Cardinality quantifiers in MLO over trees*

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Abstract. We study an extension of monadic second-order logic of order with the uncountability quantifier "there exist uncountably many sets". We prove that, over the class of finitely branching trees, this extension is equally expressive to plain monadic second-order logic of order.
Additionally we find that the continuum hypothesis holds for classes of sets definable in monadic second-order logic over finitely branching trees, which is notable for not all of these classes are analytic.
Our approach is based on Shelah's composition method and uses basic results from descriptive set theory. The elimination result is constructive, yielding a decision procedure for the extended logic. Furthermore, by the well-known correspondence between monadic second-order logic and tree automata, our findings translate to analogous results on the extension of first-order logic by cardinality quantifiers over injectively presentable Rabin-automatic structures, generalizing the work of Kuske and Lohrey.

1 Introduction

Monadic second-order logic of order, MLO, extends first-order logic by allowing quantification over *subsets* of the domain. The binary relation symbol $<$ and unary predicate symbols P_i are its only non-logical relation symbols. MLO plays a very important role in mathematical logic and computer science. The fundamental connection between MLO and automata was discovered independently by Büchi, Elgot and Trakhtenbrot when the logic was proved to be decidable over the class of finite linear orders and over $(\omega, <)$. Moving away from linear orders, Rabin proved that monadic second-order theory of the full binary tree, S2S for short, is decidable [13]. This theorem, obtained using the notion of tree automata, is one of the most celebrated results in theoretical computer science, sometimes even called "the mother of all decidability results".

First-order cardinality quantifiers, also known under the name of Magidor-Malitz quantifiers, count the number of elements with a given property. These

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quantifiers have been widely investigated in mathematical logic with respect to both decidability and the possibility of elimination. The book [1] presents results on decidability and other properties of first-order logic extended with such cardinality quantifiers over various natural classes of structures.

Second-order cardinality quantifiers in MLO, which we study in this paper, have been mostly considered in the context of automata and automatic structures. The first, basic result [2, 3] shows that the quantifier "there exist infinitely many words" can be eliminated on automatic structures. By the standard correspondence between automata and MLO mentioned above, this is equivalent to eliminating the quantifier "there exist infinitely many sets" from *weak* MLO over $(\omega, <)$. The case of full MLO over $(\omega, <)$ corresponds to injectively presented ω-automatic structures and was solved by Kuske and Lohrey in [7, 8]. Let us remark that, while cardinality quantifiers are hardly ever used directly in specifications, the structural properties of ω-regular languages identified in these results gave important insights into automatic structures and their properties.

Motivated by previous work on $(\omega, <)$ that used word automata, we investigate cardinality quantifiers over finitely branching trees, in particular over the binary tree with arbitrary labelings, which corresponds to tree automata with additional parameters. The parameterless question was previously studied by Niwiński, who in [11] proved that a regular language of infinite trees is uncountable if and only if it contains a non-regular tree.

This paper deals with the expressive power of the extension of MLO by cardinality quantifiers "there exist infinitely many subsets X such that" ($\exists^{\aleph_0}$), "there exist uncountably many subsets X such that" ($\exists^{\aleph_1}$) and "there exist at least continuum many subsets X such that" ($\exists^{2^{\aleph_0}}$). We study the extension of MLO by these quantifiers, MLO$(\exists^{\aleph_0}, \exists^{\aleph_1}, \exists^{2^{\aleph_0}})$, over *simple trees*. These are finitely-branching trees every branch of which is either finite or of order type ω. Our main results are summarized in the next two theorems.

Theorem 1 (Elimination of the uncountability quantifier). *For every* MLO$(\exists^{\aleph_0}, \exists^{\aleph_1}, \exists^{2^{\aleph_0}})$ *formula* $\varphi(\overline{Y})$ *there exists an* MLO *formula* $\psi(\overline{Y})$*, computable from* φ*, that is equivalent to* $\varphi(\overline{Y})$ *over the class of simple trees.*

In addition to the above, the reduction will show that over simple trees the quantifiers $\exists^{\aleph_1} X$ and $\exists^{2^{\aleph_0}} X$ are equivalent, i.e. that the continuum hypothesis holds for MLO-definable families of sets. Though not surprising, this is not obvious for it is known that in MLO one can define non-analytic classes of sets [12] and that CH is independent of ZFC already for co-analytic sets [10].

Theorem 2. *For every* MLO *formula* $\varphi(X, \overline{Y})$*,* $\exists^{\aleph_1} X\, \varphi(X, \overline{Y})$ *is equivalent to* $\exists^{2^{\aleph_0}} X\, \varphi(X, \overline{Y})$ *over simple trees.*

These results naturally extend to cardinality quantifiers $\exists^{\aleph_0} \overline{X}$, $\exists^{\aleph_1} \overline{X}$ and $\exists^{2^{\aleph_0}} \overline{X}$ counting (finite) tuples of sets. This follows from the basic fact that for any cardinal $\kappa \geq \aleph_0$ it holds $\exists^{\kappa}(U, \overline{V})\ \varphi \ \equiv\ \exists^{\kappa} U \left(\exists \overline{V}\, \varphi\right)\ \vee\ \exists^{\kappa} \overline{V}\, (\exists U\, \varphi)$.
Note that $\exists^{\kappa} X \varphi$ means "there exist *at least* κ sets X satisfying φ".

Our results bear relevance to the theory of automatic structures. Call a structure $\mathfrak{A}$ *generalized tree-automatic* [4], or specifically $\mathfrak{T}$-automatic, if there is a *subset interpretation* of $\mathfrak{A}$ in a labeled simple tree $\mathfrak{T}$. As introduced in [4], subset interpretations differ from monadic second-order interpretations in that the free variables of their constituent formulas are set variables. A structure $\mathfrak{A}$ is thus $\mathfrak{T}$-automatic if it has a concrete representation with subsets of $\mathfrak{T}$ as elements and atomic relations given by MLO formulas, equivalently, by Rabin tree-automata, hence the name. The first-order theory of $\mathfrak{A}$ is thus interpreted in the MLO theory of $\mathfrak{T}$. Such a representation is called injective if equality is left uninterpreted [6]. Theorem 1 entails the following.

Corollary 3. *Cardinality quantifiers can be effectively eliminated from first-order logic on injectively presented generalized tree-automatic structures.*

This supersedes the previously mentioned results from [2, 3] and [7, 8] and generalizes the theorem of Niwiński from [11], which follows form a parameterless instance of our theorem. Certain structural insight gained from some of our intermediate lemmas might be of independent interest. More specifically we show that counting sets of nodes satisfying an MLO-formula on a simple tree can be effectively reduced to a combination of counting branches satisfying a certain MLO-formula, and counting chains with certain MLO-definable properties on individual branches. While the latter essentially amounts to dealing with the special case treated in [7, 8], relying on basic results from descriptive set theory we show that counting of branches can also be implemented in MLO.

Organization

We begin by noting in Section 2 some observations considered folklore regarding the second-order infinity quantifier $\exists^{\aleph_0} X$. In Section 3 we fix terminology and notation on trees and recollect some essentials of Shelah's composition method for MLO. The rest of the paper is devoted to the proof of Theorems 1 and 2.

In Section 4 we start by reducing the question of the existence of uncountably many sets X satisfying a given MLO formula $\varphi(X, \overline{Y})$ with parameters $\overline{Y}$ over a simple tree to a disjunction of three conditions: A, B and C. Condition A deals with MLO-properties of antichains; Condition C deals with a simpler version of the uncountability quantifier, namely with the quantifier "there exist uncountably many branches". Ultimately, condition B is concerned with the cardinality of chains with a specific MLO property on individual branches, but it is postulated first in a far broader form for deductive advantages.

In Section 5, we show that Condition B can be significantly weakened assuming that conditions A and C are not satisfied. Relying on the elimination results on $(\omega, <)$ from [7, 8], we formalize this weakened form of Condition B in MLO and prove, that it guarantees the existence of continuum many sets satisfying φ.

In Section 6 we consider Condition C in the special case of the complete binary tree. The key theorem that we prove there, which might be of independent interest, is that MLO-definable sets of branches of the binary tree are Borel. This opens the way to formalizing Condition C in plain MLO, first over the binary tree and finally, in Section 7, over arbitrary simple trees.

The proofs of our main theorems are summarized in Section 8, Section 9 states further results.

2 Infinity quantifier

With regard to the second-order infinity quantifier $\exists^{\aleph_0} X$ the following observations are worth making. While it clearly cannot be eliminated over all structures, it is easily expressible in monadic second-order logic (MSO) with the auxiliary predicate $\mathrm{Inf}(Z)$ asserting that the set Z is infinite, or equivalently, with the help of the first-order infinity quantifier $\exists^{\aleph_0} x$.

Proposition 4. *For every* MSO($\exists^{\aleph_0}$) *formula* $\varphi(\overline{Y})$ *there exists an* MSO(Inf) *formula* $\psi(\overline{Y})$ *equivalent to* $\varphi(\overline{Y})$ *over all structures.*

Proof. Observe that the following are equivalent:

(1) There are only finitely many X which satisfy $\varphi(X, \overline{Y})$.
(2) There is a finite set Z such that any two different sets X_1, X_2 which both satisfy $\varphi(X_i, \overline{Y})$ differ on Z, i.e.

$$\exists Z \Big(\neg \mathrm{Inf}(Z) \;\wedge\; \forall X_1 X_2 \, \big((\varphi(X_1, \overline{Y}) \wedge \varphi(X_2, \overline{Y}) \wedge X_1 \neq X_2) \;\rightarrow \\ \exists z \in Z \, (z \in X_1 \leftrightarrow z \notin X_2) \big) \Big).$$

Item (2) implies (1) as a collection of sets pairwise differing only on a finite set Z has cardinality at most $2^{|Z|}$. Conversely, if $X_1, \ldots, X_k$ are all the sets that satisfy $\varphi(X_i, \overline{Y})$, then choose for every pair of distinct sets X_i, X_j an element $z_{i,j}$ in the symmetric difference of X_i and X_j and define Z as the set of these chosen elements. ☐

Over simple trees $\mathrm{Inf}(Z)$ can of course be expressed in MLO. Indeed, with König's Lemma in mind, Z is infinite iff there is an infinite chain, equivalently, an unbounded set of nodes each having an element of Z below it:

$$\psi_{\mathrm{Inf}}(Z) = \exists C \forall v \in C \exists w \in C, z \in Z : \; v < w \wedge v \leq z$$

Corollary 5. MLO($\exists^{\aleph_0}$) *collapses effectively to* MLO *over the class of simple trees.*

Observe that the converse of Proposition 4 holds as well. In fact, the predicate $\mathrm{Inf}(Z)$ can be defined over all structures by the formula $\exists^{\kappa} Y \, (Y \subseteq Z)$ for any $\aleph_0 \leq \kappa \leq 2^{\aleph_0}$. Therefore, by Proposition 4, any of the quantifiers $\exists^{\kappa} Y$ with $\aleph_0 < \kappa \leq 2^{\aleph_0}$ can be used to define $\exists^{\aleph_0} X$ over arbitrary structures.

3 Preliminaries

For a given set A we denote by A^* the set of all finite sequences of elements of A, by A^ω the set of all infinite sequences of elements of A (i.e. functions $\omega \to A$), and $A^{\leq\omega} = A^* \cup A^\omega$. For any sequence $s = s_0 s_1 s_2 \ldots \in A^{\leq\omega}$ we denote by $|s|$ the length of s (either a natural number or ω) and by $s|_n = s_0 \ldots s_{n-1}$ the finite sequence composed of the first n elements of s, with $s|_0 = \varepsilon$, the empty sequence. We write $s[n]$ for the $(n+1)$st element of s (we count from 0), so $s[n] = s_n$ for $n \in \mathbb{N}$. Given a finite sequence s and a sequence $t \in A^{\leq\omega}$ we denote by $s \cdot t$ (or just st) the concatenation of s and t. Moreover, we write $s \preceq t$ if s is a prefix of t, i.e. if there exists a sequence r such that $t = sr$. A subset B of $A^{\leq\omega}$ is said to be prefix-closed if for every $t \in B$ and $s \preceq t$ it holds that $s \in B$.

3.1 Trees

For a number $l \in \mathbb{N}$, $l > 0$, an l-tree is a structure $\mathfrak{T} = (T, <, P_1, \ldots, P_l)$, where the P_i's are unary predicates and $<$ is the irreflexive and transitive binary *ancestor* relation with a least element called the *root of $\mathcal{T}$* and such that for every $v \in T$ the set $\{u \in T \mid u < v\}$ of ancestors of v is linearly ordered by $<$. Elements of a tree are referred to as *nodes*, maximal linearly ordered sets of nodes are called *branches*, ancestor-closed and linearly ordered sets of nodes are called *paths*, whereas *chains* are arbitrary linearly ordered subsets. An *antichain* is a set of pairwise incomparable nodes. Given a node v, the subtree of $\mathfrak{T}$ rooted in v is obtained by restricting the structure to the domain $T_v = \{u \in T \mid u \geq v\}$ and is denoted $\mathfrak{T}_v$.

Given a finite set A, we denote by $\mathfrak{T}(A)$ the full tree over A, which is a structure with the universe A^*, $<$ interpreted as the prefix ordering and unary predicates $P_a = A^*a$ for each $a \in A$. For finite A with $|A| = n$, this structure is axiomatizable in MLO and its MLO theory is the same as SnS, the monadic second-order theory of n successors (modulo trivial MLO-interpretations in $\mathfrak{T}(n)$).

We identify a path B of $\mathfrak{T}(A)$ with the sequence $\beta = a_0 a_1 a_2 \ldots \in A^{\leq\omega}$ such that $B = \{a_0 \ldots a_s \mid s \leq |\beta|\}$. Conversely, given a sequence $\beta \in A^{\leq\omega}$ we write $\mathrm{Pref}(\beta)$ for the corresponding path B.

Ordered sums of trees are defined as follows.

Definition 6. *Let $l > 0$, $\mathfrak{I} = (I, <^{\mathcal{I}})$ be an unlabeled tree and let $\mathfrak{T}_i = (T_i, <^i, P_1^i, \ldots, P_l^i)$ be an l-tree for each $i \in I$. The* tree sum *of $(\mathfrak{T}_i)_{i \in \mathfrak{I}}$, denoted $\sum_{i \in \mathfrak{I}} \mathfrak{T}_i$, is the l-tree*

$$\mathfrak{T} = (\bigcup_{i \in I} \{i\} \times T_i, <^{\mathfrak{T}}, \bigcup_{i \in I} \{i\} \times P_1{}^i, \ldots, \bigcup_{i \in I} \{i\} \times P_l{}^i),$$

such that $(i, a) <^{\mathfrak{T}} (j, b)$ for $i, j \in I$, $a \in T_i$, $b \in T_j$ iff:

$$i <^{\mathfrak{I}} j \text{ and } a \text{ is the root of } \mathfrak{T}_i, \text{ or } i = j \text{ and } a <^i b\,.$$

Unless explicitly noted, we will not make a distinction between $\mathfrak{T}_i$ and the isomorphic subtree $\{i\} \times \mathfrak{T}_i$ of $\mathfrak{T}$.

A particular special case of the sum we will be using is when the index structure $\mathfrak{I}$ consists of a single branch, i.e. is a linear ordering. For every linear order $(I, <)$ and chain $\langle \mathfrak{T}_i \mid i \in I \rangle$ of trees, the sum $\mathfrak{T} = \sum_{i \in I} \mathfrak{T}_i$ is well defined, and $(I, <)$ forms a path (not necessarily maximal) of $\mathfrak{T}$.

We remark that not every tree can be decomposed as a sum along an arbitrarily chosen path. Such discrepancies can be ruled out by requiring that every two nodes possess a greatest common ancestor, i.e. an infimum. In this paper we consider only *simple trees*, which trivially fulfill this requirement.

Definition 7. *A* simple tree *is a finitely branching tree every branch of which is either finite or of order type ω.*

3.2 MLO and the composition method

We will work with labeled trees in the relational signature $\{<, P_1, \ldots, P_l\}$ where $<$ is a binary relation symbol denoting the ancestor relation of the tree, and the P_i's are unary predicates representing a labeling.

Monadic second-order logic of order, MLO for short, extends first-order logic by allowing quantification over *subsets* of the domain. MLO uses first-order variables $x, y, \ldots$ interpreted as elements, and set variables $X, Y, \ldots$ interpreted as subsets of the domain. Set variables will always be capitalized to distinguish them from first-order variables. The atomic formulas are $x < y$, $x \in P_i$ and $x \in X$, all other formulas are built from the atomic ones by applying Boolean connectives and the universal and existential quantifiers for both kinds of variables. Concrete formulas will be given in this syntax, taking the usual liberties and short-hands, such as $X \cup Y, X \cap Y, X \subseteq Y$, guarded quantifiers and relativization of formulas to a set.

The quantifier rank of a formula φ, denoted $\mathrm{qr}(\varphi)$, is the maximum depth of nesting of quantifiers in φ. For fixed n and l we denote by $\mathrm{Form}_{n,l}$ the set of formulas of quantifier depth $\leq n$ and with free variables among $X_1, \ldots, X_l$. Let $n, l \in \mathbb{N}$ and $\mathfrak{T}_1, \mathfrak{T}_2$ be l-trees. We say that $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are *n-equivalent*, denoted $\mathfrak{T}_1 \equiv^n \mathfrak{T}_2$, if for every $\varphi \in \mathrm{Form}_{n,l}$, $\mathfrak{T}_1 \models \varphi$ iff $\mathfrak{T}_2 \models \varphi$.

Clearly, $\equiv^n$ is an equivalence relation. For any $n \in \mathbb{N}$ and $l > 0$, the set $\mathrm{Form}_{n,l}$ is infinite. However, it contains only finitely many semantically distinct formulas, so there are only finitely many $\equiv^n$-classes of l-structures. In fact, we can compute representatives for these classes as follows.

Lemma 8 (Hintikka Lemma). *For $n, l \in \mathbb{N}$, we can compute a* finite *set $H_{n,l} \subseteq \mathrm{Form}_{n,l}$ such that:*

- *For every l-tree $\mathfrak{T}$ there is a* unique *$\tau \in H_{n,l}$ such that $\mathfrak{T} \models \tau$.*
- *If $\tau \in H_{n,l}$ and $\varphi \in \mathrm{Form}_{n,l}$, then either $\tau \models \varphi$ or $\tau \models \neg\varphi$. Furthermore, there is an algorithm that, given such τ and φ, decides which of these two possibilities holds.*

*Elements of $H_{n,l}$ are called (n, l)-*Hintikka formulas*.*

Given an l-tree $\mathfrak{T}$ we denote by $\mathrm{Tp}^n(\mathfrak{T})$ the unique element of $H_{n,l}$ satisfied in $\mathfrak{T}$ and call it the *n-type* of $\mathfrak{T}$. Thus, $\mathrm{Tp}^n(\mathfrak{T})$ determines (effectively) which formulas of quantifier-depth $\leq n$ are satisfied in $\mathfrak{T}$.

We sometimes speak of the n-type of a tuple of subsets $\overline{V} = V_1, \dots, V_m$ of a given l-tree $\mathfrak{T}$. This is synonymous with the n-type of the $(l+m)$-tree $(\mathfrak{T}, \overline{V})$ obtained by expansion of $\mathfrak{T}$ with the predicates $P_{l+1}, \dots, P_{l+m}$ interpreted as the sets $V_1, \dots, V_m$. This type will be denoted by $\mathrm{Tp}^n(\mathfrak{T}, \overline{V})$ and often referred to as an n-type in m variables, whereby the n-type of the $(l+m)$-tree $(\mathfrak{T}, \overline{V})$ is understood. Moreover, when considering substructures, e.g. $\mathfrak{T}' \subseteq \mathfrak{T}$, and given sets $\overline{X} \subseteq \mathfrak{T}$, we write $\mathrm{Tp}^n(\mathfrak{T}', \overline{X})$ to denote $\mathrm{Tp}^n(\mathfrak{T}', \overline{X} \cap \mathfrak{T}')$.

The essence of the composition method is that certain operations on structures, such as disjoint union and certain ordered sums, can be projected to n-types. A general composition theorem for MLO from which most other follow was proved by Shelah in [14]. We only cite the composition theorem that we use [9], a more detailed presentation of the method can be found in [15, 5].

Theorem 9 (Composition Theorem for Trees). *For every* MLO*-formula $\varphi(\overline{X})$ in the signature of l-trees having m free variables and quantifier rank n, and given the enumeration $\tau_1(\overline{X}), \dots, \tau_k(\overline{X})$ of $H_{n,l+m}$, there exists an* MLO*-formula $\theta(Q_1, \dots, Q_k)$ such that for every tree $\mathfrak{I} = (I, <^I)$ and family $\{\mathfrak{T}_i \mid i \in I\}$ of l-trees and subsets $V_1, \dots, V_m$ of $\sum_{i \in I} \mathfrak{T}_i$,*

$$\sum_{i \in I} \mathfrak{T}_i \models \varphi(\overline{V}) \iff \mathfrak{I} \models \theta(Q_1, \dots, Q_k)$$

where $Q_r = Q_r^{I;\overline{V}} = \{i \in I \mid \mathrm{Tp}^n(\mathfrak{T}_i, \overline{V}) = \tau_r\}$ for each $1 \leq r \leq k$. Moreover, θ is computable from φ, and does not depend on the decomposition of $\mathfrak{T}$.

4 D-nodes versus U-nodes and relevant branches

To eliminate the uncountability quantifier from $\exists^{\aleph_1} X\ \varphi(X, \overline{Y})$ over an l-tree $\mathfrak{T}$, we will consider certain colorings of segments of $\mathfrak{T}$. Let us first fix m sets $\overline{Y}$, n as the quantifier rank of φ, and k as the number of n-types in $l+m+1$ variables.

An *interval* of a tree is a connected and convex set I of nodes, i.e. such that for every $u, w \in I$ if u and w are incomparable, then their greatest common ancestor is in I, and if $u < w$ then for every $u < v < w$ also $v \in I$. We denote by $\mathfrak{T}|_I$ the restriction of an l-tree $\mathfrak{T}$ to the interval I.

An interval having a minimal element is called a *tree segment*. Observe that every interval of a simple tree is a tree segment and that the summands $\mathfrak{T}_i$ of a tree sum $\mathfrak{T} = \sum_{i \in I} \mathfrak{T}_i$ are tree segments of $\mathfrak{T}$. In fact any subtree $\mathfrak{T}_z$ of a tree $\mathfrak{T}$ is a tree segment.

Let Z be a subset of a tree $\mathfrak{T}$ and z be an element of $\mathfrak{T}$. We use the notation $\mathfrak{T}_{z \setminus Z}$ for the restriction of $\mathfrak{T}$ to the set $\mathfrak{T}_z \setminus (\bigcup_{w \in Z, z < w} \mathfrak{T}_w)$. Any tree segment $\mathfrak{T}'$ with a minimal element z can be written in the form $\mathfrak{T}_{z \setminus Z}$, where Z is the set $\{u \mid u \geq z \wedge u \notin \mathfrak{T}'\}$.

Definition 10. *Let $\mathfrak{T} = (T, <, \overline{P}, X, \overline{Y})$ be an $l+m+1$-tree such that $\mathfrak{T} \models \varphi(X, \overline{Y})$ and let I be an interval of $\mathfrak{T}$.*

(1) I is a U-interval *for φ, X, $\overline{Y}$ iff*

$$\mathfrak{T}|_I \models \forall Z\, \tau(Z, \overline{Y}) \rightarrow Z = X,$$

where $\tau(X, \overline{Y})$ is the n-type of $\mathfrak{T}|_I$ in $m+1$ variables.[4]

(2) I is a D-interval *for φ, X, $\overline{Y}$ iff it is not a U-interval.*

(3) In the special case of $I = \{u \mid u \geq z\}$ we say that the subtree $\mathfrak{T}_z$ is a U-tree *or* D-tree*, respectively, and further say that z is a* U-node *or* D-node *for $\varphi, X, \overline{Y}$.*

(4) The set of D-nodes for $\varphi, X, \overline{Y}$ is denoted $D(X)$.

(5) An infinite path P is called a D-path *for $\varphi, X, \overline{Y}$ if every $v \in P$ is a D-node for $\varphi, X, \overline{Y}$, i.e. if $P \subseteq D(X)$.*

Whenever $\varphi, X, \overline{Y}$ are clear from the context, we will write "D-interval for X" instead of "D-interval for $\varphi, X, \overline{Y}$", and similarly for the other notions above.

Observe that $D(X)$ is prefix-closed since if $u < v$ and $\mathfrak{T}_v$ is a D-tree then, by composition, $\mathfrak{T}_u$ is a D-tree as well. Therefore $D(X)$ can be thought of as a tree whose infinite paths are precisely the infinite D-paths for X.

We note that each of the notions introduced in Definition 10 is formalizable in MLO. Let us start by constructing the formula $\mathrm{DINT}_\varphi(I, X, \overline{Y})$, expressing that I is a D-interval for φ, X and $\overline{Y}$. By Lemma 8, the set of n-types $H_{n,l+m+1}$ can be computed and is finite. Thus, we can write the formula

$$\psi_{\mathrm{eqtp}}(X, X', \overline{Y}) \;=\; \bigwedge_{\tau \in H_{n,l+m+1}} \tau(X, \overline{Y}) \leftrightarrow \tau(X', \overline{Y}),$$

expressing that X and X' have the same n-type on the tree $\mathfrak{T}$. Let $\psi^{\mathrm{rel}}_{\mathrm{eqtp}}(X, Z, \overline{Y}, I)$ be the relativization of $\psi_{\mathrm{eqtp}}(X, Z, \overline{Y})$ to an interval I, which expresses that X and Z have the same n-type on I. $\mathrm{DINT}_\varphi(I, X, \overline{Y})$ can now be written as

$$\varphi(X, \overline{Y}) \;\wedge\; \exists Z(\psi^{\mathrm{rel}}_{\mathrm{eqtp}}(X, Z, \overline{Y}, I) \wedge X \cap I \neq Z \cap I).$$

Using this formula we can also write the other formulas $\mathrm{DPATH}_\varphi(P, X, \overline{Y})$ and $\mathrm{DNODE}_\varphi(v, X, \overline{Y})$, expressing, respectively, that P is a D-path and that v is a D-node for $\varphi, X, \overline{Y}$, and the formula $\mathrm{DSET}_\varphi(D, X, \overline{Y})$ which holds iff $D = D(X)$.

The following lemma is the first step in eliminating the $\exists^{\aleph_1}$ quantifier from MLO over simple trees.

Lemma 11. *Let $\mathfrak{T}$ be a simple l-tree and $\varphi(X, \overline{Y})$ an* MLO*-formula in the signature of l-trees. Then, for every tuple of subsets $\overline{V}$ of $\mathfrak{T}$,*

$$\mathfrak{T} \models \exists^{\aleph_1} X\; \varphi(X, \overline{V})$$

if and only if one of the following conditions is satisfied.

[4] As set before, n is the quantifier rank of φ and m is the length of $\overline{Y}$.

A. *There is a set U satisfying $\mathfrak{T} \models \varphi(U, \overline{V})$ and there is an infinite antichain A of D-nodes for $\varphi, U, \overline{V}$.*
B. *There is an infinite branch B of $\mathfrak{T}$ which is a D-path for uncountably many U satisfying $\mathfrak{T} \models \varphi(U, \overline{V})$.*
C. *The following set of branches of $\mathfrak{T}$*

$$\{B \mid \text{ there exists a set } U \text{ such that } B \text{ is a D-path for } \varphi, U, \overline{V}\}$$

is uncountable.

Proof. Note first that over simple trees, where König's Lemma applies, condition A is properly subsumed by, in other words implies B and is enlisted here for deductive reasons only.

Indeed, A is arguably the most natural (easily expressible) condition sufficient for the existence of continuum many sets U satisfying $\mathfrak{T} \models \varphi(U, \overline{V})$. To see that, let U and A be as in A and let v_0 denote the root of $\mathcal{T} = (\mathfrak{T}, U, \overline{V})$. Then $\mathcal{T}$ can be decomposed as $\mathcal{T} = \mathcal{T}_{v_0 \setminus A} + \sum_{w \in A} \mathcal{T}_w$. Applying the Composition Theorem (Th.9) to this decomposition, we get that $\mathfrak{T} \models \varphi(U', \overline{V})$ for every U' such that $U' \cap \mathfrak{T}_{v_0 \setminus A} = U \cap \mathfrak{T}_{v_0 \setminus A}$ and $\mathrm{Tp}^n(\mathfrak{T}_w, U', \overline{V}) = \mathrm{Tp}^n(\mathfrak{T}_w, U, \overline{V})$ for all $w \in A$. By the choice of A, U can be modified independently on each subtree $\mathcal{T}_w$ without changing its type $\mathrm{Tp}^n(\mathcal{T}_w)$. Hence there are continuum many different sets U' as above.

Furthermore, $\neg$A amounts to saying that for each U satisfying $\varphi(U, \overline{V})$ the set $D(U)$ induces a tree comprised of only finitely many branches. In particular, there are only finitely many infinite D-paths for each such U.

Condition B explicitly requires the existence of uncountably many sets satisfying $\varphi(X, \overline{V})$, so it too is sufficient for $\exists^{\aleph_1} X\ \varphi(X, \overline{V})$ to hold. Hence it remains to be shown that when B fails then C is both sufficient and necessary hereto.

Assuming B does not hold in some $\mathfrak{T}$ then A fails there as well, hence, as pointed out above, in this case there are only finitely many infinite D-paths for each U satisfying $\mathfrak{T} \models \varphi(U, \overline{V})$. Also by the failure of B every branch is a D-path for at most countably many U satisfying $\mathfrak{T} \models \varphi(U, \overline{V})$. It follows that every such set U shares its set of D-nodes with at most countably many other such U'. Indeed, this is clear from the above whenever $D(U)$ contains an infinite D-path. If on the other hand $D(U)$ is finite then U is fully determined by $U \cap D$ and the n-types of all those U-nodes that are successors of some D-node, which only allows for a finite number of choices of U given that $\mathfrak{T}$ is simple.

Thus we have established that whenever B fails in some $\mathfrak{T}$ then there are uncountably many U satisfying $\mathfrak{T} \models \varphi(U, \overline{V})$ iff there are uncountably many sets $D(U)$ with $\mathfrak{T} \models \varphi(U, \overline{V})$ iff condition C holds. □

Let us note again that if condition A holds then there are in fact continuum many sets X satisfying the formula $\varphi(X, \overline{Y})$. The description of Condition A can be directly formalized in MLO(Inf), hence, over simple trees, also in MLO as follows:

$$\psi_{\mathsf{A}}(\overline{Y}) = \exists U \exists A \big(\varphi(U, \overline{Y}) \wedge \mathrm{Inf}(A) \wedge \mathrm{antichain}(A) \wedge \\ (\forall w \in A\ \mathrm{DNODE}_{\varphi}(w, U, \overline{Y}))\big),$$

where antichain$(A) = \forall x, y \in A\ \neg(x < y \vee y < x)$.

5 Condition B

In this section, we show that a branch B is a witness for Condition B if and only if this branch satisfies a disjunction of three sub-conditions: Ba, Bb and Bc. Moreover, if both Condition A and Condition C fail, then already the sub-conditions Ba and Bc are sufficient. Finally, we express both Ba and Bc in MLO and show, that in fact both these sub-conditions guarantee the existence of continuum many sets X satisfying the formula $\varphi(X, \overline{Y})$ in consideration.

As in the previous section, we assume that the formula $\varphi(X, \overline{Y})$ of quantifier rank n is fixed together with a simple l-tree $\mathfrak{T}$ and m parameters $\overline{Y}$, and let k be the number of n-types in $l + m + 1$ variables. Additionally, we fix a branch B and introduce the formula $\psi(X, \overline{Y}, P)$ stating that P is an infinite D-path for X and that $\varphi(X, \overline{Y})$ holds:

$$\psi(X, \overline{Y}, P) = \mathrm{DPATH}_{\varphi}(P, X, \overline{Y}) \wedge \mathrm{Inf}(P) \wedge \varphi(X, \overline{Y}).$$

Note that the branch B witnesses Condition B if and only if $\exists^{\aleph_1} U\ \psi(U, \overline{Y}, B)$.

To break up Condition B, we decompose $\mathcal{T} = (\mathfrak{T}, X, \overline{Y})$ along the branch B, $\mathcal{T} = \sum_{w \in B} \mathcal{T}_{w \setminus B}$, and apply the Composition Theorem (Th.9) to this decomposition and the formula ψ. This yields a formula θ such that

$$\mathcal{T} \models \psi(X, \overline{Y}, B) \iff (B, <) \models \theta(P_1, \ldots, P_r),$$

where r is the number of qr(ψ)-types in $l+m+2$ variables, which we enumerate as $\tau_1, \ldots, \tau_r$, and

$$P_i = \{w \in B \mid (\mathcal{T}_{w \setminus B}, \{w\}) \models \tau_i\}.$$

Note that we use the expansion of $\mathcal{T}_{w \setminus B}$ by $\{w\}$ as w is the only element of $\mathcal{T}_{w \setminus B}$ that belongs to B.

With this reformulation it is clear that a branch B satisfies condition B if and only if either there are uncountably many different $\overline{P}$ satisfying θ, or some $\overline{P}$ satisfying θ has uncountably many X corresponding to it. Along these lines one obtains the following breakdown of condition B.

Lemma 12. *There are uncountably many $X \subseteq \mathfrak{T}$ satisfying the formula $\psi(X, \overline{Y}, B)$ in $\mathfrak{T}$ iff one of the following sub-conditions holds.*

(Ba) *There exists a set X such that $\mathfrak{T}_{w \setminus B}$ is a D-interval for $\varphi, X, \overline{Y}$ for infinitely many $w \in B$.*

(Bb) *There exists a set X satisfying ψ and a $w \in B$ so that*

$$\mathfrak{T}_{w \setminus B} \models \exists^{\aleph_1} X'\ \tau_i(X', \overline{Y} \cap \mathfrak{T}_{w \setminus B}, \{w\}),$$

where $\tau_i = \mathrm{Tp}^{\mathrm{qr}(\psi)}(\mathfrak{T}_{w \setminus B}, X, \overline{Y}, \{w\})$.

(Bc) *It holds that*

$$(B,<) \models \exists^{\aleph_1} \overline{P} \left(\theta(\overline{P}) \wedge \bigwedge_{i=1}^{r} P_i \subseteq Q_i \wedge \forall x \left(\bigvee_{i=1}^{r} (x \in P_i \wedge \bigwedge_{j \neq i} x \notin P_j \right) \right),$$

where for each $1 \leq i \leq r$, Q_i *is the set of nodes on the branch* B *in which the type* τ_i *is satisfied by some set* X, *i.e.*

$$Q_i = \{w \in B \mid \mathfrak{T}_{w \setminus B} \models \exists X \, \tau_i(X, \overline{Y} \cap \mathfrak{T}_{w \setminus B}, \{w\})\}$$

Observe that (Ba) already subsumes A in the sense that if condition A holds then there is a branch satisfying (Ba). Also observe that Condition (Bb) is itself just another instance of our initial problem. It is important to note, however, that the above cases classify conditions under which an *individual branch* may satisfy B. At closer inspection we find that if no branch satisfies either (Bc) or (Ba) (so that in particular A fails) and moreover condition C fails too, then B cannot hold either.

Lemma 13. *If over a simple tree* $\mathfrak{T}$ *both Conditions* A *and* C *fail, then Condition* B *holds iff some branch satisfies Condition* (Ba) *or Condition* (Bc).

One intuitive way to see this is that if all the conditions A, (Ba), (Bc) and C fail on a tree, and thereby also on every tree segment of that tree, then for (Bb) to hold for a proper tree segment that tree segment would have to contain a proper tree segment on which (Bb) holds, and so on indefinitely. This would ultimately trace an infinite branch witnessing (Ba) contrary to the initial assumption.

Next we will construct MLO formulas $\psi_{\mathsf{Ba}}(B, \overline{Y})$ and $\psi_{\mathsf{Bc}}(B, \overline{Y})$ formalizing sub-conditions (Ba) and (Bc), respectively. By the above, we can then use the formula $\psi_{\mathsf{B}}(\overline{Y}) = \exists B(\psi_{\mathsf{Ba}}(B, \overline{Y}) \vee \psi_{\mathsf{Bc}}(B, \overline{Y}))$ in place of Condition B in Lemma 11.

5.1 Formalization of Condition Ba

Much like condition A, (Ba) is naturally expressible in MLO(Inf) and thus, over simple trees, in pure MLO as well by the formula

$$\psi_{\mathsf{Ba}}(B, \overline{Y}) = \exists X \; \exists^{\aleph_0} w \; \mathrm{DINT}(T_{w \setminus B}, X, \overline{Y}),$$

where $T_{w \setminus B}$ is just a notation for the set defined by

$$x \in T_{w \setminus B} \iff w \leq x \wedge \neg \exists b \in B \, (b > w \wedge b \leq x).$$

The fact that Condition (Ba) is sufficient for the existence of continuum many sets U satisfying $\varphi(U, \overline{V})$ can be arrived at by appealing to the Composition Theorem in the same manner as for Condition A in the proof of Lemma 11, because the set X can be left intact or changed to another one with the same type on any of the infinitely many trees $\mathfrak{T}_{w \setminus B}$ which are D-intervals for X.

5.2 Formalization of Condition Bc

In order to eliminate the explicit use of the uncountability quantifier from Condition (Bc) over $(B,<) \cong (\omega,<)$, we use Proposition 2.5 from [8] reformulated using the standard equivalence of automata and MLO on $(\omega,<)$, as stated in the following proposition.

Proposition 14 (cf. [7, 8]). *For every* MLO *formula* $\varphi(\overline{X},\overline{Y})$ *there exists an effectively constructible formula* $\psi(\overline{Y})$ *such that over* $(\omega,<)$

$$\psi(\overline{Y}) \;\equiv\; \exists^{\aleph_1}\overline{X}\,\varphi(\overline{X},\overline{Y}) \;\equiv\; \exists^{2^{\aleph_0}}\overline{X}\,\varphi(\overline{X},\overline{Y}).$$

Applying this result to the formula on the right hand side of Condition (Bc), with $\overline{Q}$ as parameters, we obtain a formula $\vartheta(\overline{Q})$ such that Condition (Bc) holds iff $(B,<) \models \vartheta(\overline{Q})$, with $\overline{Q}$ as specified there.

By Proposition 14, if $\vartheta(\overline{Q})$ holds, then there are even continuum many sets $\overline{P}$ satisfying Condition (Bc). This in turn ensures the existence of continuum many sets X satisfying $\varphi(X,\overline{Y})$, because for each $\overline{P}$ accounted for in $\vartheta(\overline{Q})$ a corresponding X satisfying $\psi(X,\overline{Y},B)$ can be found and this association is necessarily injective.

To formalize Condition (Bc) in MLO over the tree $\mathfrak{T}$, we first define the sets Q_i on $\mathfrak{T}$. As the set of types is computable, we can compute each τ_i and thus effectively construct the formula $\alpha_i(w,B,\overline{Y})$ expressing that w is a node on the branch B such that $\mathfrak{T}_{w\setminus B} \models \exists X\,\tau_i(X,\overline{Y}\cap\mathfrak{T}_{w\setminus B},\{w\})$, i.e. $w \in Q_i$. Using this formula we can express Condition (Bc) as $\psi_{\mathsf{Bc}}(B,\overline{Y}) =$

$$\exists\overline{Q}\left(\bigwedge_{i=1}^{r}\big(w\in Q_i \leftrightarrow \alpha_i(w,B,\overline{Y})\big) \;\wedge\; \vartheta^B(\overline{Q})\right),$$

where ϑ^B is a relativization of ϑ to the branch B.

6 The full binary tree and the Cantor space

In order to formalize Condition C in MLO over simple trees, we first analyze the problem only on the full binary tree and identify and prove the following key topological property that distinguishes counting branches from counting arbitrary sets.

On the full binary tree $\mathfrak{T}(2) = (\{0,1\}^*, \prec, S_0, S_1)$ where $\prec$ is the prefix-order and $S_i = \{0,1\}^*i$, we show that the set of branches satisfying any given MLO formula is a Borel set in the Cantor topology and hence it has the *perfect set property*: it is uncountable iff it contains a perfect subset iff it has the cardinality of the continuum. A *perfect set* is a closed set without isolated points.

The Cantor-Bendixson Theorem states that closed subsets of a Polish space have the *perfect set property*: they are either countable or contain a perfect subset and thus have cardinality continuum. A set P is *perfect* if it is closed and if every point $p \in P$ is a condensation point of P, i.e. if every neighborhood of p contains another point from P. We shall rely on the following fundamental result of Souslin.

Theorem 15 (cf. e.g. in [10]). *A subset of a Polish space is Borel if and only if it is both analytic and co-analytic. Moreover, every uncountable analytic set contains a perfect subset.*

Note that whether co-analytic sets, or all sets on higher levels of the projective hierarchy, satisfy the continuum hypothesis is independent of ZFC [10].

A key observation that our formalization will exploit is that, even though there are non-Borel sets of trees definable in MLO, sets of definable paths are Borel. Recall that for a sequence $\pi \in \{0,1\}^*$ we denote by $\mathrm{Pref}(\pi)$ the path through $\mathfrak{T}(2)$ that corresponds to this sequence, which formally is the set of prefixes of π.

Theorem 16 (MLO definable sets of branches are Borel). *Let $U_1, \ldots, U_m$ be subsets of $\mathfrak{T}(2)$ and let $\psi(X, \overline{Y})$ be an MLO formula over $\mathfrak{T}(2)$. Then the set*

$$\mathcal{X} = \{\, \pi \in \{0,1\}^\omega \mid \mathfrak{T}(2) \models \psi(\mathrm{Pref}(\pi), \overline{U}) \,\}$$

of branches of the binary tree satisfying $\psi(X, \overline{U})$ is Borel and therefore it has the perfect set property.

Proof. Note that the complement of $\mathcal{X}$ is also definable by $\neg\psi(X, \overline{U})$. We will show that every definable set of branches is analytic. Therefore, by Souslin's Theorem, it is Borel. To prove this, we will use the following variation of the Composition Theorem (cf. [9]).

Lemma 17. *Let $\psi(X, Y_1, \ldots, Y_m)$ be an MLO formula with quantifier rank $n \geq 2$, and let k be the number of $(n+2)$-types in $m+1$ variables. Then there exists an MLO formula $\theta(I, Z_1, \ldots, Z_k)$ such that*

$$\mathfrak{T}(2) \models \psi(\mathrm{Pref}(\pi), \overline{U}) \iff (\omega, <) \models \theta(\{n \mid \pi[n] = 1\}, \overline{Q}),$$

where for each $1 \leq i \leq k$ we define $Q_i = Q_i^{\pi, \overline{U}}$ as

$$Q_i = \{j \in \omega \mid \mathrm{Tp}^{n+2}(\mathfrak{T}(2)_{\pi|_j}, \overline{U}) = \tau_i\}.$$

Let θ be the formula obtained by applying the above lemma to ψ. Then, by the well-known correspondence of MLO and finite automata on ω-words, there is an ω-regular language $\mathcal{L}_\theta \subseteq (\{0,1\}^{k+1})^\omega \cong \{0,1\}^\omega \times (\{0,1\}^k)^\omega$, such that $\mathcal{L}_\theta$ consists of those pairs of sequences (π, ρ) for which $(\omega, <) \models \theta(P, \overline{Q})$, where P and $\overline{Q}$ are subsets of ω with characteristic sequences $\pi \in \{0,1\}^\omega$ and $\rho \in (\{0,1\}^k)^\omega$. By McNaughton's theorem [**?**], $\mathcal{L}_\theta \in \mathbf{\Sigma}^0_3$.

Let $\mathcal{T}$ be the extension of $\mathfrak{T}(2)$ with each node w labeled by $(\sigma, \overline{q})$ such that w is the σ-th successor of its parent (i.e. $w \in S_\sigma$) and $\overline{q} = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in position i if $\mathrm{Tp}^{n+2}(\mathfrak{T}(2)_w, \overline{U}) = \tau_i$. The set $[\mathcal{T}]$ of labeled infinite branches of $\mathcal{T}$ is closed in the Cantor topology.

By construction, $\mathcal{X}$ is the projection of $\mathcal{L}_\theta \cap [\mathcal{T}]$ to its first component, and is analytic as $\mathcal{L}_\theta \in \mathbf{\Sigma}^0_3$ and $[\mathcal{T}] \in \mathbf{\Pi}^0_1$. □

7 Formalizing Condition C

The perfect set property established in Theorem 16 provides an MLO-definable characterization of Condition C of Lemma 11 over the full binary tree (with arbitrary labeling). Via interpretations, this can be extended to all simple trees to yield the following characterization.

Proposition 18 (Eliminating uncountably-many-branches quantifier). *For every* MLO *formula* $\varphi(X, \overline{Y})$ *the assertion "*$\exists^{\aleph_1} B\ branch(B) \wedge \varphi(B, \overline{Y})$*" is equivalent over all simple trees to the existence of a perfect set of branches* B*, each satisfying* $\varphi(B, \overline{Y})$*. The latter ensures that there are in fact continuum many such branches.*

Towards an MLO formulation, note that the collection of nodes of a perfect set of branches induces a perfect tree, and vice versa. A perfect tree is one without isolated branches, equivalently, one in which for every node u there are incomparable nodes $v, w > u$. Perfectness is thus first-order definable.

Corollary 19. *Over simple trees Condition* C *is expressible in* MLO *as*

$$\psi_{\mathsf{C}}(\overline{Y}) = \exists P\, perfect(P)\ \forall B \subset P, branch(B)\ \exists X\ \mathrm{DPATH}_{\varphi}(B, X, \overline{Y})$$

Hence if Condition C *holds then there are continuum many D-paths altogether for all sets* U *satisfying* $\varphi(U, \overline{Y})$*.*

8 Summary of the proofs

As we have shown above, the conditions of Lemma 11 can be formalized in MLO over simple trees, thus we can again state the conclusion of this Lemma: $\mathfrak{T} \models \exists^{\aleph_1} X\ \varphi(X, \overline{Y})$ holds if and only if

$$\mathfrak{T} \models \psi_{\mathsf{A}}(\overline{Y}) \vee \exists B(\psi_{\mathsf{Ba}}(B, \overline{Y}) \vee \psi_{\mathsf{Bc}}(B, \overline{Y})) \vee \psi_{\mathsf{C}}(\overline{Y}).$$

Using the above, we can reduce any formula of MLO($\exists^{\aleph_1}$) to an MLO formula equivalent over the class of simple trees by inductively eliminating the inner-most occurrence of a cardinality quantifier. Theorem 1 follows. Moreover, as we have shown in the corresponding sections, each of the conditions of Lemma 11 implies the existence of continuum many sets X satisfying $\varphi(X, \overline{Y})$, whence Theorem 2.

9 Further results

The technique we used here can be applied to linear orders and leads to the following generalization of the theorem of Kuske and Lohrey (c.f. Proposition 14).

Theorem 20 (Eliminating uncountability quantifier on linear orders).

(1) For every MLO($\exists^{\aleph_1}$) *formula* $\varphi(\overline{Y})$ *there exists an* MLO *formula* $\psi(\overline{Y})$ *that is equivalent to* $\varphi(\overline{Y})$ *over the class of all ordinals.*

(2) For every MLO($\exists^{\aleph_1}$) *formula* $\varphi(\overline{Y})$ *there exists an* MLO *formula* $\psi(\overline{Y})$ *that is equivalent to* $\varphi(\overline{Y})$ *over the class of all countable linear orders. Moreover,* $\exists^{\aleph_1} X\ \varphi(X, \overline{Y})$ *is equivalent to* $\exists^{2^{\aleph_0}} X\ \varphi(X, \overline{Y})$ *over the class of countable linear orders.*

Furthermore, in all these cases ψ *is computable from* φ*.*

The proof will be provided in an extension of this paper. Note that the elimination result in (2) cannot be obtained simply by interpretation of countable linear orders in the full binary tree.

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A Expressing Condition B – proofs

Lemma 12. *There are uncountably many $X \subseteq \mathfrak{T}$ satisfying the formula $\psi(X, \overline{Y}, B)$ in $\mathfrak{T}$ iff one of the following sub-conditions holds.*

(Ba) *There exists a set X such that $\mathfrak{T}_{w \setminus B}$ is a D-interval for $\varphi, X, \overline{Y}$ for infinitely many $w \in B$.*

(Bb) *There exists a set X satisfying ψ and a $w \in B$ so that*

$$\mathfrak{T}_{w \setminus B} \models \exists^{\aleph_1} X' \ \tau_i(X', \overline{Y} \cap \mathfrak{T}_{w \setminus B}, \{w\}),$$

where $\tau_i = \mathrm{Tp}^{\mathrm{qr}(\psi)}(\mathfrak{T}_{w \setminus B}, X, \overline{Y}, \{w\})$.

(Bc) *It holds that*

$$(B, <) \models \exists^{\aleph_1} \overline{P} \ \left(\ \theta(\overline{P}) \ \wedge \ \bigwedge_{i=1}^{r} P_i \subseteq Q_i \ \wedge \forall x \left(\bigvee_{i=1}^{r} (x \in P_i \wedge \bigwedge_{j \neq i} x \notin P_j \right) \right),$$

where Q_i is the set of nodes on the branch B in which the type τ_i is satisfied by some set X, i.e.

$$Q_i = \{ w \in B \mid \mathfrak{T}_{w \setminus B} \models \exists X \, \tau_i(X, \overline{Y} \cap \mathfrak{T}_{w \setminus B}, \{w\}) \}$$

for each $1 \leq i \leq r$.

Proof. By the application of the Composition Theorem done above, $\mathcal{T} \models \psi(X, \overline{Y}, B)$ iff $(B, <) \models \theta(P_1, \ldots, P_r)$. Let us consider the following cases.

Case 1: There exists a tuple $\overline{P}$ such that $(B, <) \models \theta(\overline{P})$ and there are uncountably many sets X for which $P_i = \{w \in B \mid (\mathfrak{T}_{w \setminus B}, X, \overline{Y}, \{w\}) \models \tau_i\}$ for each $1 \leq i \leq r$.

In this case the branch B witnesses Condition B, so we only need to show that one of the sub-conditions holds. By contradiction, assume that sub-condition (Ba) does not hold. Then, for every set X satisfying $\psi(X, \overline{Y}, B)$, the segment $\mathfrak{T}_{w \setminus B}$ is a D-interval only for finitely many $w \in B$. Consider one of the uncountably many sets X which have $\mathrm{qr}(\psi)$-types on $\mathfrak{T}_{w \setminus B}$ described by $\overline{P}$. Since $\mathrm{qr}(\psi) \geq \mathrm{qr}(\varphi)$ and $\mathfrak{T}_{w \setminus B}$ is a U-interval for X for all but finitely many w's, all of the continuum many sets that share $\overline{P}$ must be equal to X on all but finitely many $\mathfrak{T}_{w \setminus B}$. Thus, there is as well a single w for which there are continuum many different sets sharing the types with X on $\mathfrak{T}_{w \setminus B}$, and thus Condition (Bb) is satisfied.

Case 2: For each tuple $\overline{P}$ such that $(B, <) \models \theta(\overline{P})$ there are only countably many sets X for which $P_i = \{w \in B \mid (\mathfrak{T}_{w \setminus B}, X, \overline{Y}, \{w\}) \models \tau_i\}$.

In this case, we show that Condition (Bc) is both necessary and sufficient for the existence of uncountably many sets X satisfying ψ.

Necessity of Condition (Bc).

As a direct consequence of the application of the Composition Theorem above and the condition of this case, if there are uncountably many sets X satisfying ψ then there are uncountably many corresponding tuples $\overline{P}$ for which $(B, <) \models$

$\theta(\overline{P})$. By definition, P_i is the set of w's for which $(\mathfrak{T}_{w\setminus B}, X, \overline{Y}, \{w\}) \models \tau_i$. Taking the X above we get $\mathfrak{T}_{w\setminus B} \models \exists X \tau_i(X, \overline{Y} \cap \mathfrak{T}_{w\setminus B}, \{w\})$, and therefore $P_i \subseteq Q_i$ holds. Since Hintikka formulas are mutually exclusive, each two sets P_i, P_j for $i \neq j$ are disjoint. This guarantees that the remaining conjunct $\forall x \big(\bigvee_{i=1}^{r}(x \in P_i \wedge \bigwedge_{s\neq r} x \notin P_s\big)$ of Condition (Bc) is satisfied, and thus Condition (Bc) holds.

Sufficiency of Condition (Bc).
By definition of the sets Q_i, for each $w \in Q_i$ there is a set $X_{w,i}$ which makes the type τ_i satisfied on the extension of $\mathfrak{T}_{w\setminus B}$. Assuming that Condition (Bc) holds, let $\mathcal{P}$ be the uncountable set of tuples $\overline{P}$ that witness this condition. For each such tuple $\overline{P}$ and each $w \in B$ the last conjunct of Condition (Bc) guarantees that there is a unique $i = i(w)$ for which $w \in P_i$. Construct $X_{\overline{P}}$ as the sum of $X_{w,i(w)}$ over all $w \in B$. Since $P_i \subseteq Q_i$, the tuple $\overline{P}$ indeed describes the types of the set $X_{\overline{P}}$. Therefore for different tuples $\overline{P}_1, \overline{P}_2$ the sets $X_{\overline{P_1}}, X_{\overline{P_2}}$ are different as well. Moreover, since $\theta(\overline{P})$ holds, the above application of the Composition Theorem guarantees that $\psi(X_{\overline{P}}, \overline{Y}, B)$ holds. Thus $\{X_{\overline{P}} \mid \overline{P} \in \mathcal{P}\}$ constitutes an uncountable family of sets satisfying ψ. □

Lemma 13. *If over a finitely branching tree $\mathfrak{T}$ both Condition* A *and Condition* C *fail, then Condition* B *holds if and only if there exists a branch that satisfies Condition* (Ba) *or Condition* (Bc)*.*

Proof. If conditions A and C fail, then, as we have already seen, the set $\mathcal{D} = \{D(X) \mid \mathfrak{T} \models \varphi(X, \overline{Y})\}$ is countable. Moreover, each $D \in \mathcal{D}$ is a union of finitely many paths.

If Condition B holds then there are uncountably many sets X satisfying $\varphi(X, \overline{Y})$ and thus, as $\mathcal{D}$ is countable, there is a set D such that $D = D(X)$ for uncountably many X satisfying φ. Fix such a set D and consider all its labelings by the types of X on the partial trees $\mathfrak{T}_{w\setminus D}$, i.e. the set $\mathcal{L} = \{\overline{L}^X \mid D(X) = D\}$ where $\overline{L}^X = \langle L_1^X \dots L_k^X \rangle$ and

$$L_j^X = \{w \in D \mid \mathrm{Tp}^n(\mathfrak{T}_{w\setminus D}, X, \overline{Y}, \{w\}) = \tau_j\}.$$

We are going to show that the failure of Condition (Bc) guarantees that the set $\mathcal{L}$ is countable.

First, D is the union of a finite set of branches, therefore there is a finite set $E = \{e_1, \dots, e_s\}$ of maximal branching points of D. For $i = 1 \dots s$, let $\mathrm{Path}_i = \{v \in D \mid v > e_i\}$, let B_i be the unique branch of D that contains Path_i and let $T_{\mathrm{fin}} = D \setminus \cup_i \mathrm{Path}_i$. Note that T_{fin} is a finite subtree of D and hence there are only finitely many possible labelings of T_{fin}. Note also that B_i are infinite branches.

If $\mathcal{L}$ was uncountable then there would exist an i with uncountably many different labelings of Path_i, i.e. the set $\mathcal{H} = \{\overline{H}^X \mid D(X) = D\}$ where $\overline{H}^X = \langle H_1^X \dots H_k^X \rangle$,

$$H_j^X = \{w \in Path_i \mid \mathrm{Tp}^n(\mathfrak{T}_{w\setminus D}, X, \overline{Y}, \{w\}) = \tau_j\},$$

would be uncountable. However, for $w \in \mathrm{Path}_i$, $\mathfrak{T}_{w\setminus D} = \mathfrak{T}_{w\setminus \mathrm{Path}_i} = \mathfrak{T}_{w\setminus B_i}$. Therefore, $\mathcal{Q} = \{\overline{Q}^X \mid D(X) = D\}$ where $\overline{Q}^X = \langle Q_1^X, \dots Q_k^X\rangle$ and

$$Q_j^X = \{w \in B_i \mid \mathrm{Tp}^n(\mathfrak{T}_{w\setminus B_i}, X, \overline{Y}, \{w\}) = \tau_j\}$$

would be uncountable. Since $\mathrm{qr}(\psi) \geq n$, different n-types induce different $\mathrm{qr}(\psi)$-types, so the set $\mathcal{P} = \{\overline{P}^X \mid D(X) = D\}$, with $\overline{P}^X = \langle P_1^X, \dots P_r^X\rangle$ and

$$P_j^X = \{w \in B_i \mid \mathrm{Tp}^{\mathrm{qr}(\psi)}(\mathfrak{T}_{w\setminus B_i}, X, \overline{Y}, \{w\}) = \tau_j\},$$

is uncountable as well. (Note that here τ_j is an $\mathrm{qr}(\psi)$-type.) As shown in the part on necessity of Condition (Bc) in the proof of Lemma 12, each such $\overline{P}^X$ satisfies the formula in Condition (Bc), so this condition holds for B_i.

As shown above, $\mathcal{L}$ is countable. Since there are uncountably many X with $D(X) = D$, there exists a single type labeling $\overline{L}$ such that $\overline{L} = \overline{L}^X$ for uncountably many of these sets X. Thus each of these uncountably many sets X has the same type $\mathrm{Tp}^n(\mathfrak{T}_{w\setminus D}, X, \overline{Y}, \{w\})$ for each $w \in B$, which we denote $\tau_{(w)}$.

If Condition (Ba) is not satisfied either, all but finitely many of these $\tau_{(w)}$ uniquely define X on the respective tree segments $\mathfrak{T}_{w\setminus D}$.

Thus, there exists a $w \in D$ such that there are uncountably many X as above pairwise differing on the tree segment $\mathfrak{T}_{w\setminus D}$. However, by definition, every subtree of $\mathfrak{T}_{w\setminus D}$ is a U-tree relative to every of these X, because $D(X) = D$. Hence if $\mathfrak{T}$ is finitely branching, i.e. if $\mathfrak{T}_{w\setminus D} \setminus \{w\}$ is a finite union of such U-trees, then there can be only finitely many X as above pairwise differing on $\mathfrak{T}_{w\setminus D}$, which is a contradiction. □

B Overview of topological notions

The argument we present is based on basic results of descriptive set theory and the theory of finite automata on infinite words in connection with monadic second-order logic and the Borel hierarchy of the Cantor space. Let us recall a few basic notions from descriptive set theory. A thorough introduction to descriptive set theory can be found in [10], we only mention a few basic facts.

The Cantor space is the topological space with the product topology on $\{0,1\}^\omega$. It is a Polish space with the topology generated by basic neighborhoods $w\{0,1\}^\omega$ with the prefix $w \in \{0,1\}^*$. Alternatively, it can be defined by the metric $d(\alpha,\beta) = 2^{-\min\{n\,:\,\alpha[n]\neq\beta[n]\}}$.

The hierarchy of Borel sets is generated starting from open sets, i.e. unions of basic neighborhoods, denoted $\mathbf{\Sigma}_1^0$, and closed sets, which are complements of open sets and denoted $\mathbf{\Pi}_1^0$. Further on by transfinite induction for any countable ordinal α, $\mathbf{\Sigma}_\alpha^0$ is defined as $\{\bigcup_{i\in\omega} A_i \mid \forall i\ \exists \beta_i < \alpha\ A_i \in \mathbf{\Pi}_{\beta_i}^0\}$ and the $\mathbf{\Pi}_\alpha^0$-sets are the complements of $\mathbf{\Sigma}_\alpha^0$-sets. The projective hierarchy is built on top of the Borel hierarchy, starting with $\mathbf{\Sigma}_0^1 = \mathbf{\Pi}_0^1$ as the class of Borel sets. On the first level one has the class $\mathbf{\Sigma}_1^1$ of *analytic sets*, which are projections of Borel sets, and the class $\mathbf{\Pi}_1^1$ of *co-analytic sets*, whose complements of analytic. The hierarchy is built in this manner with sets in $\mathbf{\Sigma}^1_{\alpha+1}$ being projections of $\mathbf{\Pi}^1_\alpha$-sets, and $\mathbf{\Pi}^1_{\alpha+1}$ sets being complements of $\mathbf{\Sigma}^1_\alpha$ sets.

The connection between the topological complexity of MLO-definable tree languages and the complexity of tree-automata recognizing them is well understood. By Rabin's complementation theorem, all MLO-definable tree languages are in $\mathbf{\Sigma}^1_2 \cap \mathbf{\Pi}^1_2$. There are $\mathbf{\Sigma}^1_1$-complete as well as $\mathbf{\Pi}^1_1$-complete regular tree languages. For instance, the set of $\{a, b\}$-labeled binary trees, which have on every path only finitely many a's, is $\mathbf{\Pi}^1_1$-complete.There also exist regular tree languages not contained in $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$, however languages accepted by deterministic tree automata are contained in $\mathbf{\Pi}^1_1$. In contrast, by McNaughton's theorem, ω-regular languages, i.e. MLO-definable sets of ω-words, are boolean combinations of $\mathbf{\Pi}^0_2$ sets.

C Proof of Lemma 17

Lemma 17 is weaker than the full Composition Theorem for trees (Th. 9) of Lifsches and Shelah [9], as the index structure on which the tree is decomposed is a single branch and we consider a specific labeling. However, even if it is not very likely to be useful for other applications, we need this particular version for our proof.

Lemma 17. *Let $\psi(X, Y_1, \ldots, Y_m)$ be an* MLO *formula with quantifier rank $n \geq 2$, and let k be the number of $(n+2)$-types in $m+1$ variables. Then there exists an* MLO *formula $\theta(I, Z_1, \ldots, Z_k)$ such that*

$$\mathfrak{T}(2) \;\models\; \psi(\mathrm{Pref}(\pi), \overline{U}) \qquad \Longleftrightarrow \qquad (\omega, <) \models \theta(\{n \mid \pi[n] = 1\}, \overline{Q}),$$

where for each $1 \leq i \leq k$ we define $Q_i = Q_i^{\pi,\overline{U}}$ as

$$Q_j = \{j \in \omega \mid \mathrm{Tp}^{n+2}(\mathfrak{T}(2)_{\pi|_j}, \overline{U}) = \tau_i\}.$$

Proof. To construct θ, we first apply the Composition Theorem (Th.9) to $\psi(X, \overline{Y})$ on the full binary tree $\mathfrak{T}(2)$ decomposed along any branch B. This yields an MLO formula $\theta_0(\overline{T})$ such that, for every branch B of $\mathfrak{T}(2)$,

$$\mathfrak{T}(2) \models \psi(\mathrm{Pref}(\pi), \overline{U}) \iff (B, \prec) \models \theta_0(\overline{P}).$$

Here, by definition of $P_r = P_r^{B;\mathrm{Pref}(\pi),\overline{U}}$, holds for each n-type τ_r, each $\iota \in \{0, 1\}$ and $v \in P_r$ that $v\iota \in B$ if and only if τ_r is the n-type of $(\mathrm{Pref}(\pi), \overline{U})$ on the tree segment $\mathfrak{T}(2)_v \setminus \mathfrak{T}(2)_{v\iota}$.

As a first step we refine $\theta_0(\overline{P})$ to a formula $\theta_1(I, \overline{P})$ such that $(B, \prec) \models \theta_1\left(I, \overline{P}^{B;\mathrm{Pref}(\pi),\overline{U}}\right)$ if and only if all of the following three conditions hold:

– $(B, \prec) \models \theta_0\left(\overline{P}^{B,\mathrm{Pref}(\pi),\overline{U}}\right)$,
– $B = \mathrm{Pref}(\pi)$, and
– $I = B \cap S_1$.

Observe that a node $v \in B$ lies on the path π or is a 1-successor precisely if the n-type $\tau_r(X, \overline{Y})$ such that $v \in P_r$ stipulates that X is not empty, or that the root belongs to S_1, respectively. As we assumed that $n \geq 2$, let H and G be the sets of those n-types $\tau_r(X, \overline{Y})$ from which $\exists x\, (x \in X)$, respectively, $\exists x\ \forall y(x \leq y) \wedge x \in S_1$, can be inferred. Then we set $\theta_1(I, \overline{P})$ to be

$$\theta_0(\overline{P}) \wedge \forall v \left(\bigvee_{\tau_r \in H} v \in P_r \wedge (v \in I \leftrightarrow \bigvee_{\tau_r \in G} v \in P_r) \right),$$

and it indeed has the above property, i.e.

$$\mathfrak{T}(2) \ \models\ \psi(\mathrm{Pref}(\pi), \overline{U}) \qquad \Longleftrightarrow \qquad (\omega, <) \models \theta_1(\{n \mid \pi[n] = 1\}, \overline{T}^{(\pi,\overline{U})}),$$

with $T_r = \{i \in \omega \mid \tau_r = \mathrm{Tp}^n(\mathfrak{T}(2)_{\pi|_i \setminus \pi|_{i+1}}, \{\pi|_i\}, \overline{U})\}$ for each n-type τ_r.

Finally, for each $i \in \{0, 1\}$ and $(n+2)$-type $\sigma_s(\overline{Y})$ and n-type $\tau_r(X, \overline{Y})$ we define the relationship $\sigma_s \vdash_i \tau_r$, meaning that σ_s ensures that τ_r is the n-type of the tree segment obtained by removing the subtree of the i-th successor of the root. This condition is expressible with a formula of quantifier rank $n+2$ as follows: (This explains the need for $(n+2)$-types.)

$$\begin{aligned} \sigma_s(\overline{Y}) \ \models\ \exists z \exists Z \Big(& \forall x(z \leq x) \wedge \\ & \exists y \Big(\ y \in S_i \ \wedge\ \forall x(x < y \to x = z) \ \wedge\ \forall x\, (x \in Z \leftrightarrow y \not\leq x) \Big) \wedge \\ & \tau_r^Z(\{z\}, \overline{Y}|_Z) \ \Big), \end{aligned}$$

where the superscript Z denotes relativization to Z. Finally, θ can be defined as promised by $\theta(I, \overline{Q}) =$

$$\exists \overline{P}\ \forall n \bigwedge_{\sigma_s \vdash_i \tau_r} (n \in Q_s \wedge \mathrm{s}(n) \in S_i \to n \in P_r) \ \wedge\ \theta_1(I, \overline{P}).$$

where $\mathrm{s}(n)$ refers to the immediate successor of n, which is of course definable, but used here in functional notation for brevity. □

D Formalizing Condition C – proofs

Proposition 18 *For every* MLO *formula $\varphi(X, \overline{Y})$ the assertion "$\exists^{\aleph_1} B$ branch$(B) \wedge \varphi(B, \overline{Y})$" is equivalent over all simple trees to the existence of a perfect set of branches B, each satisfying $\varphi(B, \overline{Y})$. The latter ensures that there are in fact continuum many such branches.*

Proof. Let $\psi(\overline{Y})$ be the MLO formula expressing that there is a prefix-closed set of nodes Λ, such that $(\Lambda, <)$ is a perfect tree and every infinite branch $B \subset \Lambda$ satisfies $\varphi(B, \overline{Y})$.

By definition of perfectness, ψ implies that there are uncountably many branches B satisfying $\varphi(B, \overline{Y})$ over any tree. As we have shown in Theorem 16, over the full binary tree with arbitrary additional unary predicates, ψ is equivalent to this condition. We transfer the result to all simple trees using an encoding of any simple tree in $\mathfrak{T}(2)$ with appropriate predicates, as follows.

Every simple l-tree $\mathfrak{T}$ is isomorphic to some $(T, \prec, P_1, \ldots, P_l)$ where $T \subseteq \mathbb{N}^*$ is a prefix-closed subset of finite sequences of natural numbers and $\prec$ is the prefix relation. Consider the following encoding $\mu : \mathbb{N}^* \to \{0,1\}^*$

$$(n_0, n_1, \ldots, n_s) \mapsto 0^{n_0}10^{n_1}1 \ldots 0^{n_s}1,$$

and set $S = \mu(T)$ and $Q_i = \mu(P_i)$ for each $i = 1 \ldots l$.

Given that $v \prec w$ in $\mathfrak{T}$ iff $\mu(v) \prec \mu(w)$ in $\mathfrak{T}(2)$, this defines an interpretation of $\mathfrak{T}$ inside $(\mathfrak{T}(2), S, Q_1, \ldots, Q_l)$. In particular, for every MLO-formula $\vartheta(\overline{X})$ of l-trees

$$\mathfrak{T} \models \vartheta(\overline{U}) \iff (\mathfrak{T}(2), S, Q_1, \ldots, Q_l) \models \vartheta^*(\mu(\overline{U})),$$

where ϑ^* is obtained from ϑ by interpreting each P_i with Q_i and relativizing all quantifiers to subsets/elements of S.

Observe that μ induces a function μ^* mapping each infinite branch B of $\mathfrak{T}$ to the unique infinite branch $\mu^*(B)$ of $\mathfrak{T}(2)$ containing $\mu(w)$ for all $w \in B$. Conversely, every infinite branch of $\mathfrak{T}(2)$ containing the μ-image of infinitely many nodes of $\mathfrak{T}$ is the μ^* image of the unique infinite branch of $\mathfrak{T}$ containing all of these nodes. Hence μ^* is injective (but not surjective).

Consider the formula $\varphi(B, \overline{Y})$ defining, with parameters $\overline{V}$ over $\mathfrak{T}$, an uncountable set of branches. Thus, over simple trees, it defines an uncountable set of infinite branches $\mathcal{D} = \{B \mid \mathfrak{T} \models \varphi(B, \overline{V})$ and B is an infinite branch$\}$.

Then, according to earlier remarks, $\mathcal{D}^* = \{\mu^*(B) \mid B \in \mathcal{D}\}$ is an uncountable set of branches of $\mathfrak{T}(2)$ and it is defined by "branch$(B) \wedge \exists$ infinite $P \subseteq B\ \varphi^*(P, \mu(\overline{V}))$" over $(\mathfrak{T}(2), S, Q_1, \ldots, Q_l)$.

Thus, by Theorem 16, there is a $\Lambda^* \subseteq \mathfrak{T}(2)$ inducing a perfect tree $(\Lambda^*, <)$, every infinite branch of which is in $\mathcal{D}^*$.

We claim that $\Lambda = \mu^{-1}(\Lambda^*)$ induces a perfect tree in $\mathfrak{T}$, every infinite branch of which is then in $\mathcal{D}$.

For one, because Λ^* is prefix-closed, so is Λ, therefore it induces a tree in $\mathfrak{T}$. We know moreover, as Λ^* is perfect, that the image $\mu^*(B)$ of every infinite branch $B \subset \Lambda$ is not isolated in Λ^*. Hence for every $w \in B$ there is an infinite branch $C^* \subset \Lambda^*$ different from $\mu^*(B)$ and such that $\mu(w) \in C^*$. Therefore $w \in \mu^{-1}(C^*)$, which is a branch through Λ and is different from B. This shows that B is not isolated in Λ, and so Λ is perfect. $\square$