $b \in B_i$ that is an infix of some $a \in A_i$, the pair (A_i, B_i) can safely be removed from $\mathcal{S}$ without altering the language $\mathcal{W}$, since Pl. 0 cannot win focusing on this pair. Furthermore, if for some pair (A_i, B_i) we have that A_i is uncountable, the pair can also be removed from $\mathcal{S}$ without changing $\mathcal{W}$, since any infinite word can only contain countably many sequences (a sequence in a word can be characterized by the positions where it starts and ends, hence for the set $X(\alpha) := \{\mathsf{seq}(\pi) : \pi \text{ infix of } \alpha\}$ we have that $|X(\alpha)| \leq |\omega \times \omega| = \omega$).

3.3.1 Sequential winning conditions with countably many colors

We only discuss such sequential winning conditions where both A_i and B_i are countable, for every i. As said above, this is not a real restriction for the A_i, but it is for the B_i. To achieve this restriction in a natural way, we - as we have done while discussing Muller winning conditions - restrict the set of colors to be no larger than countably infinite. Accordingly, we can always use $\Omega \colon V \to \omega$ in the following. Since ω has only countably many finite subsets, and each subset has only finitely many permutations, the set of possible sequences ω^* is still countable, hence all B_i are as well.

We name two properties and show that in general, one of them implies the other, and then focus on cases in which this implication is valid both ways. More precisely, we introduce a property which implies that Pl. 0 has a winning strategy, and then show for certain classes of sequential winning conditions that if Pl. 0 has a winning strategy, the property holds. The property expresses that there is a vertex from where on one pair (A, B) can be successfully enforced, hence all $a \in A$ can be seen infinitely often, while each $b \in B$ can be made impossible to be seen again:
Property (S): There exists a $v \in v_0E^*$ and some $i \in I$ such that

(a) for all $v' \in vE^*$ and all $w \in A_i$: $w \in C^*_{\mathsf{seq}}(v')$, and

(b) for all $v' \in vE^*$ and all $w \in B_i$, there exists a $u \in v'E^*$ such that $w \notin C^*_{\mathsf{seq}}(u)$.

As already said, we show first that whenever (S) holds for a Banach-Mazur game with a sequential winning condition where all A_i, B_i are countable, then Pl. 0 has a winning strategy. We do so by describing such a strategy. According to this strategy, Pl. 0 moves to the vertex v from property (S) in his first move. In every later move, at vertex u after π has already been played, he sees the first $|\pi|$ sequences from A_i (with respect to some well-ordering on A_i) and end the path in some u' for which $w \notin C^*_{\mathsf{seq}}(u')$ for the smallest $w \in B_i \cap C^*_{\mathsf{seq}}(u)$ (with respect to a well-ordering on B_i). This is always possible, because every $a \in A_i$ can always be seen in vE^*, and every $b \in B_i$ can be prevented from being seen again. Since A_i is countable, the above strategy ensures that every $a \in A_i$ is seen infinitely often. Since B_i is countable, every $b \in B_i$ is eventually rendered impossible to be seen again. Thus, the strategy is a winning strategy for Pl. 0.

Classes of such sequential winning conditions The first class of sequential winning conditions we are going to discuss in more detail is the class of those winning conditions where there is only one pair, i.e. $\mathcal{S} := \{(A, B)\}$. As we already know, if (S) holds for a game with such a winning condition, then Pl. 0 has a winning strategy. We will now show that for singleton $\mathcal{S}$, the reverse also holds, i.e. if Pl. 0 has a winning strategy, then property (S) holds. But before that, we show that such games are always determined.

Lemma 3.3.1. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, \mathcal{S} = \{(A, B)\})$ be a Banach-Mazur game with a sequential winning condition over countably many colors. Then $\mathcal{G}$ is determined.*

Proof. To show that $\mathcal{G}$ is determined, we show that the set of paths whose Ω-word is in the language $\mathcal{W}$ as defined by $\mathcal{S}$ is a Borel set. We therefore construct Borel sets for the set of paths on which a sequence w is seen infinitely often, and then combine these sets to the overall required set.

Let $X^n_w := \{\pi = v_0 \cdots v_m : w \text{ appears at least } n \text{ times in } \mathsf{seq}(\pi)\}$. Since X^n_w is a set of finite words, the set $X^n_w \cdot V^\omega$ is open, from which it follows that the set

$$X_w := \bigcap_{n \in \omega} X^n_w \cdot V^\omega \cap \mathsf{Paths}(G) \in \Pi^0_2$$

of words in whose sequence w appears infinitely often is a countable intersection of open sets. The desired set $\mathsf{Win} := \{\alpha : \mathsf{w}(\alpha) \in \mathcal{W}\}$ of all paths winning for Pl. 0 can then be written as

$$\mathsf{Win} = \underbrace{\bigcap_{w \in A} X_w}_{\in \Pi^0_2} \cap \underbrace{\bigcap_{w \in B} X_w^{\mathcal{C}}}_{\in \Sigma^0_2} \in \Pi^0_3$$

and is hence a Borel set. By Martin's Theorem, $\mathcal{G}$ is determined. □

As such games are always determined, we are ready to prove the lemma below.

Lemma 3.3.2. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, \mathcal{S} = \{(A, B)\})$ be a Banach-Mazur game with a sequential winning condition for some $A, B \subseteq \omega$. Then the following are equivalent:*

- *Pl. 0 has a winning strategy.*
- *Property (S) holds.*

Proof. As the other direction has already been shown above, let $\mathcal{G}$ be a game as required, and assume that (S) does not hold. We show that in this case, Pl. 1 has a winning strategy. By Lemma 3.3.1 it follows that if Pl. 0 has a winning strategy, property (S) holds.

If (S) does not hold, it follows that no reachable v exists from where all of A can be seen infinitely often, while every sequence from B can be forced to be seen at most finitely many times. This however means that for every reachable v, one can move to some vertex v' from where on not all sequences from A can be seen infinitely often (regardless of both players' moves), or a v'' is reachable such that afterwards some $w \in B$ can always be seen again. In both cases Pl. 1 clearly has a winning strategy, thus the lemma is proven. □

After having talked about singleton $\mathcal{S}$-sets, we now show similar lemmas for games where $\mathcal{S}$ is countable. Again, we begin with showing that such games are always determined.

Lemma 3.3.3. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, \mathcal{S} = \{(A_i, B_i) : i \in I\})$ - for some I with $|I| \leq \omega$ and $A_i, B_i \subseteq \omega$ - be a Banach-Mazur game with a sequential winning condition. Then $\mathcal{G}$ is determined.*

Proof. As in Lemma 3.3.1, we show that the set $\mathsf{Win} := \{\alpha : \mathsf{w}(\alpha) \in \mathcal{W}\}$ is a Borel set. As already said, the set $\mathsf{Win}_{(A,B)}$ of paths winning for one pair (A, B) is on level Π^0_3 of the Borel hierarchy. Thus we have

$$\mathsf{Win} = \bigcup_{i \in I} \mathsf{Win}_{(A_i,B_i)} \in \Sigma^0_4.$$

Again, by Martin's Theorem it follows that $\mathcal{G}$ is determined. □

Having shown that games with a sequential winning condition over countably many colors and a countable set $\mathcal{S}$ are always determined, we can - with a similar proof as in the case for singleton $\mathcal{S}$ - obtain that Pl. 0 having a winning strategy and property (S) being valid are equivalent for such games.

Lemma 3.3.4. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, \mathcal{S} = \{(A_i, B_i) : i \in I\})$ be a Banach-Mazur game with a sequential winning condition for some countable index set I and $A_i, B_i \subseteq \omega$ for all $i \in I$. Then the following are equivalent:*

- *Pl. 0 has a winning strategy.*
- *Property (S) holds.*

Proof. Again, we only have to prove that Pl. 0 having a winning strategy implies (S). Therefore, assume that (S) does not hold for such a game $\mathcal{G}$. This means that for every $v \in v_0E^*$ and every $i \in I$,

(i) there exists a $v' \in vE^*$ and some $w \in A_i$ such that $w \notin C^*_{\mathsf{seq}}(v')$, or

(ii) there exists a $v' \in vE^*$ and some $w \in B_i$ such that for all $u \in v'E^*$, $w \in C^*_{\mathsf{seq}}(u)$.

Whenever, for any $i \in I$, a vertex v' as in (i) is reached, it is easy to see that Pl. 0 cannot win anymore using this pair (A_i, B_i), so Pl. 1 can eliminate possible winning pairs of Pl. 0 if (i) holds for these pairs at the current vertex. Whenever (ii) holds, he can however enforce that some $w \in B_i$ is seen infinitely often. Using this, one obtains the following winning strategy for Pl. 1. Let $<_I$ be a well-ordering on I. At a position v, let $P_0(v) := \{i \in I : A_i \subseteq C^*_{\mathsf{seq}}(v)\}$. Let further $\mathcal{B}$ be the set of pairs of indices and sequences $(i, w \in B_i)$ for which Pl. 1 chose to play a move according to (ii) before. Pl. 1 moves as follows: he picks the smallest $i \in P_0(v)$ such that $(i, w) \notin \mathcal{B}$, for any w. If (i) holds for this i, then Pl. 1 moves to a closest v' such that $A_i \not\subseteq C^*_{\mathsf{seq}}(v')$. Otherwise, Pl.

1 chooses a $w \in B_i$ for which there exists a v' as in (ii) and moves there. Furthermore, (i, w) is added to $\mathcal{B}$. If no such i exits, set $v' := v$. Either way, from v', a shortest path π is picked for which w is an infix of $\mathsf{seq}(\pi)$ for all w such that $(j, w) \in \mathcal{B}$ for some j. (Note that $\mathcal{B}$ is always finite and the w have been picked in such a way that such a path always exists.)

By using the above strategy, Pl. 0 cannot win by any pair $(A_i, B_i) \in \mathcal{S}$, hence the strategy is winning for Pl. 1, which proves the lemma since $\mathcal{G}$ is determined. □

In the case where $\mathcal{S}$ is uncountable, $\mathcal{G}$ need not be determined anymore. This directly follows from the fact that there exists a nondetermined Banach-Mazur game with a Muller winning condition over countably many colors where both $\mathcal{F}_0$ and $\mathcal{F}_0$ are uncountable. If the winning condition from Lemma 3.2.10 is written as a sequential one, $\mathcal{S}$ is uncountable and the game is obviously not determined, since the set of winning paths as well as the graph do not change.

Chapter 4

Simple winning strategies

A very important question, maybe the most important one when it comes to applications of Banach-Mazur games on graphs, is what kinds of winning strategies are required or sufficient for certain classes of winning conditions. (Notice for example that in Chapter 2, the proof that topological and probabilistic largeness for ω-regular properties coincide only works because of the existence of *positional* strategies.) In general, strategies for Banach-Mazur games are complex objects, namely functions mapping tuples of finite paths whose length increases during the course of a play to next moves. It was shown in Theorem 1.3.3 that this can be simplified in the way that instead of storing the actual moves, the concatenation of all moves suffices. Still, as this path is also of strictly increasing length during a play, storing it as memory is inappropriate as well.

Therefore, other concepts of strategies need to be considered or developed, namely what we call *simple strategies*. By simple strategies we mean strategies where the information needed to implement the strategy is somewhat restricted, e.g. only a single natural number is stored, or only tuples of a fixed dimension. As such strategies are weaker (with respect to games determined via such strategies) than general ones, for each type of simple winning strategy one has to investigate which classes of winning conditions guarantee determinacy via strategies of that type.

We use the results about classes of winning conditions obtained in the previous chapter to propose different kinds of simple strategies. We start with an examination of strategies without memory and explain why finite memory can be ignored. We then investigate two different classes of strategies using infinite memory, namely counting strategies and strategies using Finite Appearance Records.

From time to time, we use results known from classical graph games, but mostly introduce proofs that directly operate on Banach-Mazur games. To keep things as simple as possible, recall that we can restrict the general notion of strategies to decomposition invariant ones (cf. Theorem 1.3.3).

4.1 Positional strategies and finite memory

The most basic kind of a simple strategy is a strategy that does not use any memory. Such strategies are called *positional*. We discuss the strength of positional strategies, illustrate for which classes of winning conditions they suffice and explain why they do not suffice for other interesting classes of winning conditions. We continue with finite memory strategies and recite a theorem stating that any finite memory can be eliminated from a strategy, resulting in a positional one.

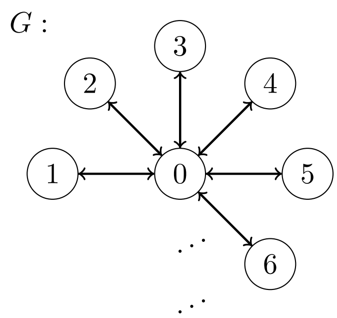
4.1.1 Positional strategies

Positional strategies are in a sense strategies of the simplest possible form. Instead of determining the next move taking into account the history of the game (i.e. the prefix played so far), they only depend on the current vertex in the graph.

Definition 4.1.1. A *positional* strategy for a Banach-Mazur game $\mathcal{G} = (G, v_0, \mathsf{Win})$ is a function $f\colon V \to \mathsf{FinPaths}(G)$ satisfying $f(v) \in \mathsf{FinPaths}(G, v)$.

It can easily be seen that there exist determined Banach-Mazur games that do not allow positional winning strategies, i.e. games where one of the players has a winning strategy, but no positional one.

Example 4.1.1. Let $\mathcal{G} = (G, v_0, \mathsf{Win})$ be a Banach-Mazur game, with the arena being $G = (\omega, \{(0, n) : n > 0\} \cup \{(n, 0) : n > 0\})$.



Let Win be the set of all infinite paths such that each $n \in \omega$ is seen at least once, and let $v_0 := 0$. (Notice that if we required every n to be seen infinitely often, the winning condition could be written as a Muller winning condition with a singleton $\mathcal{F}_0 := \{\omega\}$ and the coloring function $\Omega \colon V \to \omega, n \mapsto n$.) Pl. 0 obviously has a winning strategy - e.g. go to n in his n-th move - but no positional one: for any positional strategy f of Pl. 0, let $N_0(f) := \{n : n \text{ appears in } f(0)\}$ be the set of all vertices visited in $f(0)$. Obviously, $N_0(f)$ is finite, thus a simple strategy for Pl. 1 with which he wins against f would be to move to 0 from every $n \in \omega$ with $n \neq 0$, and to play $1 \cdot 0$ at position 0. This way, in the resulting play, only a finite number of colors is seen at all, hence no positional strategy is winning for Pl. 0, which proves that the above game is not positionally determined.

There are, however, winning conditions that guarantee positional determinacy. Using the characterizations of closed sets as presented in the section on topology (Section 1.2), the following proposition can be shown.

Proposition 4.1.1 ([5]). *Let $\mathcal{G} = (G, v_0, \mathsf{Win})$ be a Banach-Mazur game such that* Win *is on level Σ_2^0 of the Borel hierarchy. If Pl. 0 has a winning strategy $f \colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$, then he also has a positional one.*

Proof. By the definition of the Borel hierarchy, we know that Win is of the form $\mathsf{Win} := \bigcup_{n\in\omega} \Gamma_n$ for some closed sets Γ_n. Recall that a closed set can be characterized as being the set of infinite paths in a tree, thus let T_n be this tree for Γ_n. Let furthermore $f \colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ be a winning strategy for Pl. 0. We inductively construct a finite path π such that $\mathcal{BO}(\pi) \subseteq \mathsf{Win}$. Thus set $\pi_0 := f(v_0)$. For any $i \neq 0$ such that $\pi_i \notin \bigcup_{n<i} T_n$, we distinguish between two cases: if $\pi_i \cdot \rho \in T_i$ for all $\rho \in \mathsf{FinPaths}(G, \mathsf{last}(\pi_i))$, then $\mathcal{BO}(\pi_i) \subseteq \mathsf{Win}$ already holds, hence the construction terminates with $\pi := \pi_i$. Otherwise choose some finite path ρ such that $\pi_i \cdot \rho \notin T_i$, and set $\pi_{i+1} = \pi_i \cdot \rho \cdot f(\pi_i \cdot \rho)$.

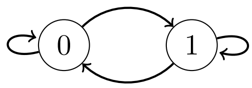
It remains to show that the construction always terminates. Therefore assume that it does not. In this case the process results in an infinite path α, which is obviously consistent with f. From the construction it follows that $\alpha \notin \Gamma_n$ for any $n \in \omega$, which entails $\alpha \notin \mathsf{Win}$, contradicting f being a winning strategy.

Since Pl. 0 will win any play that opens with π, he has a positional winning strategy $f' \colon V \to \mathsf{FinPaths}(G)$, namely set $f'(v_0) := \pi$, and set $f'(v)$ to an arbitrary but fixed successor, for any $v' \neq v_0$. □

Σ_2^0 is the highest level in the Borel hierarchy for which the above proposition holds. In [5], a game with a winning condition in Π_2^0 is provided, for which Pl. 0 has a winning strategy, but no positional one. As we will refer to this example in later sections, we introduce it at this point:

Example 4.1.2. Let $\mathcal{G} := (G, v_0, \mathsf{Win})$ be the following game:

$$G = (V := \{0, 1\}, E := V \times V)$$



$$v_0 := 0$$

$$\mathsf{Win} := \{\alpha : \text{ for inf. many prefixes } \pi \text{ of } \alpha,\ |\pi|_1 > |\pi|_0\}$$

Pl. 0 wins exactly those plays where, for infinitely many i, the prefix of length i contains more 1s than 0s. It is easy to see that he has a winning strategy, since when he moves after a prefix π, he only needs to know the result of $|\pi|_1 - |\pi|_0$ to ensure that the play contains another prefix as required, thus eventually infinitely many prefixes of this kind.

To see that he does not have a positional winning strategy, let $f\colon V \to \mathsf{FinPaths}(G)$ be an arbitrary positional strategy for $\mathcal{G}$. We show that Pl. 1 can play in such a way that he wins although the play is consistent with f, hence f cannot be winning. To see that Pl. 1 can win, let $m := |f(0)|_1$ be the number of 1s in the strategy's move at position 0. Pl. 1 will play paths 0^l in such a way that, after the first move of Pl. 1, the prefix played so far will always contain more 0s than 1s. In his first move Pl. 1 gets to move after $f(0)$ has been played. Let $k := |f(0)|_1 - |f(0)|_0$, i.e. k more 1s than 0s have been seen so far. Set now $l := k + m + 1$. Then after Pl. 1's first move, any prefix will contain more 0s than 1s.

It remains to show that $\mathsf{Win} \in \Pi_2^0$, i.e. that Win can be written as a countable intersection of open sets. For any $n \in \omega$, let $P_n \subseteq \mathsf{FinPaths}(G, v_0)$ be the set of finite paths that have at least n prefixes that have more 1s than 0s. Win can be written as

$$\mathsf{Win} = \bigcap_{n \in \omega} P_n \cdot V^\omega \cap \mathsf{Paths}(G, v_0).$$

Muller winning conditions

Although many determined Banach-Mazur games are not positionally determined, there are important classes of winning conditions that guarantee determinacy via positional strategies. One of these classes is the class of Muller winning conditions over a finite set of colors, which we examine first. From Theorem 1.3.4 and from a reduction to parity games as in [10, Theorem 3.6] we know that Banach-Mazur games with a Muller winning condition are determined via finite memory strategies. It turns out that such games are in fact positionally determined - as are all other games determined via finite memory strategies (cf. Section 4.1.2) - which directly follows from a lemma in a previous chapter.

Theorem 4.1.1 ([5]). *Let $\mathcal{G} = (G, v_0, \Omega \colon V \to C, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition over a finite set C of colors. Then $\mathcal{G}$ is positionally determined.*

Proof. From Lemma 3.2.1 we know that Pl. 0 has a winning strategy if and only if a stable vertex v with $C(v) \in \mathcal{F}_0$ is reachable from v_0.

If Pl. 0 now has a winning strategy, he also has a positional one, namely move to a stable v with $C(v) \in \mathcal{F}_0$ from position v_0, and for every $v' \in vE^*$ play a shortest path π with $\Omega(\pi) = C(v)$.

If Pl. 1 has a winning strategy, he again also has a positional one, namely move to a stable vertex if the current position is not yet stable, and then see all reachable colors in every move. This strategy is winning for him, since we know from the lemma that $C(v) \in \mathcal{F}_1$ for every stable v. □

On certain classes of graphs - e.g. graphs with a countable vertex set - positional strategies also suffice for Banach-Mazur games with a Muller winning condition over a countable set of colors, and also for some sequential winning conditions. This is shown in more detail in Section 4.2.1, namely in Corollary 4.2.2, Corollary 4.2.3 and Theorem 4.2.8.

Omega-regular winning conditions

Another important class of winning conditions that guarantee positional determinacy is the class of ω-regular winning conditions, which are e.g. characterized by the following theorem:

Theorem 4.1.2 ([7])**.** *Let C be a finite set of colors. A language $\mathcal{W} \subseteq C^\omega$ is* ω-regular *if and only if it is definable in monadic second order logic on infinite words (S1S).*

With the help of results from automata theory and the theorem stating that finite memory can be eliminated in Banach-Mazur games, we now explicate why Banach-Mazur games with such winning conditions are positionally determined.

Definition 4.1.2. A *deterministic Muller automaton* is a tuple $\mathcal{A} = (Q, C, q_0, \delta, \mathcal{F})$, where Q is a finite set of states, C is a finite set of colors (or *alphabet*), $q_0 \in Q$ is the initial state, $\delta\colon Q \times C \to Q$ is the transition function and $\mathcal{F} \subseteq \mathcal{P}(Q)$ is the acceptance condition. A *run* of $\mathcal{A}$ on an infinite word $\gamma = a_1 a_2 a_3 \cdots \in C^\omega$ is a sequence $\rho(\gamma) = q_0 q_1 q_2 \cdots$ of states such that $q_i = \delta(q_{i-1}, a_i)$ for any $i > 0$. A word γ is *accepted by $\mathcal{A}$*, if

$$\mathsf{Inf}(\rho(\gamma)) = \{q \in Q : q \text{ appears infinitely often in } \rho(\gamma)\} \in \mathcal{F}.$$

In this case we say that γ is in the language $\mathcal{L}(\mathcal{A}) = \{\gamma \in C^\omega : \gamma \text{ is accepted by } \mathcal{A}\}$ recognized by the automaton.

Using the theorem of McNaughton [18], one obtains that a language $\mathcal{W}$ is ω-regular if and only if it is the language recognized by some deterministic Muller automaton (for more details see [11]).

To show that Banach-Mazur games with an ω-regular winning condition are positionally determined, we use the existence of a deterministic Muller automaton recognizing the winning condition and construct the product of the game and the automaton. Once this product game is constructed, it is easy to show that it is positionally determined (as the product game is a game with a Muller winning condition), and by a theorem from the next section we infer that the game itself is also positionally determined. We already use a notion of strategies that is introduced in Section 4.1.2 in the proof, but the theorem itself speaks only about positional strategies.

Theorem 4.1.3 ([5])**.** *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to C, \mathcal{W})$ be a Banach-Mazur game such that C is finite and $\mathcal{W} \subseteq C^\omega$ is ω-regular. Then $\mathcal{G}$ is positionally determined.*

Proof. Let $\mathcal{A} = (Q, C, q_0, \delta, \mathcal{F})$ be a deterministic Muller automaton such that $\mathcal{L}(\mathcal{A}) = \mathcal{W}$. The product game $\mathcal{G} \times \mathcal{A} = (G', v'_0, \Omega', (\mathcal{F}_0, \mathcal{F}_1))$ is defined in the following way:

$$G' = \begin{pmatrix} V' := (V \times Q), \\ E' := \{((v, q), (v', q')) : (v, v') \in E, q' = \delta(q, \Omega(v'))\} \end{pmatrix}$$

$$v_0' := (v_0, \delta(q_0, \Omega(v_0)))$$
$$\Omega' : V \times Q \to Q, (v, q) \mapsto q$$
$$\mathcal{F}_0 := \mathcal{F}, \mathcal{F}_1 := \mathcal{P}(Q) \setminus \mathcal{F}_0.$$

$\mathcal{G}'$ is a Banach-Mazur game with a Muller winning condition over a finite set of colors, since Q is finite. Furthermore, both $\mathcal{G}$ and $\mathcal{G}'$ are in a sense equivalent: since $\mathcal{A}$ is deterministic, there is a bijection $b\colon \mathsf{Paths}(G', v_0') \to \mathsf{Paths}(G, v_0)$ between infinite paths in $\mathcal{G}'$ and $\mathcal{G}$, namely the projection $b(\alpha)$ of a path α in $\mathcal{G}'$ to the vertices' first component, and $b(\alpha)$ is won by Pl. 0 in $\mathcal{G}$ if and only if α is won by Pl. 0 in $\mathcal{G}'$.

Because of Theorem 4.1.1, $\mathcal{G}'$ is positionally determined. A positional strategy for $\mathcal{G}'$ can easily be transformed into a strategy using $\mathcal{A}$ as a finite memory (cf. Section 4.1.2) for $\mathcal{G}$. Thus $\mathcal{G}$ is determined via strategies using finite memory. From Theorem 4.1.4 it immediately follows that $\mathcal{G}$ is also positionally determined. □

4.1.2 Finite memory strategies

In the above section we discussed positional strategies. Now, we consider the notion of a strategy using a memory structure, as this is a natural extension of positional strategies. In classical games on graphs, strategies with finite memory guarantee determinacy for a larger class than positional ones, but it turns out that for Banach-Mazur games they are only as strong as strategies without memory. In order to demonstrate this, we give a definition of strategies with memory (which we also use in later sections on strategies with infinite memory) and present the result from [9] that finite memory can be eliminated.

Definition 4.1.3. A *memory structure* for a game $\mathcal{G} = (G, v_0, \Omega, \mathcal{W})$ is a tuple $\mathfrak{M} = (M, \mathsf{init}, \mathsf{update})$ that consists of a set of memory states M, an initialization function $\mathsf{init}\colon \{v_0\} \to M$ and a function $\mathsf{update}\colon M \times V \to M$ to change the memory state on moving to another vertex. We say that a memory is *finite*, if $|M| < \omega$.

We also use the implicitly defined function $\mathsf{memory}\colon M \times \mathsf{FinPaths}(G) \to M$, which returns, for a memory state and any possible finite path, the new memory state after the path has been played. Formally,

$$\mathsf{memory}\colon m \times \pi \mapsto \begin{cases} \mathsf{update}(m, v) & , \pi = v \\ \mathsf{update}(\mathsf{memory}(m, v_1 \cdots v_{n-1}), v_n) & , \pi = v_1 \cdots v_n. \end{cases}$$

Instead of writing $\mathsf{memory}(\mathsf{init}(v_0), v_1 \cdots v_n)$ for a $\pi = v_0 \cdots v_n$ that starts in v_0, we also write $\mathsf{memory}(\pi)$.

Definition 4.1.4. A *strategy using memory* $\mathfrak{M}$ for a game $\mathcal{G} = (G, v_0, \Omega, \mathcal{W})$ is a function $f\colon V \times M \to \mathsf{FinPaths}(G)$ satisfying $f(v, m) \in \mathsf{FinPaths}(G, v)$ for any $v \in V, m \in M$. If a vertex v has been reached via a finite path $\pi \in \mathsf{FinPaths}(G, v_0)$, the next move according to f at v is $f(v, \mathsf{memory}(\pi))$.

A game $\mathcal{G}$ is *determined via strategies using memory* $\mathfrak{M}$, if the player who has a winning strategy also has one using memory $\mathfrak{M}$.

The idea behind the elimination of finite memory is that at any given vertex, the set of memory states that appear after any finite prefix ending in this vertex is finite. This can be used in such a way that a consistent move for each such possible m is done by the player of the strategy. Since no memory is needed to do so, the resulting strategy is positional, and winning if and only if the original one is.

Theorem 4.1.4 (Elimination of finite memory [9])**.** *Let $\mathcal{G} = (G, v_0, \Omega, \mathcal{W})$ be a Banach-Mazur game. If $\mathcal{G}$ is determined via strategies using finite memory, then $\mathcal{G}$ is also positionally determined.*

Proof. For simplicity assume first that Pl. 1 has a winning strategy $f\colon V \times M \to \mathsf{FinPaths}(G)$ using a finite memory $\mathfrak{M}$ for the game $\mathcal{G}$. Let $M(v)$ be the set of all memory states that may occur at vertex v in any play:

$$M(v) := \{\mathsf{memory}(\pi) : \mathsf{last}(\pi) = v, \pi \in \mathsf{FinPaths}(G, v_0)\}.$$

Let furthermore $<_M$ be a linear ordering on M, with $\mathsf{init}(v_0) \le m$ for any $m \in M$. With the above, we define a positional strategy $f'\colon V \to \mathsf{FinPaths}(G)$ and argue that it is winning for Pl. 1.

Let thus $v \in V$ be an arbitrary vertex. Consider the ordered set of possible memory states at v,

$$M(v) = \{m_1 <_M \cdots <_M m_n\}.$$

$f'(v)$ is created iteratively by prolonging finite paths π_i. We start with $\pi_1 := f(v, m_1)$, and set $\pi_{i+1} := \pi_i \cdot f(\mathsf{last}(\pi_i), \mathsf{memory}(m_{i+1}, \pi_i))$. At last, we define $f'(v) := \pi_n$.

It remains to show that f' is winning for Pl. 1. To see this, we rewrite a consistent play α in the following way:

$$\alpha = \vartheta_0 \cdot \underbrace{\rho_1^0 \cdot \rho_2^0 \cdot \rho_3^0}_{f'(\mathsf{last}(\vartheta_0))} \cdot \vartheta_1 \cdot \underbrace{\rho_1^1 \cdot \rho_2^1 \cdot \rho_3^1}_{f'(\mathsf{last}(\vartheta_1))} \cdots$$
$$= \vartheta_0 \cdot \rho_1^0 \cdot \underbrace{\rho_2^0}_{f(\mathsf{last}(\rho_1^0), \mathsf{memory}(\vartheta_0 \cdot \rho_1^0))} \cdot \rho_3^0 \cdot \vartheta_1 \cdot \rho_1^1 \cdot \underbrace{\rho_2^1}_{f(\mathsf{last}(\rho_1^1), \mathsf{memory}(\vartheta_0 \cdots \rho_1^1))} \cdot \rho_3^1 \cdots$$

It follows from the definition of f' that such a rewriting of every move of Pl. 1 always exists, and since the ρ_1- and ρ_3-parts could also be part of Pl. 0's moves, any α consistent with f' is also consistent with f, thus f' is a winning strategy.

Assume now that Pl. 0 has a winning strategy using a finite memory $\mathfrak{M}$ for the game $\mathcal{G}$. The argument in this case is analogous to the above one, just that $M(v)$ in this case only contains those memory states that derive from finite paths that start with the opening move of the strategy. This is mainly due to the importance of the opening move in strategies of Pl. 0, which is implicit in the Banach-Mazur Theorem. □

4.2 Counting strategies

In the previous section it was explained why finite memory strategies do not provide any additional strength in comparison to positional ones. Therefore, we now consider classes of strategies using infinite memory. The first one we propose is the class of strategies that use a counter which is increased by predefined rules.

Definition 4.2.1. A *counting strategy* is a strategy of the form $f\colon V \times \omega \to \mathsf{FinPaths}(G)$ that has access to a counter, i.e. a sequence of natural numbers, such that during a consistent play, the counter is strictly increasing and automatically updated according to a fixed rule.

We present two kinds of counting strategies and discuss what classes of winning conditions they guarantee determinacy for. It turns out that the first kind is useful for many prefix independent winning conditions, but fails to be of use for even very simple winning conditions that are not. The second kind overcomes this problem and is indeed stronger than the first, but also has shortcomings for more complex non-prefix independent winning conditions.

4.2.1 Move-counting strategies

The first class of counting strategies we examine in more detail are strategies where the counter stores the number of the current move. The moves according to such a strategy therefore have access to a strictly increasing infinite counter, but the counter does not provide any specific information about the finite path played so far.

Definition 4.2.2. A *move-counting strategy* is a function $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ that provides a next move depending only on the current vertex and the number of moves played so far (by the player using the strategy). In a consistent play, the moves of a player as generated by the strategy are thus $g(\_,0)$, $g(\_,1)$, $g(\_,2)$, $\dots$ for the respective vertices.

Obviously, any game that is already positionally determined is also determined via move-counting strategies. Similarly, every class of winning conditions that guarantees positional determinacy also guarantees determinacy via move-counting strategies.

Muller winning conditions

In Theorem 4.1.1 it was shown that Muller winning conditions over a finite set of colors guarantee positional determinacy, hence we do not have to consider such winning conditions again. One major reason why Banach-Mazur games with a Muller winning condition over an infinite set of colors are not in general positionally determined is that the single winning sets of the respective winning player may be infinite. Thus it is no longer possible to see a complete set in a single move, but the player has to find another way of seeing it infinitely often. One natural way to do so is to see larger and larger initial subsets. As a means to determine the size of the next initial subset, access to an increasing sequence of numbers is one option, which motivates the investigation of move-counting strategies, especially for Muller winning conditions.

In order to organize the discussion and to be able to reuse certain results, we begin with simple Muller winning conditions over a countably infinite set of colors, namely where either $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton, and then increase the complexity of the winning condition to finite and countably infinite $\mathcal{F}_0$ and $\mathcal{F}_1$. For all three classes, we present theorems stating that they guarantee determinacy via move-counting strategies.

For singleton $\mathcal{F}_\sigma$, the theorem mainly follows the motivation for move-counting strategies mentioned above, i.e. constructing a strategy where larger and larger initial subsets are seen. However, the strategy also has to take into account that other colors need to be prevented from being seen infinitely often.

Theorem 4.2.1. *Let* $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ *be a Banach-Mazur game with a Muller winning condition such that either* $\mathcal{F}_0$ *or* $\mathcal{F}_1$ *is a singleton. Then* $\mathcal{G}$ *is determined via move-counting strategies.*

Proof. The case where $\mathcal{F}_1$ is a singleton is similar to the one where $\mathcal{F}_0$ is, apart from the fact that the importance of the opening move has to be regarded differently in the respective strategies. Thus, we restrict the proof to games with $\mathcal{F}_0 = \{A\}$ for some $A \subseteq \omega$. Our proof consists of two parts. First, we show that if Pl. 0 has a winning strategy, he also has a move-counting one, and then we argue that the same is true for Pl. 1.

Assume that Pl. 0 has a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$. By Lemma 3.2.2 we know that for any v reachable after the opening move according to f, it holds that all colors in A can still be seen and for every $b \notin A$ a position from where b cannot be seen infinitely often anymore is reachable. We define a move-counting strategy $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ in the following way: as an opening move $g(v_0, 0) := f(v_0)$ we copy the opening move of f. For any later position v and move number n we set $g(v, n) := \vartheta_1 \cdot \vartheta_2$, where ϑ_1 is a shortest path from v such that $\mathsf{first}(A, n) \subseteq \Omega(\vartheta_1)$, and ϑ_2 is a shortest path from $v' := \mathsf{last}(\vartheta_1)$ such that $b := \min\{m : \{m\} \in C^\infty(v')\}$ can no longer be seen infinitely often afterwards, i.e. $b \notin C^\infty(\mathsf{last}(\vartheta_2))$. Both ϑ_1 and ϑ_2 always exist because of the lemma cited above, thus g is a strategy (i.e. a well-defined function). It is also winning since for any consistent play α it holds that $A \subseteq \mathsf{Inf}(\alpha)$ (because of the ϑ_1-parts) and $\mathsf{Inf}(\alpha) \subseteq A$ (because of the ϑ_2-parts).

Assume that Pl. 1 has a winning strategy. In this case ($\mathcal{F}_0$ being a singleton), he also has a positional one, hence in particular a move-counting one. By Lemma 3.2.3 it holds for any $v \in V$ that there exists a reachable v' such that A cannot be seen infinitely often anymore, or a reachable v'' such that some $b \notin A$ can always be seen from there on. The idea is to define a positional strategy $g\colon V \to \mathsf{FinPaths}(G)$ for Pl. 1 that uses the above properties. For any v where a v' as above is reachable, we set $g(v)$ to a shortest path to a closest such v'. For any other vertex we know that there exists at least one $b \notin A$ such that for some $v_b \in vE^*$ it holds that b can always be seen after v_b has been reached. Let b be the minimal color not in A for which such a vertex is reachable from v. We then set $g(v)$ to a shortest path to a closest such v_b prolonged with a shortest path to a vertex with color b. For any consistent play α it now holds that $A \not\subseteq \mathsf{Inf}(\alpha)$ or that $b \in \mathsf{Inf}(\alpha)$ for some $b \notin A$. In either case, the play is won by Pl. 1, hence g is winning. □

The above theorem entails the following corollary.

Corollary 4.2.1. *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over countably many colors such that either $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via move-counting strategies.*

The next step is to increase the size of the smaller set of winning sets, now allowing finite $\mathcal{F}_0$ or $\mathcal{F}_1$ sets. It was shown in the previous chapter that such winning conditions are similar to the ones where one of the $\mathcal{F}$-sets is a singleton, namely that these games can in a way be reduced to ones with a singleton set of winning sets.

Theorem 4.2.2. *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over a countable set of colors such that either $\mathcal{F}_0$ or $\mathcal{F}_1$ is finite. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via move-counting strategies.*

Proof. Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a winning condition as required. Again the cases where either $\mathcal{F}_0$ is finite or $\mathcal{F}_1$ is are analogous except for the importance of the first move, so we once more restrict the proof to the case where $\mathcal{F}_0$ is finite.

Let first Pl. 0 have a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$. We construct a move-counting winning strategy $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ using Lemma 3.2.4 and Lemma 3.2.5. We know by Lemma 3.2.4 that a $\tilde{v}$ exists after which the game can, according to Lemma 3.2.5, be played like one with a singleton $\mathcal{F}'_0 = \{A\}$. Let $h\colon V \times \omega \to \mathsf{FinPaths}(G)$ be a move-counting winning strategy for Pl. 0 in the respective singleton game starting in $\tilde{v}$ (this strategy exists by the previous theorem and since $\tilde{v}$ is in the winning region of Pl. 0). The opening move of the new strategy g now consists of a shortest path ϑ from v_0 to $\tilde{v}$ and the opening move of h, hence $g(v_0, 0) := \vartheta \cdot h(\tilde{v}, 0)$. For any vertex v reachable from $\tilde{v}$, we copy h, i.e. $g(v, n) := h(v, n)$. For any other vertex v, we set $g(v, n)$ to an arbitrary but fixed successor. For any play α consistent with g we know that $\mathsf{Inf}(\alpha) = A \in \mathcal{F}_0$, so g is indeed a winning strategy.

For the other case, let Pl. 1 have a winning strategy. Furthermore, we fix a well-ordering on the finite subsets of ω. By Lemma 3.2.6, for every possible v there exists a finite set $B \subseteq \omega$ whose being seen infinitely often entails that Pl. 0 loses. This means that Pl. 1 can already win with a positional strategy $g\colon V \to \mathsf{FinPaths}(G)$. For any vertex v, we construct $g(v)$ by choosing the minimal B in the above sense for which there exists a vertex v_B from where on B can always be seen completely. (From the lemma we know that at least one such B exists.) $g(v)$ is now a shortest path to a closest such v_B for this minimal B that visits all colors in B. (Notice that $v = v_B$ is possible and will, in fact, always be true after the first move consistent with the strategy.) For any consistent play α, we hence have $B \subseteq \mathsf{Inf}(\alpha)$, so α is won by Pl. 1. □

If one of the $\mathcal{F}$-sets is countable, via results from Section 3.2.2 an analogous theorem can be proven. Still, in contrast to the above two theorems about Muller winning conditions and move-counting strategies, in this case both players need move-counting strategies (instead of positional strategies being sufficient for one of them).

Theorem 4.2.3. *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over a countable set of colors such that $\mathcal{F}_0$ or $\mathcal{F}_1$ is countable. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via move-counting strategies.*

Proof. We show that every Banach-Mazur game $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ with a winning condition as formulated in the precondition is determined via move-counting strategies. We once more restrict the proof to the case where $\mathcal{F}_0$ is countable, since the opposite case differs mainly in the importance of the opening move.

Let Pl. 0 have a winning strategy for $\mathcal{G}$. By Lemma 3.2.9 there exists a reachable v and some $A \in \mathcal{F}_0$ such that Pl. 0 can enforce $\mathsf{Inf}(\alpha) = A$ after reaching v. As was shown in Theorem 4.2.1, this can be achieved by a move-counting strategy, thus Pl. 0 has a move-counting strategy for $\mathcal{G}$.

For the remaining other direction, let Pl. 1 have a winning strategy for $\mathcal{G}$. From Lemma 3.2.7 it follows that for every reachable vertex v and every $A \in \mathcal{F}_0$, Pl. 1 can make sure with one move that A cannot be seen infinitely often, or he can force some $b \notin A$ to be seen infinitely often (by seeing it infinitely often in his later moves). In the proof of Lemma 3.2.8, a strategy is constructed that can easily be rewritten to be a move-counting one (instead of setting $n := |v_0 \cdots v|$, n is given as an input argument to the strategy function). □

For those Muller conditions where both $\mathcal{F}_0$ and $\mathcal{F}_1$ are uncountable, the situation is more complicated. If there still exists a vertex v and some A winning for the player with a winning strategy that can be forced to exactly be the set of colors seen infinitely often (i.e. all colors in A can be seen infinitely often, while all colors not in A can successively be made impossible), then this player also has a move-counting strategy. But since such winning conditions do not in general guarantee determinacy, they also do not guarantee determinacy via move-counting strategies.

Sequential winning conditions

Sequential winning conditions are a generalization of Muller winning conditions where the condition does not require colors to be seen infinitely often, but finite sequences of colors. In a Banach-Mazur game this is not so much of a difficulty since in each move a player is allowed to choose a finite path. This already hints to the fact that there is a connection between determinacy via move-counting strategies for certain classes of Muller winning conditions and determinacy via move-counting strategies for certain classes of sequential ones. As it turns out, for sequential winning conditions over a countable set of colors this connection is rather straightforward, i.e. if the sequential winning condition consists of at most countably many pairs (A, B), then the game is determined via move-counting strategies.

As for sequential winning conditions over a countable set of colors there is a strong relation between the property (S) from Section 3.3.1 and Pl. 0 having a winning strategy, we prove first that whenever the property holds, then Pl. 0 has not only a general winning strategy, but in fact also a move-counting one.

Lemma 4.2.1. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, \mathcal{S} = \{(A_i, B_i) : i \in I\})$ be a Banach-Mazur game with a sequential winning condition such that property (S) holds. Then Pl. 0 has a move-counting winning strategy.*

Proof. The move-counting winning strategy can be constructed in the same way as the winning strategy following the definition of property (S) was. Recall that property (S) states that there exists a reachable v and some $i \in I$ such that after v has been visited, all sequences from A_i can always be seen again, and for all sequences from B_i it is always possible to move somewhere where they cannot be seen again. Thus the opening move of the move-counting strategy $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ is $g(v_0, 0) := \vartheta$, a shortest path from v_0 to the v from property (S). Fixing a well-order $<$ (via a bijective mapping $\underline{n}$ to ω, i.e. $a < b \iff \underline{n}(a) < \underline{n}(b)$) on the finite sequences of colors, $g(u, n)$ - for all $u \in vE^*$ - consists of two parts, namely seeing elements from A_i and eliminating ones from B_i. We thus have $g(u, n) := \vartheta_1 \cdot \vartheta_2$, where ϑ_1 is a shortest path starting in u such that, for all $j \leq n$, the j-th element of A is an infix of $\mathsf{seq}(\vartheta_1)$. ϑ_2 is a shortest path such that the minimal (wrt. $<$) $b \in B_i$ for which $b \in C^*_{\mathsf{seq}}(\mathsf{last}(\vartheta_1))$ is true cannot be seen again from $\mathsf{last}(\vartheta_2)$ onwards. Notice that both ϑ_1 and ϑ_2 always exist because (S) holds. For all other vertices u we set $g(u, n)$ to some arbitrary but fixed successor.

Let α be a play that is consistent with g. It immediately follows from the definition of the ϑ_1-parts that all sequences from A_i are seen infinitely often. Assume that some $b \in B_i$ is also seen infinitely often. But since B_i is countable, there must exist a move where b is the minimal still possible element from B_i - $<$ is induced by a bijection to the natural numbers - and hence b cannot be seen again in any later move. But then it cannot appear infinitely often in α, so α is won by Pl. 0, which means that g is a winning strategy. □

With the above lemma we can prove two theorems about sequential winning conditions guaranteeing determinacy via move-counting strategies by just showing that if Pl. 1 has a winning strategy, he also has a move-counting one. We begin with a simple case, namely where $\mathcal{S}$ is a singleton.

Theorem 4.2.4. *Let $\mathcal{S} = \{(A, B)\}$ be a sequential winning condition over a countable set of colors. Then $\mathcal{S}$ guarantees determinacy via move-counting strategies.*

Proof. Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, \mathcal{S})$ be an arbitrary Banach-Mazur game with a sequential winning condition as required. It has to be shown that if a player has a winning strategy for $\mathcal{G}$, he also has a move-counting one. Let at first $\mathcal{G}$ be such that Pl. 0 has a winning strategy. By Lemma 3.3.2 we know that in this case property (S) holds. From Lemma 4.2.1 it follows that he has a move-counting winning strategy.

Let now $\mathcal{G}$ be such that Pl. 1 has a winning strategy. Again Lemma 3.3.2 tells us that in this case property (S) does not hold. This means that from any vertex some vertex v is reachable, from where not every $a \in A$ can be seen infinitely often, or a v' from where on some $b \in B$ can always be seen. Exploiting this, one can easily construct a positional winning strategy for Pl. 1, namely move to a v as above from every vertex where this is possible, and for all other vertices choose the minimal $b \in B$ for which a v' exists, move to v' and see the sequence b. This strategy is obviously winning, since either not all of A is seen infinitely often, or at least one sequence from B is seen infinitely often. □

The next step is to prove a similar theorem for countable $\mathcal{S}$. In this case, it is no longer possible to always have a positional winning strategy for Pl. 1, but still a move-counting one exists.

Theorem 4.2.5. *Let $\mathcal{S} = \{(A_i, B_i) : i \in I\}$ be a sequential winning condition over a countable set of colors such that I is countable. Then $\mathcal{S}$ guarantees determinacy via move-counting strategies.*

Proof. Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, \mathcal{S})$ be a Banach-Mazur game with a sequential winning condition as required. If Pl. 0 wins, it follows from Lemma 3.3.4 that property (S) holds and from Lemma 4.2.1 that he has a move-counting winning strategy.

If Pl. 1 has a winning strategy, this means - by Lemma 3.3.4 - that property (S) does not hold. Hence for any reachable v and any i, it is either possible for Pl. 1 to move somewhere from where not all of A_i can be seen infinitely often, or to move somewhere where at least one element of B_i can be forced to be seen infinitely often. We construct a move-counting winning strategy $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ that makes use of these properties. For any vertex v let $P_0(v) := \{i \in I : A_i \subseteq C^*_{\mathsf{seq}}(v)\}$. For any v and any n, the move $g(v, n)$ is constructed in the following way: let m be minimal such that $m \in P_0(v)$ and no $b \in B_m$ exists that can be seen from every vertex reachable from v, if such an index exists, and $m := n$ otherwise. If a minimal index exists, we distinguish between two cases. If there exists a v' reachable from v such that not all of A_m can be seen after visiting v', we set ϑ_1 to a shortest path from v to a closest such v'. Otherwise let $b \in B_m$ be minimal such that there exists a vertex v'' for which it is true that b can always be seen again after v'' has been visited. Then ϑ_1 is a shortest path from v to a closest such v''. Notice that one of these has to be possible since property (S) holds. If m was set to n, ϑ_1 is a path to an arbitrary but fixed successor of v.

This does not yet suffice, since all sets for which a v'' has been chosen earlier have to be taken care of in the sense that an element of the respective B-set has to be seen. For every $i \in P_0(\mathsf{last}(\vartheta_1))$ such that $i \leq m$, pick the minimal $b_i \in B_i$ which can be seen from every later vertex. Then ϑ_2 is a shortest path starting in $\mathsf{last}(\vartheta_1)$ for which it holds that every b_i like this appears in $\mathsf{seq}(\vartheta_2)$ (such a path always exists). We set $g(v, n) := \vartheta_1 \cdot \vartheta_2$.

It remains to show that g is a winning strategy for Pl. 1. Let thus α be a play consistent with g. For any $i \in I$ (since I is countable) it then holds that either A_i could not have been seen infinitely often - if a v' appears in α - or that from some vertex in α onwards some $b \in B_i$ can always be seen in G. But then Pl. 1 made sure that the minimal such b was seen in every later move, hence some $b \in B_i$ is seen infinitely often in α. This means that Pl. 0 cannot win with any pair $(A_i, B_i) \in \mathcal{S}$, so α must be won by Pl. 1. □

There may of course also be games with sequential winning conditions where $\mathcal{S}$ is not countable that are determined via move-counting strategies, but in general such winning conditions do not guarantee determinacy. Also one can conceive of games with a sequential winning condition over an uncountable set of colors that are determined via move-counting strategies, but it is not known whether this is true in general, as in this case the B_i do not have to be countable, hence the construction of the move-counting strategy for Pl. 0 using property (S) cannot be transferred to this more general case.

Move-counting strategies on acyclic graphs

The key feature of a move-counting strategy as seen so far has been the ability to access a strictly increasing sequence of natural numbers. However, it is not important that the sequence is of the form

$$0 < 1 < 2 < \cdots,$$

but it also suffices - as we will show - if it has the form $m_0, m_1, m_2, \cdots$ such that there exists an infinite subsequence

$$m_{i_0} < m_{i_1} < m_{i_2} < \cdots$$

that is strictly increasing. This is a useful observation that allows, amongst others, to simplify move-counting strategies to positional ones on certain kinds of acyclic graphs.

Lemma 4.2.2. *Let $\mathcal{G}$ be a Banach-Mazur game for which Pl. σ, $\sigma \in \{0,1\}$, has a move-counting winning strategy. Then Pl. σ also has a counting strategy that is winning for any sequence $n_0, n_1, n_2, \cdots$ of counter values, such that there exists an infinite index sequence $i_0 < i_1 < \cdots$ satisfying $n_{i_j} < n_{i_{j+1}}$ for all $j \geq 0$.*

Proof. The idea of this proof is to simulate the move-counting strategy by iterating its moves, similar to the proof of Theorem 4.1.4 on the elimination of finite memory. Starting with a move-counting winning strategy $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ of Pl. σ for $\mathcal{G} = (G, v_0, \mathsf{Win})$, we therefore define a strategy $h\colon V \times \omega \to \mathsf{FinPaths}(G)$ that uses $n_0, n_1, \cdots$ as counter values and argue by rewriting consistent paths that it is winning for Pl. σ.

For any $v \in V$ and any $n \in \omega$, we set $h(v,n)$ to be a concatenation of moves according to g for all numbers less than or equal to n:

$$h(v,n) := g(u_0, 0) \cdot g(u_1, 1) \cdot \cdots \cdot g(u_n, n)$$

for $u_0 := v$ and $u_i := \mathsf{last}(g(u_{i-1}, i-1))$ for $1 \leq i \leq n$.

To see that h is winning, let α be a consistent play. By thinking of certain parts of Pl. σ's moves as parts of Pl. $1-\sigma$'s, it turns out that α is also consistent with g. To simplify the argument, we provide the rewritten decomposition of α for $\sigma = 0$ only, since it is completely analogous for $\sigma = 1$. By ρ_i we mean the i-th move of Pl. 1.

$$\begin{aligned} \alpha &= h(v_0, n_0) \cdot \rho_0 \cdot h(\_, n_1) \cdot \rho_1 \cdot h(\_, n_2) \cdot \cdots \\ &= h(v_0, n_0) \cdot \cdots \cdot h(\_, n_{i_1}) \cdot \cdots \cdot h(\_, n_{i_2}) \cdot \cdots \\ &= g(v_0, 0) \cdot \cdots \cdot g(\_, 1) \cdot \cdots \cdot g(\_, 2) \cdot \cdots \end{aligned}$$

Hence for any α consistent with h it holds that $\alpha \in \mathsf{Win}$, since α is also consistent with the winning strategy g. □

We can use the above lemma to obtain positional determinacy for games with certain winning conditions on special kinds of acyclic graphs. The most basic class of graphs for which we show this is the class of countable acyclic graphs. Notice that a finite acyclic graph cannot be the arena of a Banach-Mazur game as it must have at least one vertex without a successor. In the theorem below we use that the graph is countable by using a natural enumeration of the vertices and that $<$ is a well-order on ω.

Theorem 4.2.6. *Let $\mathcal{G} = (G, v_0, \mathsf{Win})$ be a Banach-Mazur game on a countable acyclic graph G that is determined via move-counting strategies. Then $\mathcal{G}$ is already positionally determined.*

Proof. Since G is countable, there exists a bijection $s\colon V \to \omega$. Because G is acyclic and $<$ is a well-order on ω, for any infinite path $\alpha = v_0v_1v_2\cdots$, the sequence

$$s(v_0)s(v_1)s(v_2)\cdots = n_0n_1n_2\cdots$$

has a strictly increasing subsequence. Therefore, the fact that $\mathcal{G}$ is determined via move-counting strategies entails that $\mathcal{G}$ is determined via counting strategies that use as counter values the vertex numbers assigned by s. But any positional strategy can be viewed as a strategy which uses such a counter sequence, hence $\mathcal{G}$ is positionally determined. □

Corollary 4.2.2. *If $\mathcal{W} \subseteq C^\omega$ is a winning condition that guarantees determinacy via move-counting strategies, then it guarantees positional determinacy on countable acyclic graphs.*

For uncountable graphs we cannot argue in the same way, since (when using a well-order on the vertices as a source for the counter) we have to deal with an uncountable amount of vertex numbers which we have to map to the natural numbers. Consequently, the sequence of these natural numbers may be non-increasing. On some acyclic graphs with an uncountable vertex set one can instead use a partition of the vertex set that satisfies two conditions as a source for the counter values.

Definition 4.2.3. Let $G = (V, E)$ be an acyclic graph such that V is uncountable. A partition P of V into countably infinitely many non-empty classes $P = (P_0, P_1, \cdots)$ is called a *partition-ordering*, if it satisfies the following two constraints:

(i) For every $v \in P_i$, there exists a $v' \in vE^*$ such that $v' \in P_j$ for some $j > i$.

(ii) For any $v \in P_i$, no vertex reachable from v is in a lower class:

$$vE^* \cap \bigcup_{j<i} P_j = \emptyset.$$

Theorem 4.2.7. *Let $\mathcal{G} = (G, v_0, \mathsf{Win})$ be a Banach-Mazur game on an acyclic graph G that has a partition-ordering such that $\mathcal{G}$ is determined via move-counting strategies. Then $\mathcal{G}$ is positionally determined.*

Proof. The idea behind the proof of this theorem is that the index of the class of the partition a vertex is a member of can be used as a counter value. Because of property (i) of the partition-ordering, it can always be ensured that a larger counter value is seen later on, and because of property (ii), the counter value cannot decrease.

Formally, let $P = (P_0, P_1, \cdots)$ be a countably infinite partition-ordering on G, and let $p\colon V \to \omega$ be a function that assigns to a vertex the number of the class it is a member of, i.e. $p(v) = i \iff v \in P_i$. Let furthermore $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ be a move-counting winning strategy for the winning Pl. σ. We define a positional strategy $h\colon V \to \mathsf{FinPaths}(G)$ for Pl. σ by

$$h(v) := g(u_0, 0) \cdots g(u_{p(v)}, p(v)) \cdot \vartheta,$$

where $u_0 := v$, $u_{i+1} := \mathsf{last}(g(u_i, i))$ and ϑ is a shortest path starting in $\mathsf{last}(g(u_{p(v)}, p(v)))$ such that $p(\mathsf{last}(\vartheta)) > p(\mathsf{last}(g(u_{p(v)}, p(v))))$.

By a similar argument as in the proof of Lemma 4.2.2, and since the partition-ordering guarantees that the sequence $(p(v))$ resulting from any play has a strictly increasing subsequence, it follows that any play consistent with h is also consistent with g, so h is indeed a positional winning strategy for Pl. σ. □

Corollary 4.2.3. *If $\mathcal{W} \subseteq C^\omega$ is a winning condition that guarantees determinacy via move-counting strategies, then it guarantees positional determinacy on acyclic graphs that have a partition-ordering.*

There is also another class of games for which move-counting strategies can be reduced to positional ones, namely games with a Muller winning condition over a countable set of colors where one of the $\mathcal{F}$-sets is finite and no stable vertices exist. (Notice that in this case, we do not require the graph to be acyclic.) Recall that a vertex v in a Banach-Mazur game with a Muller winning condition is called stable, if the set of colors which are reachable from v is the same for every later vertex, i.e. $C(v) = C(v')$ for all $v' \in vE^*$. Although this notion was primarily introduced for Muller winning condition over a finite set of colors, it can also be applied to winning conditions over an infinite set of colors. Note that an example for a game with such a winning condition with stable vertices has already been given (cf. Example 4.1.1), and it has been explained why positional strategies do not suffice.

Theorem 4.2.8. *Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a Muller winning condition, such that $\mathcal{F}_\sigma$ is finite, for either $\sigma = 0$ or $\sigma = 1$. If no stable vertex v with $C(v) \in \mathcal{F}_\sigma$ is reachable in G, then $\mathcal{G}$ is positionally determined.*

Proof. It has already been shown in Theorem 4.2.2 that, if Pl. $1 - \sigma$ has a winning strategy, he also has a positional one.

Let thus $\mathcal{G}$ be as required, such that Pl. σ has a winning strategy, and assume - in order to simplify the argument - that $\sigma = 0$. (The opposite case where $\sigma = 1$ is analogous.) Let furthermore $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ be a winning strategy of Pl. 0. From Lemma 3.2.4 and Lemma 3.2.2 we know, that there exists an $A \in \mathcal{F}_0$ such that for any vertex $v \in \tilde{v}E^*$ (or $v \in f(v_0)E^*$, respectively, if $\mathcal{F}_0$ is a singleton) it holds that

(i) $A \in C^\infty(v)$, and

(ii) for all $b \notin A$ s.th. $\{b\} \in C^\infty(v)$, there exists a $v' \in vE^*$ with $\{b\} \notin C^\infty(v')$.

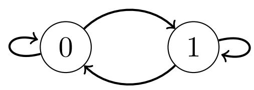
Since no stable vertex v with $C(v) \in \mathcal{F}_0$ is reachable in G, some $b \notin A$ as in *(ii)* always exists. In the move-counting strategy introduced in Theorem 4.2.2, the minimal such b

was chosen in every move, and was made impossible to be seen again. However, it is easy to see that the infinite sequence of these $b_1 < b_2 < \cdots$ can be used as a counter sequence. Since in games like this without stable positions of Pl. 0, such a sequence always exists, no further memory is needed by Lemma 4.2.2. □

Limitations of move-counting strategies

In the previous sections, games not determined via move-counting strategies have already been discussed. In most cases, this was due to the fact that the games were not determined at all. It turns out, however, that there are also very simple games where move-counting strategies do not suffice. A major reason for this can be found in the irrelevance of the decomposition of a prefix into moves for strategies for Banach-Mazur games. A move-counting strategy only has access to information about this decomposition, but not about the underlying prefix. As was stated in Theorem 1.3.3, knowing the decomposition of a prefix into moves does not strengthen a strategy, but gaining access to information about the prefix does.

A simple game that demonstrates this has been introduced in Example 4.1.2. Recall that the objective for Pl. 0 was to have infinitely many prefixes that contain more 1s than 0s on the following arena:



By a similar argument as in the mentioned example, we show that although Pl. 0 has a winning strategy for the game $\mathcal{G}$, he does not have a move-counting one. Therefore assume that Pl. 0 has a move-counting winning strategy g. Let $x_n := |g(0,n)|_1$ be the number of 1s in the n-th move according to g if the game is at vertex 0 in the arena. Pl. 1 can now play with a strategy that guarantees that after the first move of Pl. 1, every prefix always has more 0s than 1s. He can, for example, achieve this by playing $0^{x_{n+1}+|\pi|}$ in his n-th move, where π is the prefix played so far. Every time Pl. 0 gets to move after some prefix, the difference of 0s minus 1s is already larger than the number of 1s he will see in his move according to g. Hence Pl. 1 wins with the described strategy, contradicting that g is a winning strategy.

4.2.2 Length-counting strategies

In the previous section we introduced move-counting strategies and explained why they suffice for a large class of winning conditions, e.g. various interesting classes of Muller winning conditions. However, in the last part of the previous section, limitations concerning these strategies were presented, as they only have access to information about the decomposition of a prefix played so far in a play (i.e. about the number of moves), but not about the prefix itself. We now introduce another class of counting strategies which overcomes this very problem. It will turn out that both classes are equivalent for classical alternating games on graphs, but differ in the setting of Banach-Mazur games, meaning that one kind is strictly stronger than the other.

Definition 4.2.4. A *length-counting strategy* is a function $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ that provides a next move depending only on the current vertex and the length of the prefix played so far. At a vertex v after a prefix π has already been played, the next move according to g is hence determined by $g(v, |\pi|)$.

Given the definition it is obvious that for classical alternating games on graphs, i.e. games where in a single move a player only chooses one edge, move-counting and length-counting strategies are equivalent, since after any prefix, the length of this prefix can easily be computed from the number of moves and vice versa. This is no longer true for Banach-Mazur games. We thus start this section with a theorem stating that any game that is determined via move-counting strategies is also determined via length-counting strategies. We then illustrate that length-counting strategies are more powerful than move-counting ones by providing an example of a game that is determined via length-counting, but not via move-counting strategies. At last, we also present an example of a game that is not determined via length-counting strategies.

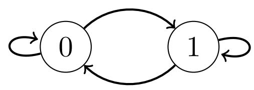
Theorem 4.2.9. *Let $\mathcal{G}$ be a Banach-Mazur game that is determined via move-counting strategies. Then $\mathcal{G}$ is also determined via length-counting strategies.*

Proof. Since the length of a prefix is strictly increasing during a play, the theorem directly follows from Lemma 4.2.2. ☐

Corollary 4.2.4. *Let $\mathcal{W} \subseteq C^\omega$ be a winning condition that guarantees determinacy via move-counting strategies. Then $\mathcal{W}$ also guarantees determinacy via length-counting strategies.*

To see that length-counting strategies are (in a sense) stronger than move-counting strategies, consider the example that has been given in the section on the limitations of move-counting strategies.

Example 4.2.1. Let again $\mathcal{G}$ be the game on the following arena,



where Pl. 0 wins exactly those plays where there exist infinitely many prefixes which contain more 1s than 0s. Recall that although Pl. 0 has a winning strategy, he does not have a move-counting one. It was already hinted to the fact that a major reason for this was that using any move-counting strategy, one does not have any insight on what has been played so far (e.g. the length of the prefix, or how many times a certain color has been seen), only on how this has been done. With a length-counting strategy, this problem does not exist in this form anymore. There in fact exist very simple length-counting winning strategies for Pl. 0 for the above game. Consider for example the length-counting strategy $g\colon V \times \omega \to \mathsf{FinPaths}(G)$ where $g(v, |\pi|) := 1^{|\pi|+1}$. In any play consistent with g it will always be true that after a move of Pl. 0, the prefix played so far has more 1s than 0s, simply because Pl. 0 plays more 1s than the prefix is long, which obviously is an upper bound for the difference of 1s minus 0s.

Information about the length of the current prefix does not always suffice to win. We now give an example in which this is due to the fact that for all vertices, all paths leading there from the start vertex are of the same length, so a length-counting strategy is in no way stronger than a positional one for this game. As it turns out, knowing the current position in the arena is not enough information in this game, but some knowledge about how one got there is necessary to prevent the opponent from being able to win by anticipating one's move.

Example 4.2.2. Consider the following game $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega \cup \{\bot\}, \mathcal{W})$:

$$V := \omega \cup (\omega \times \omega)$$

$$E := \{(n, (n, j)) : n, j \in \omega\} \cup \{((n, j), n + 1) : n, j \in \omega\}$$

$$v_0 := 0$$

$$\Omega(n) = \bot, \Omega(n, j) = j$$

where $\mathcal{W}$ is the set of all words of the form

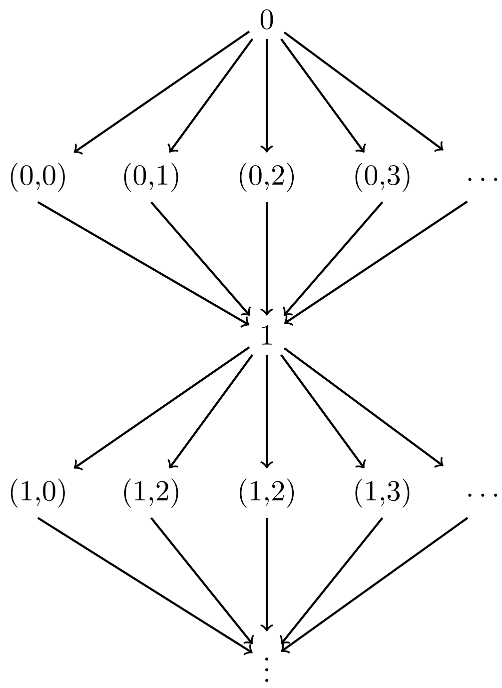
$$\bot \cdot n_0 \cdot \bot \cdot m_1^0 \cdot \bot \cdots m_{k_0}^0 \cdot \bot \cdot n_1 \cdot \bot \cdot m_1^1 \cdot \bot \cdots m_{k_1}^1 \cdot \bot \cdot n_2 \cdots$$

such that

$$n_i = \prod_{j=1}^{k_i} p_j^{m_j^i}, m_{k_i}^i \neq 0$$

for all i, with p_i being the i-th prime number.

In words, Pl. 0 wins exactly those plays in the graph (of which an initial fragment is provided in the figure below) that consist of natural numbers followed by the exponents of their prime factors, omitting the intermediate $\bot$-symbols.



It is obvious that Pl. 1 has a winning strategy for this game, he can in fact win with his first move by either destroying the sequence Pl. 0 might have started or by seeing numbers that do not belong to a valid sequence. Still, he does not have a length-counting strategy, which can be seen by contraposition: assume he has a length-counting winning strategy g. For every n, the vertex n is at distance $2n$ from $v_0 = 0$, regardless of which path was chosen to get there. Thus, at any such vertex n, Pl. 1 will always play $g(n, 2n)$. But then Pl. 0 can move in such a way that he reaches n ending any previous sequence of factors in the pre-previous copy of ω and choosing the one $(n-1, k)$ such that $g(n, 2n)$ consists - neglecting the $\perp$-symbols - of the sequence of exponents of prime factors of k. Pl. 0 wins any such play although it is consistent with g, which contradicts the assumption that g is a winning strategy.

4.3 Strategies using FAR-memory

In the previous section we proposed counting strategies, which are strategies using an infinite memory, but one that is somewhat independent of the actual course of the play. We now revisit the concept of a strategy with memory (as introduced in Section 4.1.2 on finite memory strategies), but this time use infinite memory structures. The memory structures we investigate in the following are Finite Appearance Records (FAR), which were established in [10] by Grädel and Kaiser, motivated by the inapplicability of the Latest Appearance Records (LAR) of Gurevich and Harrington [13] towards Muller games with infinitely many colors.

While LAR-memory is useful for Muller games with finitely many colors, it is of no further use for Banach-Mazur games since LAR structures are always finite and can thus be eliminated in a strategy for Banach-Mazur games. Although Banach-Mazur games with infinitely many colors are no longer always positionally determined, in many cases they still are via simple counting strategies. In this section we establish that they often are also determined via strategies using FAR-memory. We further examine the relation between counting strategies and strategies using FAR-memory and provide examples demonstrating that the classes of winning conditions they guarantee determinacy for are in a sense incomparable.

Definition 4.3.1. A *Finite Appearance Record (FAR)* of dimension d for a set of colors C and a game $\mathcal{G} = (G, v_0, \Omega\colon V \to C, \mathcal{W})$ is a memory structure $\mathfrak{M} = (M, \mathsf{init}, \mathsf{update})$, where $M = (C \cup \Sigma)^d$ is the universe of the structure, for some finite alphabet Σ, $\mathsf{init}\colon \{v_0\} \to M$ is the initialization function and $\mathsf{update}\colon M \times V \to M$ updates a memory state given a vertex, satisfying for any tuples $(m_1, \cdots, m_d), (m'_1, \cdots, m'_d) \in M$ and any vertex v for which

$$\mathsf{update}((m_1, \cdots, m_d), v) = (m'_1, \cdots, m'_d),$$

that for all $1 \leq i \leq d$, the i-th element of the updated tuple meets the condition

$$m'_i \in \{m_1, \cdots, m_d\} \cup \Sigma \cup \{\Omega(v)\}.$$

A *strategy using FAR-memory* is a strategy with a memory $\mathfrak{M}$ for some dimension d.

Muller winning conditions

As FAR-memory was introduced for Muller games, we begin the investigation by examining strategies using FAR-memory for Banach-Mazur games with a Muller winning condition. In [10], the concept of *FAR-reductions* for Muller games was introduced and it was shown that Muller games over a countable set of colors where $\mathcal{F}_0$ is finite reduce via FAR-memory to parity games with colors in ω; hence these games are determined via strategies using FAR-memory. This concept can directly be transferred to Banach-Mazur games and the results about determinacy can be as well. We do not go into detail about this technique here, but rather present new proofs for this and other results, using the fact that in Banach-Mazur games moves are finite paths. As for counting strategies, the first class of Muller winning conditions we consider are those with singleton $\mathcal{F}$-sets. Instead of then extending this result to countable ones, we formulate a theorem that describes a connection between determinacy via move-counting strategies and determinacy via strategies that use FAR-memory.

Theorem 4.3.1. *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over a countable set of colors such that either $\mathcal{F}_0$ or $\mathcal{F}_1$ is a singleton. Then $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via strategies using a 1-dimensional FAR-memory.*

Proof. Let $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ be a Banach-Mazur game with a winning condition such that $\mathcal{F}_0$ is a singleton, i.e. $\mathcal{F}_0 = \{A\}$ for some $A = \{a_0 < a_1 < \cdots\} \subseteq \omega$. (The case where $\mathcal{F}_1$ is a singleton is again symmetric, but for the importance of the first move.) There are again two directions to the proof, namely under the assumption that Pl. σ has a winning strategy, we need to show that he also has one using a 1-dimensional FAR-memory, both for $\sigma = 0$ and $\sigma = 1$.

The simple direction is the one where we assume that Pl. 1 has a winning strategy. It was shown in Theorem 4.2.1 that he then also has a positional one, hence trivially also one using a 1-dimensional FAR-memory.

So assume that Pl. 0 has a winning strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$. We have to construct a 1-dimensional FAR-structure and a winning strategy using this memory. The idea is to remember the largest color from A that has been seen so far (with $\bot < n$ for all $n \in \omega$). Formally, we have $\mathfrak{M} = (M, \mathsf{init}, \mathsf{update})$ with

$$M := A \cup \{\bot\}$$

$$\mathsf{init}\colon v_0 \mapsto \bot$$

$$\mathsf{update}\colon (m, v) \mapsto \begin{cases} \max(m, \Omega(v)) & , \Omega(v) \in A \\ m & , \text{otherwise.} \end{cases}$$

For the strategy $g\colon V \times M \to \mathsf{FinPaths}(G)$ we define

$$g(v_0, \bot) := \begin{cases} f(v_0) & , A \cap \Omega(f(v_0)) \neq \emptyset \\ f(v_0) \cdot \vartheta & , \text{otherwise} \end{cases},$$

where ϑ is a shortest path from $\mathsf{last}(f(v_0))$ such that $A \cap \Omega(\vartheta) \neq \emptyset$. For all v reachable after $f(v_0)$ and all $a_n \in A$ we set

$$g(v, a_n) := \vartheta_1 \cdot \vartheta_2,$$

where ϑ_1 is a shortest path from v such that $\mathsf{first}(A, n+1) \subseteq \Omega(\vartheta_1)$ and ϑ_2 is either an empty path - if $b := \min\{b \notin A : \{b\} \in C^\infty(\mathsf{last}(\vartheta_1))\}$ does not exist - or a shortest path such that $\{b\} \notin C^\infty(\mathsf{last}(\vartheta_2))$; by Lemma 3.2.2 we know that both ϑ_1 and ϑ_2 always exist. For all other v and all $m \in M$ we set $g(v, m)$ to an arbitrary fixed successor.

It remains to show that g is a winning strategy for Pl. 0. Therefore assume that it is not. Since for any consistent play α, we obviously have $A \subseteq \mathsf{Inf}(\alpha)$ because of the ϑ_1-parts, there has to be some $b' \notin A$ with $b' \in \mathsf{Inf}(\alpha)$. But this b', since $<$ is a well-order on ω, has to be less than or equal to the minimal b chosen for the ϑ_2-part in some move. This means that b can no longer be seen infinitely often after the respective move, which contradicts $b \in \mathsf{Inf}(\alpha)$. It follows that $A = \mathsf{Inf}(\alpha)$. $\square$

Notice that the size of the FAR-memory as defined in the proof of the above theorem depends only on the size of the set A. This means that Muller winning conditions such that one of the $\mathcal{F}$-sets is a singleton containing only a finite set guarantee determinacy via strategies using finite memory. Combined with Theorem 4.1.4 about the elimination of finite memory we obtain the following corollary:

Corollary 4.3.1. *Let* $(\mathcal{F}_0, \mathcal{F}_1)$ *be a Muller winning condition over countably many colors such that* $\mathcal{F}_\sigma = \{A\}$ *is a singleton with* $A \subseteq \omega$ *being finite, for* $\sigma = 0$ *or* $\sigma = 1$*. Then* $(\mathcal{F}_0, \mathcal{F}_1)$ *guarantees determinacy via positional strategies.*

After having seen how FAR-strategies for Banach-Mazur games with a Muller winning condition where one of the sets of winning sets is a singleton can be constructed, in a similar way - with the lemmas provided by Section 3.2.2 - strategies for games with a Muller winning condition where one of the $\mathcal{F}$-sets is finite can also be constructed. The same is true if one of the sets is countable, although the construction is more involved in this case. Instead of spelling this out in detail, we prove a theorem stating that for Muller winning conditions over a countable set of colors, move-counting strategies can be simulated by strategies using FAR-memory. We will later see that this is not true in general, i.e. for arbitrary classes of winning conditions.

Theorem 4.3.2. *Let* $\mathcal{G} = (G, v_0, \Omega\colon V \to \omega, (\mathcal{F}_0, \mathcal{F}_1))$ *be a Banach-Mazur game with a Muller winning condition that is determined via move-counting strategies. Then it is also determined via strategies using FAR-memory.*

Proof. The idea of the proof is to use the maximal color seen so far as a counter value (using the natural order on ω). This can, however, not be done in a completely straightforward manner since it may happen that a play reaches a component of the graph that cannot be left again, but only has finitely many colors. To overcome this problem, we exploit that Banach-Mazur games with a Muller winning condition over a finite set of colors are positionally determined (cf. Theorem 4.1.1).

Since there may be many closed subgraphs (i.e. subgraphs that cannot be left again) with only finitely many colors, we have to combine the positional strategies for all of them. Recall that in a finitely colored graph, stable vertices are always reachable. Thus whenever a vertex v is reached such that vE^* is finitely colored, there exists a reachable vertex v' (that is not necessarily different from v) which is stable. From Theorem 4.1.1 we know that at any stable vertex of this kind, any strategy that sees all possible colors in every move is winning for a player, if there exists a winning strategy for the respective player at all. We thus define a function f that assigns to any vertex $v \in V$ with

$C(v) = \{\Omega(u) : u \in vE^*\}$ being finite a shortest path to a closest stable v' (possibly $v = v'$), prolonged with a shortest path from v' on which all colors from $C(v')$ are seen.

Assume now that $\mathcal{G}$ is such that Pl. 0 has move-counting winning strategy $g\colon V \times \omega \to \mathsf{FinPaths}(G)$. Then it holds for all stable v with $|C(v)| < \omega$ which can be reached from $\mathsf{last}(g(v_0, 0))$ that Pl. 0 wins the game $(G|_{vE^*}, v, \Omega|_{vE^*}, (\mathcal{F}_0, \mathcal{F}_1))$. Furthermore, the function f defined above is a winning strategy in the respective game.

Let $\mathfrak{M} = (M := \omega, \mathsf{init}\colon v_0 \mapsto \Omega(v_0), \mathsf{update}\colon m, v \mapsto \max(m, \Omega(v)))$ be a 1-dimensional FAR-memory for $\mathcal{G}$, and let $h\colon V \times M \to \mathsf{FinPaths}(G)$ be defined by

$$h(v,m) := \begin{cases} f(v) & , C(v) \text{ is finite} \\ g(v,0) \cdot g(\_,1) \cdots g(\_,m) \cdot \vartheta & , \text{otherwise} \end{cases}$$

where ϑ is either the empty path (if the memory state after $g(v,0) \cdots g(\_,m)$ is already larger than m or the play has reached a vertex v with $|C(v)| < \omega$), or a shortest path to a closest v' with $\Omega(v') > m$ (which must always exist in this case, since otherwise the play would have reached a finitely colored closed subgraph).

For any play α consistent with h it now holds that

(i) either some stable vertex is reached, hence Pl. 0 wins with his positional strategy there,

(ii) or the play is - by the same argument as in the proof of Lemma 4.2.2 - also consistent with g, and since g is a winning strategy, Pl. 0 wins.

If $\mathcal{G}$ is such that Pl. 1 has a winning strategy, then he has a positional winning strategy from any stable vertex v reachable from v_0. Analogous to the above, one can construct a winning strategy using a 1-dimensional FAR-memory from these positional strategies and the move-counting winning strategy. □

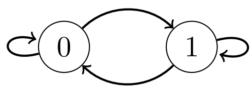
Corollary 4.3.2. *Let $(\mathcal{F}_0, \mathcal{F}_1)$ be a Muller winning condition over a countable set of colors that guarantees determinacy via move-counting strategies. Then it also guarantees determinacy via strategies using FAR-memory.*

Relation to counting strategies

It was shown above that for games with a Muller winning condition over a countable set of colors, move-counting strategies can essentially be converted to strategies using

FAR-memory. This raises the question if such a transformation is possible for all winning conditions. It turns out that this is not the case. There are in fact very simple games which are determined via move-counting strategies, but not via strategies using FAR-memory. One easy way to find such an example is considering sequential winning conditions. For such winning conditions, in contrast to Muller winning conditions, the condition being formulated over a finite set of colors does not guarantee positional determinacy, but it has been shown that many sequential winning conditions guarantee determinacy via move-counting strategies.

Example 4.3.1. Recall the game from Example 3.3.1, $\mathcal{G} = (G, v_0, \Omega\colon v \mapsto v, \mathcal{S} = \{(1^+, \emptyset)\})$ on the arena G:



We already argued in the mentioned example that the game is not positionally determined. Since any FAR-memory for $\mathcal{G}$ is finite (there are only two colors), and finite memory can always be eliminated, the game is also not determined via strategies using FAR-memory. But from Theorem 4.2.4 it follows that $\mathcal{G}$ is determined via move-counting strategies (which can also easily be seen directly).

So far we have seen games determined via move-counting strategies but not via strategies using FAR-memory. Still, move-counting strategies are not stronger than ones using FAR-memory. In fact, there are games which are not even determined via length-counting strategies (for which we already know that they are stronger than move-counting ones), but which allow winning strategies using FAR-memory.

Example 4.3.2. Let $\mathcal{G}$ be the game from Example 4.2.2. Recall that Pl. 1 obviously has a winning strategy for $\mathcal{G}$, but no length-counting one.

Let $\mathfrak{M} = (\omega, \mathsf{init}\colon v_0 \mapsto 0, \mathsf{update}\colon m, v \mapsto \max(m, \Omega(v)))$, with $\perp < n$ for all $n \in \omega$, be a 1-dimensional FAR-memory. Then Pl. 1 has a winning strategy using $\mathfrak{M}$. For reasons of simplicity, we only explain the basic idea of the strategy instead of spelling it out in detail. The strategy uses the fact that any exponent of a prime factor of some number is always smaller than the number itself. If now at any position in the game the highest number seen so far is n and Pl. 1 sees two numbers $n_1 < n_2$ such that $n < n_1$, the resulting infinite play cannot be won by Pl. 0 anymore.

To round things up we present another example demonstrating that there also are games which are determined via length-counting strategies (and not via move-counting ones) for which none of the players has a winning strategy using FAR-memory. The example we use has already been introduced above to establish that length-counting strategies are stronger than move-counting ones.

Example 4.3.3. Let $\mathcal{G}$ be the game from Example 4.2.1, where Pl. 0 wins a play if it contains infinitely many prefixes with more 1s than 0s. Then Pl. 0 has a length-counting, but no move-counting winning strategy.

Since $\mathcal{G}$ is played on the same graph as Example 4.3.1, any FAR-memory for $\mathcal{G}$ is finite (as there are only finitely many colors). But since $\mathcal{G}$ is not determined via move-counting strategies, it is also not positionally determined, thus not determined via strategies using FAR-memory.

Last but not least, there are also games which are determined, but neither via length-counting strategies, nor via strategies using FAR-memory. An example of such a game is obtained from the game in Example 4.3.2 by replacing vertices (n, k) with chains $(n, k, i_i) \to \cdots \to (n, k, i_l)$ such that $i_1 \cdots i_l$ is a binary encoding of k. The coloring function is adjusted in such a way that $\Omega(n, k, i) = i$, i.e. we have $\Omega\colon V \to \{0, 1, \bot\}$, and the winning condition is rewritten, replacing every number in a winning word by its binary encoding. With the same argument as in Example 4.2.2, the game is not determined via length-counting strategies. Since it is not positionally determined and the number of colors is finite, it follows that it also cannot be determined via strategies using FAR-memory.

Chapter 5

Bounded strategies

A great amount of the strength of the strategies introduced in the previous chapter - and in fact of all strategies for Banach-Mazur games - is based on the fact that moves of unbounded, increasing length can be provided. Especially in regard of possible applications of Banach-Mazur games, this might not always seem realistic (recall that the boundedness of a winning strategy was for example used in the main result from Chapter 2) and the question arises whether the strength of a class of strategies is limited when a restriction on move lengths is enforced.

In this chapter, we propose two variants of restricting the length of moves, on the one hand bounding from above the length of a move by a function dependent on the duration of the play, and on the other hand by a function that assigns a bound to every vertex. We discuss both types, and show that different definitions of the way a function is restricted lead to different limitations. Furthermore, the actual arena of a game is important, as for the same winning condition a game on one graph might be determined, while one on another graph is not.

Before introducing the two different approaches, we give a general definition of what we mean by bounded strategies.

Definition 5.0.2. A strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ for a Banach-Mazur game is called *bounded*, if the length of its moves is bounded from above by a function $l\colon \mathfrak{P} \to \omega$, where $\mathfrak{P} = \{p_i : i \in I\}$ is a partition of the set of finite paths in G that start in v_0, i.e. for each finite path $\pi = v_0 \cdots v_n \in p_i$ it holds that $|f(\pi)| < l(p_i)$.

For simplicity, we used decomposition invariant strategies in the above definition, but of course the definition can be adapted to other types, e.g. bounded move-counting strategies or positional strategies that are bounded.

This chapter is split into two parts, as both approaches towards restricting lengths of moves are investigated separately. For both approaches, we give a precise definition of how the strategies are bounded, and then examine the effects such bounds have. Furthermore, we propose bounded Banach-Mazur games, i.e. games where both players independently choose their respective bounding function before a regular Banach-Mazur game commences (in which both players are restricted to strategies consistent with their chosen bounds). The independent choice can be understood as imperfect information, and it turns out that this element of imperfect information is already sufficient to forbid simply concluding that games are determined based on properties of winning conditions. (We present examples of games with a simple Muller winning condition that are no longer determined if the game is changed to a bounded Banach-Mazur games.) Furthermore, we provide examples from which it follows that the two different approaches lead to different limitations, thus both are worth exploring.

5.1 $\mathcal{L}$-bounded strategies

We begin the discussion of bounded strategies with strategies where the length of a next move is bounded by a function that uses the length of the prefix played so far as an input. Formally, there are two equivalent characterizations of such strategies:

Definition 5.1.1. A strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ is $\mathcal{L}$-bounded, if for all $n \in \omega$, the set

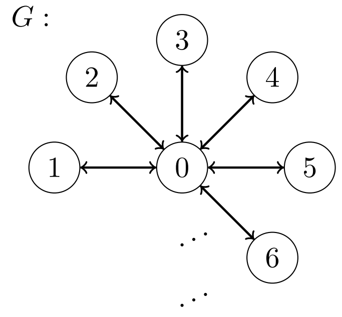
$$\{|f(\pi)| : |\pi| = n, \pi \in \mathsf{FinPaths}(G, v_0)\}$$

has a maximum, or equivalently, if there exists a function $l\colon \omega \to \omega \setminus \{0\}$ such that

$$|f(\pi)| < l(|\pi|)$$

for all finite paths π that start in v_0. If such a function l is given for a strategy f, the strategy f is called l-bounded.

To require a strategy to be bounded in the above sense does not always have significant impact. In fact, it is still possible that the length of the moves according to the strategy increases during the play, i.e. the strategy can be bounded while simultaneously providing longer and longer moves. (Obviously, this depends on the function l, e.g. if l was constant, this would not be true.) To see that the restriction is not strict for all games, recall the game from Example 4.1.1, i.e. the game on the arena G as below, which is won by Pl. 0 if every $n \in \omega$ is seen infinitely often.



As was explained earlier, Pl. 0 has a winning strategy for the above game, namely see larger and larger initial subsets $\{0, \cdots, n\}$ of ω in every move. This, however, can be done with an $\mathcal{L}$-bounded strategy, e.g. for the function $l\colon n \mapsto n+1$. (Note that a constant l is also sufficient, which is explained in more detail in the next section on $\mathcal{V}$-bounded strategies.)

This raises the question of whether there are functions l for which restricting players to such l-bounded strategies never results in a true restriction. We thus examine possible bound-functions in more detail, and it turns out that no l is "sufficient", i.e. no l exists that suffices for all Banach-Mazur games where a player has an $\mathcal{L}$-bounded winning strategy. To see this we construct, for each possible l, a game $\mathcal{G}_l$, where Pl. 0 has an $\mathcal{L}$-bounded winning strategy, but none that is bounded by l.

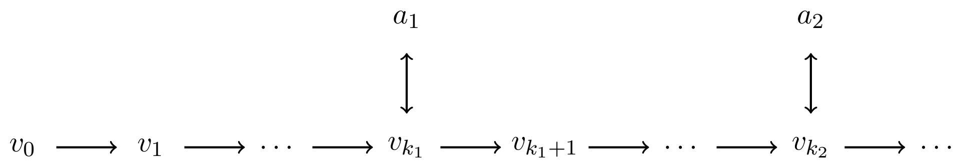
Theorem 5.1.1. *For every function $l\colon \omega \to \omega \setminus \{0\}$ there exists a game $\mathcal{G}_l$ such that Pl. 0 has a bounded winning strategy, but no l-bounded one.*

Proof. Let $l\colon \omega \to \omega \setminus \{0\}$ be an arbitrary function. We construct a Banach-Mazur game $\mathcal{G}_l$ with a Muller winning condition over a finite set of colors, and do so in such a way that - although he could win with a bounded winning strategy - Pl. 0 cannot see the color he needs to see infinitely often in any move because of the path-length restrictions due to l. Let $\mathcal{G}_l := (G, v_0, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$ be the game with

- $G = (V, E)$, where

$$V := \{a_1, a_2, a_3, \cdots\} \cup \{v_0, v_1, v_2, \cdots\}$$

  and E as indicated in the picture below, such that $k_1 = l(|v_0|) + 1$ and $k_{j+1} = l(|v_0 v_1 \cdots v_{k_j+1}|) + 1$.



- $\Omega\colon V \to \{0,1\}$ with $\Omega(v_i) = 0$, $\Omega(a_i) = 1$.
- $\mathcal{F}_0 := \{\{1\},\{0,1\}\}, \mathcal{F}_1 = \mathcal{P}(\{0,1\}) \setminus \mathcal{F}_0$.

Pl. 0 does not have an l-bounded winning strategy, since in his first move he cannot reach a_1, and Pl. 1 can then directly move to v_{k_1+1}. Afterwards, a_1 is not reachable anymore and Pl. 0 cannot directly reach a_2. If Pl. 1 continues like that, he is able to prevent any a-vertex from being seen, hence he wins the corresponding play. However, Pl. 0 has an $\mathcal{L}$-bounded winning strategy, even a positional one, but for an l' different from l that allows longer moves. □

The game in the above proof can also be understood as a reachability game. In fact, Pl. 0 cannot win with an l-bounded strategy if his objective is to see color 1 at least once. Since reachability and Muller winning conditions are in a sense winning conditions of a very basic kind, this is a strong argument towards the non-existence of a single function l that suffices to capture $\mathcal{L}$-bounded strategies for any class of winning conditions, i.e. that no function l exists such that each $\mathcal{L}$-bounded strategy for a game with a certain winning condition can be transformed into an l-bounded one.

In the above game $\mathcal{G}_l$, if Pl. 0 is restricted to l-bounded strategies, the outcome of the game is changed, i.e. Pl. 1 now has a winning strategy (even an l-bounded one). This can be generalized, as it turns out that not only for every function l there exists a game which is no more determined for the same player, if he is restricted to such bounded functions, but there also is a game for which Pl. σ does not have an $\mathcal{L}$-bounded winning strategy for any function l, despite having a general winning strategy. Thus the following theorem implies that the restriction to $\mathcal{L}$-bounded strategies is a true restriction, as, essentially, the underlying game is changed.

Theorem 5.1.2. *There exists a Banach-Mazur game for which Pl. σ has a winning strategy, but if he is restricted to $\mathcal{L}$-bounded strategies, the opponent can win.*

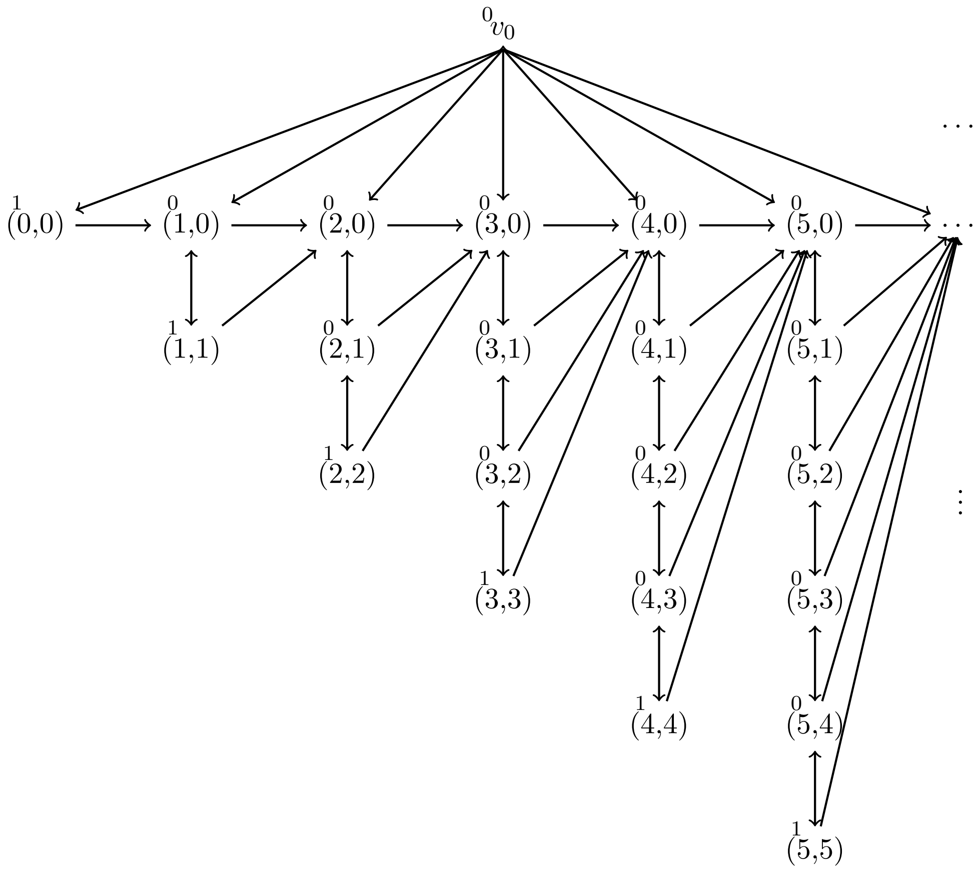
Proof. We define a game with a simple Muller winning condition over a finite set of colors for which Pl. 1 has a winning strategy, but no bounded one for any l. Let thus $\mathcal{G} = (G, v_0, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$ be the following game:

- $G = (V, E)$ with

$$V := \{v_0\} \cup \{(n,i) : n \in \omega, i \leq n\}$$

$$\begin{aligned} E := &\{(v_0,(n,0)) : n \in \omega\} && (=: E_1)\\ &\cup \{((n,i),(n,i+1)) : n \in \omega, i < n\} && (=: E_2)\\ &\cup \{((n,i),(n,i-1)) : n \in \omega, 0 < i \leq n\} && (=: E_3)\\ &\cup \{((n,i),(m,0)) : n \in \omega, i \leq n, n < m\} && (=: E_4)\end{aligned}$$

  Notice that in the picture below not all edges from E_4 are drawn in order to assure better readability.



- $\Omega\colon v_0 \mapsto 0, (n,j) \mapsto \begin{cases} 0 & ,j < n \\ 1 & ,j = n \end{cases}$
- $\mathcal{F}_0 := \{\emptyset, \{0\}\}$, $\mathcal{F}_1 := \{\{1\}, \{0,1\}\}$

Pl. 1 wins the above game $\mathcal{G}$ if he manages to see color 1 infinitely often. Obviously, this can even be done with a positional winning strategy, since Pl. 1 wins if, at any vertex $(n,i), i < n$, he moves to (n,n), and arbitrarily otherwise. To see that he does not have an $\mathcal{L}$-bounded winning strategy, assume the contrary, i.e. that $f\colon \mathsf{FinPaths}(G, v_0,) \to \mathsf{FinPaths}(G)$ is an l-bounded winning strategy for Pl. 1, for some function $l\colon \omega \to \omega \setminus \{0\}$. In the following, we describe a strategy for Pl. 0 with which he wins although the resulting play is consistent with f, contradicting that f is a winning strategy of Pl. 1.

This strategy of Pl. 0 works in such a way that, after a prefix $\pi = v_0 \cdots (m,k)$ has been played so far, Pl. 0 moves to $(n,0)$, where

$$n := \max\{l(|\pi| + 1) + 1, m + 1\}.$$

(Notice that Pl. 0 can reach $(n,0)$ using only a single edge.) In any play consistent with this behavior and in which Pl. 1 uses an l-bounded strategy, Pl. 1 cannot reach a vertex with color 1 in any of his moves. Thus there exist plays which are consistent with f, but in which 1 is not seen infinitely often, hence f cannot be a winning strategy, and furthermore, no l-bounded strategy can be winning for Pl. 1, for any function l.

It follows that Pl. 1 does not have an $\mathcal{L}$-bounded winning strategy, in spite of having an unbounded positional winning strategy. □

5.1.1 $\mathcal{L}$-bounded Banach-Mazur games

So far it was shown that the winner of a Banach-Mazur game possibly changes if one of the players is restricted to $\mathcal{L}$-bounded strategies. In the following we argue that if both players are restricted to such bounded strategies - meaning that before the actual play begins, both players independently choose their respective l_σ - the game might not even be determined anymore, i.e. no player has a bounded winning strategy with which he wins against all bounded strategies of his opponent. This can be explained by looking at the reduction of such games to standard games on graphs, through which one learns that if both players are restricted to the use of $\mathcal{L}$-bounded strategies, there is no single such reduction, but in fact for every pair of functions (l_0, l_1) such that Pl. σ uses an

l_σ-bounded strategy, the game is - after the reduction - a different one. In essence, if for a game $\mathcal{G}$ both players are restricted to $\mathcal{L}$-bounded strategies, there is no longer a single game $\mathcal{G}$, but a family of games $\{\mathcal{G}(l_0, l_1)\}$. If $\mathcal{G}$ has a winning condition that guarantees determinacy in the game without restrictions, every $\mathcal{G}(l_0, l_1)$ is determined as well, but no player has a strategy that is winning for the complete family. Essentially, this is due to the introduced imperfect information, as the players have no information on the opponent's choice of a bound prior to their own decision.

However, notice first that the strategy of Pl. 0 in the proof of the above Theorem 5.1.2 depends on Pl. 1's function l, but is itself bounded by $l' \colon n \mapsto 1$. It is evident that if both Pl. 0 and Pl. 1 are restricted to use $\mathcal{L}$-bounded strategies, then Pl. 0 can win every instance of the game where the pair of functions (l_0, l_1) has been fixed (with $l_0 = l'$), but he does not have a general winning strategy. In other words, for every function l that bounds Pl. 1's moves, Pl. 0 has an l'-bounded strategy that is winning for him, but he does not have a winning strategy that is winning regardless of which l is chosen by Pl. 1. Although no winning strategy exists, the $\mathcal{L}$-bounded game is still in a way determined, as the same player can always win (using a simple description of what his strategy should be like, i.e. an algorithm that constructs an actual winning strategy respecting his own bound as soon as the opponent's bound is known). Before we present a theorem expressing that this is not necessarily true in general, we formally introduce $\mathcal{L}$-bounded games.

Definition 5.1.2. An *$\mathcal{L}$-bounded Banach-Mazur game* $\mathcal{G}_\mathcal{L}$ is a Banach-Mazur game $\mathcal{G}$ in which both players are restricted to $\mathcal{L}$-bounded strategies. A play consists of two steps: both players independently choose a respective function $l_\sigma \colon \omega \to \omega \setminus \{0\}$, and then the Banach-Mazur game $\mathcal{G}$ is played, such that Pl. σ is only allowed to use l_σ-bounded strategies (for the l_σ he has chosen before).

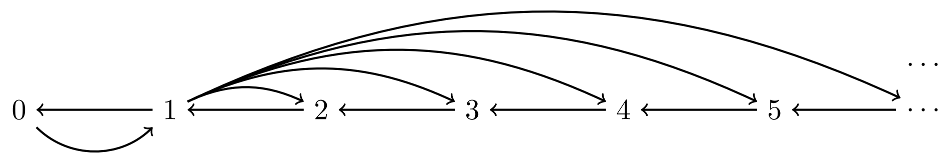
We say that a game $\mathcal{G}_\mathcal{L}$ is determined if one of the players can choose a function l such that for any l' the opponent may have chosen, he has an l-bounded winning strategy $f_{l'}$.

In the above sense, the game introduced in the previous proof is determined, as such a function l exists for Pl. 0. As already said, not all $\mathcal{L}$-bounded Banach-Mazur games are determined, even if the underlying unbounded Banach-Mazur game is. The idea of the proof of the next theorem is to construct a game in which it is true that for every player and every function l he might have chosen, there exists a function l' such that he cannot win against one of the opponent's l'-bounded strategies.

Theorem 5.1.3. *There exists a determined Banach-Mazur game such that the $\mathcal{L}$-bounded variant of this game is not determined.*

Proof. We begin the proof with introducing the game $\mathcal{G}$. We then show that, for all $\sigma \in \{0,1\}$ and every function l_σ, there exists a function $l_{1-\sigma}$ which is sufficient for Pl. $1-\sigma$ to win against all of Pl. $\sigma's$ l_σ-bounded strategies.

Let $\mathcal{G} := (G, 1, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$ be the Banach-Mazur game on the arena



$$G := \left(\begin{array}{l} V := \omega \\ E := \{(0,1)\} \cup \{(1,n) : n > 1\} \cup \{(n, n-1) : n > 0\} \end{array} \right),$$

with the coloring function

$$\Omega \colon v \mapsto \begin{cases} 0 & , v = 0 \\ 1 & , \text{otherwise} \end{cases}$$

and a Muller winning condition ($\mathcal{F}_0 := \{\emptyset, \{1\}\}, \mathcal{F}_1 := \{\{0\}, \{0,1\}\}$).

As all Banach-Mazur games with a Muller winning condition over a finite set of colors are determined, so is the above one. It is in fact positionally determined, with the winning strategy of Pl. 1 being - at a vertex $n > 0$ - to play the path $n-1 \cdots 0$, and to play $1 \cdot 0$ at vertex 0.

However, for any function l_1 such that Pl. 1 is restricted to l_1-bounded strategies, Pl. 0 has an $\mathcal{L}$-bounded winning strategy. Therefore let $l_1 \colon \omega \to \omega \setminus \{0\}$ be an arbitrary function. The idea of the strategy for Pl. 0 is to use edges of the kind $(1,n)$ to move so far away from vertex 1 that Pl. 1 cannot reach it in his next move. This strategy $f \colon \mathsf{FinPaths}(G,1) \to \mathsf{FinPaths}(G)$ can formally be described by

$$f(\pi = 1 \cdot n_1 \cdot n_2 \cdots n_m) := n_m - 1 \cdot n_m - 2 \cdot \cdots \cdot 1 \cdot n',$$

where $n' = l_1(|\pi| + n_m) + 1$. If Pl. 0 uses the above strategy, then Pl. 1 can - in every move - only reach vertices $1, 2, \cdots, n'$ for the respective n'. This means that he is never able to reach vertex 0 (which he needs to see infinitely often in order to win), but Pl. 0

can always move to an n' as required. The strategy f is hence winning for Pl. 0 against all l_1-bounded strategies of Pl. 1. It is also $\mathcal{L}$-bounded, e.g. by a function

$$l_0\colon n \mapsto \max\{l_1(i) : i \leq n\} + 2,$$

since Pl. 0 makes sure with his strategy that no vertex further away than this could be reached so far, thus he can always return to vertex 1 and choose one additional edge to some new n'.

To complete the proof, it remains to show that, for any function l_0 that Pl. 0 is restricted to, Pl. 1 can win by an $\mathcal{L}$-bounded strategy. Let $l_0\colon \omega \to \omega\backslash\{0\}$ be an arbitrary function. Since Pl. 0 starts the play, we cannot argue in the same straightforward manner as above, because simply increasing the bound for Pl. 1 could be faced with Pl. 0 choosing larger n to move to from vertex 1. Still, Pl. 1 can choose a function l_1 that allows moves significantly longer than Pl. 0's, so that Pl. 1 may not be able to reach vertex 0 in a single move, but Pl. 0 cannot reach vertex 1 in his next move either. To simplify things, we set l_1 to $l_1(n) := 2^m + 1$, where $m := \max\{l_0(j) : 0 \leq j \leq n+1\}$. By this definition, it is always guaranteed that Pl. 1 can choose a significantly longer finite path than Pl. 0 can choose in his next move. We argue that the only way for Pl. 0 to win the above game is to choose n that are so far away from vertex 0 that Pl. 1 cannot reach 0 in his following move. But then n is also so far away from vertex 1 that Pl. 1 can make sure with his next move that Pl. 0 cannot choose a new distant n. In fact, he only needs to make sure that the length of the remaining path to 1 is larger than the maximal length of Pl. 0's next move. By our choice of l_1, this is always possible (with e.g. a move of length 1). It follows that eventually it falls to Pl. 1 to be able to reach 0, and possibly also to go back to 1 and choose a new distant n. Once this has happened, he will proceed analogously to the way Pl. 0 did in the other case, hence he is able to see 0 in every later move.

The above arguments entail that $\mathcal{G}_\mathcal{L}$ is indeed not determined, while $\mathcal{G}$ is. (Notice that although $\mathcal{G}_\mathcal{L}$ is not determined, this is solely due to the independent choice of the l-functions. As soon as both l_0 and l_1 are fixed, the game is determined.) □

We conclude that although sometimes using $\mathcal{L}$-bounded strategies is not a restriction, it usually is, and in fact one that can have major effects on the outcome of the game. Furthermore, the existence of winning strategies is no longer a sensible universal characterization of determinacy for such bounded games, as sometimes they would not only have to depend on the arena and the winning condition, but also on the bounding function of the opponent. Additionally, when introducing independent choices of the respective bounding function, determinacy even in the sense proposed above is sometimes lost, which raises the question whether additional restrictions, e.g. on the arena, exist such that determinacy is regained.

5.2 $\mathcal{V}$-bounded strategies

We continue the examination of bounded strategies by proposing another approach towards restricting move lengths, namely to restrict the length based on the current position in the arena. In other words, for every vertex there exists a certain number n, such that all moves the strategy produces for prefixes ending there have a length of less than n. When formulated precisely, the definition of this new variant of bounded strategies only slightly differs from the one in the previous section, but nevertheless, the two types of strategies behave differently in many games.

Definition 5.2.1. A strategy $f\colon \mathsf{FinPaths}(G, v_0) \to \mathsf{FinPaths}(G)$ is $\mathcal{V}$-bounded, if for all $v \in V$ the set

$$\{|f(\pi)| : \pi = v_0 \cdots v \in \mathsf{FinPaths}(G, v_0)\}$$

has a maximum, or again equivalently, if there exists a function $d\colon V \to \omega$ such that for all vertices v,

$$|f(v_0 \cdots v)| < d(v)$$

for all finite paths $v_0 \cdots v$ that end in v.

Although at first this seems to be a very strong restriction especially on cyclic graphs, this does not necessarily have to be true, but depends once more on the actual game. As an example, recall the game from Example 4.1.1 (which was reintroduced in the previous section). It was already explained why $\mathcal{L}$-bounded strategies suffice for this game, and that in fact a constant l would do. A $\mathcal{V}$-bounded strategy somewhat corresponds to such an $\mathcal{L}$-bounded strategy for a constant l, since throughout a whole play the maximal length of a move starting in vertex 0 does not increase. Still, Pl. 0 has a $\mathcal{V}$-bounded winning strategy: the idea of this strategy is to see certain intervals of a fixed size. In order to do this, consider the following enumeration s of intervals of natural numbers of size k:

n	0	1	2	3	4	5	$\cdots$
s(n)	$[1,k]$	$[2,k+1]$	$[1,k]$	$[2,k+1]$	$[3,k+2]$	$[1,k]$	$\cdots$

Using this enumeration, we can easily define even a move-counting winning strategy of Pl. 0 (for any fixed $k > 0$), which will turn out to be $\mathcal{V}$-bounded. For every $v \in V, n \in \omega$, we set

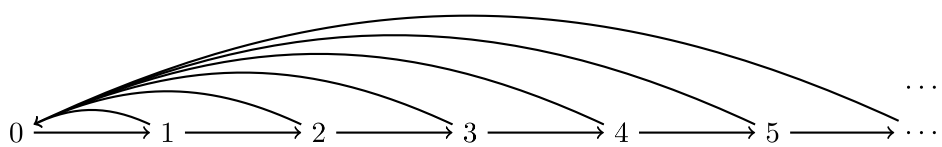
$$g(v,n) := \begin{cases} s_1 \cdot 0 \cdot s_1 + 1 \cdot 0 \cdots 0 \cdot s_2 & , v = 0 \\ 0 \cdot s_1 \cdot 0 \cdot s_1 + 1 \cdot 0 \cdots 0 \cdot s_2 & , \text{otherwise} \end{cases}$$

where $[s_1, s_2] := s(n)$. It can be seen directly that this strategy g is $\mathcal{V}$-bounded by the function $d\colon v \mapsto 2k+1$.

However, this does not generalize, and restricting one of the players to $\mathcal{V}$-bounded strategies might produce similar results as restricting him to $\mathcal{L}$-bounded strategies (although possibly in different games).

Theorem 5.2.1. *There exists a Banach-Mazur game for which one of the players has an unbounded winning strategy as well as an $\mathcal{L}$-bounded one, but no $\mathcal{V}$-bounded one, i.e. the opponent can win against all $\mathcal{V}$-bounded strategies.*

Proof. A game as required can be found by altering the game discussed above. Formally, we set $\mathcal{G} := (G, v_0, \Omega, (\mathcal{F}_0, \mathcal{F}_1))$, where



$$G := (\omega, \{(n,0) : n > 0\} \cup \{(n-1,n) : n > 0\})$$

$$\Omega\colon v \mapsto v$$

$$\mathcal{F}_0 := \omega, \mathcal{F}_1 := \mathcal{P}(\omega) \setminus \{\omega\}$$

By similar arguments as in previous sections, Pl. 0 has an $\mathcal{L}$-bounded winning strategy for $\mathcal{G}$. But he does not have a $\mathcal{V}$-bounded one, as $d(0) = n$ for one fixed n, which means that whenever Pl. 0 gets to move at vertex 0, he can see at most the numbers $0, \cdots, n$. Pl. 1 thus obviously wins with the positional $\mathcal{V}$-bounded strategy

$$f : v \mapsto \begin{cases} 0 & , v > 0 \\ 1 \cdot 0 & , v = 0 \end{cases}$$

□

For other winning conditions it becomes even more obvious that $\mathcal{V}$-bounded strategies are a true restriction, i.e. that such strategies often do not suffice for Banach-Mazur games. For instance, recall the game with a sequential winning condition from Example 3.3.1, where the objective for Pl. 0 was to see all of 1^+ infinitely often on the completely connected graph with vertices 0 and 1. As there are only finitely many vertices, it follows immediately that for every $\mathcal{V}$-bounded strategy there exists a number n such that Pl. 0 cannot play paths of a length greater than n. The only way for Pl. 0 to win the game would thus be that Pl. 1 helps him play all these sequences 1^m where $m > n$. Obviously, Pl. 1 would never do that, hence he this time even has an overall positional $\mathcal{V}$-bounded winning strategy (i.e. one that is winning against all $\mathcal{V}$-bounded strategies of Pl. 0), namely $f : v \mapsto 0$.

Though it may look like $\mathcal{V}$-bounded strategies are weaker than $\mathcal{L}$-bounded strategies, this is not true. They are rather thoroughly different. To see this more clearly, we revisit the games in the proofs of the section on $\mathcal{L}$-bounded strategies, and it turns out that, despite not allowing $\mathcal{L}$-bounded winning strategies, they do allow $\mathcal{V}$-bounded ones, even without a change in the winner. We begin with the game from the proof of Theorem 5.1.1. Recall that the game was essentially a line where from time to time it was possible to move left and return. Notice that Pl. 0 would win every play in which the step to the left was taken infinitely many times. Of course this can be done with a $\mathcal{V}$-bounded strategy, as the bound can, for every vertex v, be set to be slightly longer than the distance to the next possible vertex one can move left from.

For the game from the proof of Theorem 5.1.2, this is even more straightforward, as at any vertex (n, i) the next vertex with color 1 is (n, n), thus at a distance of no more than n. Obviously, since he wins if he reaches such a vertex infinitely many times, Pl. 1 can win with a $\mathcal{V}$-bounded strategy, e.g. for the function $d : (n, i) \mapsto n + 1$.

At last, for the game from Theorem 5.1.3 - which was not determined as an $\mathcal{L}$-bounded game - Pl. 1 again has a simple winning strategy that is $\mathcal{V}$-bounded. We simply set $d : v \mapsto v+1$, and then define a positional strategy bounded by this d by $f(v) := v-1 \cdots 0$.

With this strategy, Pl. 1 makes sure that vertex 0 is visited in every one of his moves, hence seen infinitely often.

5.2.1 $\mathcal{V}$-bounded Banach-Mazur games

In a similar way as $\mathcal{L}$-bounded Banach-Mazur games were defined, we do the same for $\mathcal{V}$-bounded ones. With the definition we then examine whether the imperfect information causes some kind of nondeterminism or not. It turns out that nondetermined games exist (with a similar notion of determinacy as for $\mathcal{L}$-bounded ones).

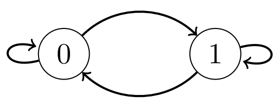
Definition 5.2.2. A $\mathcal{V}$-bounded Banach-Mazur game $\mathcal{G}_{\mathcal{V}}$ is a Banach-Mazur game $\mathcal{G}$, in which both players - before the game starts - independently choose functions d_σ, such that later they are allowed to only use $\mathcal{V}$-bounded strategies, bounded by their respective d_σ. Again, a game $\mathcal{G}_{\mathcal{V}}$ is determined whenever one of the players can choose a function d such that he can win with d-bounded strategies regardless of his opponent's choice.

To show that not all $\mathcal{L}$-bounded Banach-Mazur games are determined, we constructed a game in which the winner depends on the choice of the functions l_0 and l_1. We pursue a similar course here, only that for $\mathcal{V}$-bounded strategies it is no longer of use to construct large graphs where the required length of a move depends on the vertex. Thus it suffices to concentrate on small graphs, and we use sequential winning conditions, as a game where a player was not able to preserve his winning strategy after being restricted to $\mathcal{V}$-bounded strategies was already described.

Theorem 5.2.2. *There exists a determined Banach-Mazur game such that the corresponding $\mathcal{V}$-bounded one is not determined.*

Proof. The proof is by construction of a game as required. We use the same scheme as in the proof of Theorem 5.1.3, i.e. present the game, show that it is determined, and show that for every d_σ there exists a $d_{1-\sigma}$ against which Pl. σ looses when having to use strategies bounded by d_σ.

Let $\mathcal{G} := (G, v_0, \Omega, \mathcal{S})$ be as follows:



$G := (\{0,1\}, \{0,1\} \times \{0,1\})$

$$v_0 := 0$$

$$\Omega\colon v \mapsto v$$

$$\mathcal{S} := \{(\{1^n\}, \{0^m : m > n\}) : n \in \omega\}$$

As the game has a sequential winning condition over a finite set of colors, it is determined. In fact, Pl. 1 has a winning strategy, he wins if he always makes sure that he sees a word 0^n for some n that is larger than the length of the longest 1-word seen so far. If played like that, for every 1^m seen infinitely often, there exists some m' such that $0^{m'}$ is also seen infinitely often.

To show that the corresponding game $\mathcal{G}_\mathcal{V}$ is not determined, we start with assuming that there exists a function d_0 such that Pl. 0 can always win with a strategy bounded by d_0, regardless of the actual d_1. However, it can easily be seen that this cannot be true. Let $m := \max\{d_0(0), d_0(1)\}$. This means that Pl. 0 is unable to play moves of a length greater than m. Consider now $d_1\colon v \mapsto 2m+1$. Then Pl. 1 can see 0^{2m} in every move, hence Pl. 0 cannot win with a strategy bounded by d_0, because he has to make sure that some 1^n for $n > 2m$ is seen infinitely often.

By a similar argument, no d_1 can exist with which Pl. 1 can win against all $\mathcal{V}$-bounded strategies of Pl. 0. The idea is again to consider some d_0 which allows moves that are longer than the longest possible move of Pl. 1, and then see the longest possible 1-word in every move. □

Interestingly, if we consider the corresponding $\mathcal{L}$-bounded game, it can easily be seen that it is determined. In fact, if we set $l\colon n \mapsto n+1$, the strategy for Pl. 1 described for the unrestricted game is already l-bounded. This shows that $\mathcal{V}$-bounded games are different from $\mathcal{L}$-bounded games, both are different from unrestricted Banach-Mazur games, and the respective strategies are different as well.

What remain open is whether determined $\mathcal{V}$-bounded Banach-Mazur games allow winning strategies. For the examples of determined games considered above, there always exists a winning strategy that is $\mathcal{V}$-bounded, but whether this is true in general is not yet clear, especially since it is not true for $\mathcal{L}$-bounded games.

Conclusion

In this work we discussed Banach-Mazur games on graphs. Despite having been studied in the field of topology for half a century, it was only recently that Banach-Mazur games were reintroduced as a special variant of games on graphs [5, 9, 19]. It turned out that the natural idea of a path game on a graph and the notion of Banach-Mazur games on the natural topology on infinite plays in such games coincide, leaving topological Banach-Mazur games as an excellent starting point for the exploration of properties of such path games. Furthermore, as Banach-Mazur games on graphs are a special case of their topological variant, known results, e.g. the famous Banach-Mazur theorem [17], can easily be transferred and expressed in graph terms.

After briefly examining various results carried over from topology, this thesis concentrated on an investigation of what kinds of simple winning strategies prove useful, and which classes of winning conditions guarantee determinacy via such strategies. In order to simplify and organize this discussion, we started with examining different classes of winning conditions, namely - as such winning conditions are of great importance in Computer Science - Muller winning conditions over finite and countably infinite sets of colors, as well as a generalization to so-called sequential winning conditions requiring words of colors to be seen infinitely often instead of just colors. As it was already known that Muller winning conditions over a finite set of colors guarantee positional determinacy [5], the emphasis in this work was laid upon winning conditions over a countably infinite set of colors. We were able to show in Chapter 3.2.2 and 3.3 that for several classes of winning conditions of this kind, the existence of winning strategies can be related to the existence of vertices from where on certain colors or sequences of colors can always be seen, while others can be prevented from being seen again.

Having obtained properties of the above sort for these winning conditions, we focused the investigation on different classes of strategies. Since strategies without or with only finite memory are equivalent [9], and despite working well for e.g. Muller winning conditions over a finite set of colors, they are not strong enough for even simple games in which the arena is colored with infinitely many colors, the major part of Chapter 4 was dedicated to strategies using infinite memory. Of such strategies we closely examined two different variants.

The first class of strategies using infinite memory discussed was the class of counting strategies (Chapter 4.2), of which two different subclasses were analyzed - one where the number of moves is counted, and one where the length of the path created so far is. We were able to show that move-counting strategies cover many interesting classes of winning conditions, e.g. Muller winning conditions over a countably infinite set of colors with a countable $\mathcal{F}_\sigma$ guarantee determinacy via move-counting strategies, and so do many sequential winning conditions over a countable set of colors. However, move-counting strategies are not strong enough for even simple non-prefix independent winning conditions, for which in turn length-counting strategies sometimes suffice. Although they are strong enough for some non-prefix independent winning conditions, length-counting strategies also fail for certain simple Banach-Mazur games with such winning conditions.

The next class of strategies with infinite memory - treated in Chapter 4.3 - was the class of strategies using FAR-memory. We provided new proofs showing that certain classes of Muller winning conditions over a countably infinite set of colors guarantee determinacy via strategies using FAR-memory in the setting of Banach-Mazur games on graphs, and also began to take a closer look on the relation of such strategies towards counting strategies. We found that the two are in a sense incomparable, as there are games determined via the one class of strategies, but not via the other, and vice versa. However, for certain winning conditions and on certain graphs, move-counting strategies can be simulated by strategies using FAR-memory.

In the last chapter, we introduced the notion of a bounded strategy, i.e. a strategy where the maximal length of a move is restricted in some way. Two approaches to such restrictions were presented, one basing the restriction on the duration of the play, while in the other one it was based on the current position in the graph. Regardless of which variant was used, restricting players results in a change of the winner in some games. We then introduced the notion of a bounded Banach-Mazur game, in which the players independently select their bounding function before the actual Banach-Mazur game commences. This element of imperfect information has grave impact on the outcome of some games, as e.g. there not necessarily have to be winning strategies in the usual sense anymore, or determinacy is lost at all (meaning that the outcome depends on the choice of bounding functions, but no player can safely know which one to choose).

Future work

A large part of this work was dedicated to examining counting strategies. Although several winning conditions for which they work well were discovered, we lack a precise characterization of the classes of winning strategies that guarantee determinacy via different variants of counting strategies, including strict upper bounds. For move-counting strategies, it is furthermore unknown whether there exist non-prefix independent winning conditions they are useful for.

For strategies using FAR-memory, the situation is similar. Until now, only rough bounds on their strength are known, but no detailed classification has been discovered. What is more, the role of the dimension of such memory is not fully understood. For the games considered so far, small dimensions were sufficient; it remains open whether this is always true, or whether there exists a sequence of games for which the dimension cannot be bounded from above. Additionally, further investigations on the relations towards different variants of counting strategies should be conducted, in order to provide knowledge on where the classes of winning conditions guaranteeing determinacy via either kind of strategies differ, especially as sometimes move-counting strategies can be simulated by ones using FAR-memory.

Although Muller winning conditions are generally well understood, there remain subclasses for which important questions are still open. For example, for such conditions over countably many colors where both sets of winning sets are uncountable, it is known that the games need not always be determined anymore, but it is not known whether these conditions reduce to simpler ones in games that actually are determined. Additionally, a theory of Muller winning condition over an uncountable set of colors has not yet been developed for Banach-Mazur games.

For sequential winning conditions, even less is known. We were able to show that for some subclasses of these winning conditions, counting strategies are useful. However, many other classes of such conditions remain uncharacterized, and the question of what other classes of strategies prove useful is unanswered.

As we mainly focused on prefix independent winning conditions, this leaves a similar discussion of non-prefix independent ones for the future.

Another class of strategies that has not been considered in this work are probabilistic strategies, i.e. positional strategies where the actual move is randomly chosen from a fixed set of possible moves from the current vertex along with their respective probabilities.

We suspect that in many cases, move-counting winning strategies can be transferred to probabilistic ones by guessing the move-number. Provided that each such counter value has a probability greater than 0, in an infinite play each value will almost surely be chosen at least once, and thus an infinite increasing subsequence is created. Using this and the construction explained in Lemma 4.2.2, any play where such a probabilistic version of a move-counting winning strategy is used will be won with probability 1. Whether this always works and is also possible for other kinds of winning strategies is an open question that requires further research.

Last but not least, especially with regard to possible practical applications of Banach-Mazur games, the theory of bounded strategies should be further developed. For the two variants introduced in this work, many open questions remain, e.g. whether there is something like guaranteed determinacy via some specific class of such strategies in bounded games and what actual impact the introduced imperfect information has, i.e. when nondeterminism really occurs. In addition, different approaches towards restricting the length of moves should be considered, such that in the end the different variants are properly distinguished, and the results of the restrictions are described completely.

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