

Is Polynomial Time Choiceless?

Erich Grädel and Martin Grohe

For Yuri Gurevich, on the occasion of his 75th birthday

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Is there a logic that captures PTIME?

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Informal definition: A logic L captures PTIME if it defines precisely those properties of finite structures that are decidable in polynomial time:

- (1) For every sentence $\psi \in L$, the set of finite models of ψ is decidable in polynomial time.
- (2) For every PTIME-property S of finite τ -structures, there is a sentence $\psi \in L$ such that $S = \{\mathfrak{A} \in \text{Fin}(\tau) : \mathfrak{A} \models \psi\}$.

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The precise definition, **due to Yuri**, is more subtle. It includes effectiveness requirements to exclude pathological ‘solutions’.

Is there a logic for PTIME?

Yuri has made many important contributions to this problem, including numerous studies on expressive power, model-theoretic properties, complexity of logics that arise in this context.

His precise formulation of what would really constitute a logic for PTIME is an indispensable prerequisite to all attempts to prove

Yuri's Conjecture: There is no logic for PTIME !

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We discuss the state of the art and survey some recent results on CPT.

First-Order Logic

First-order logic (FO) is far too weak to capture PTIME.

- FO can express only local properties of finite structures

Theorems of Gaifman and Hanf

Global properties (e.g. planarity of graphs) are not expressible.

- FO has no mechanism for recursion or unbounded iteration.

Transitive closures, reachability or termination properties, winning regions in games, etc. are not FO-definable.

- FO can only express properties in AC^0

AC^0 is constant parallel time with polynomial hardware. In particular, $FO \subseteq LOGSPACE$.

Second-Order Logic

Second-order logic (SO) is probably too strong to capture P_{TIME} .

Fagin's Theorem. Existential SO captures NP.

Corollary. SO captures the polynomial hierarchy.

Thus SO captures polynomial time if, and only if, $P = NP$.

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Monadic second-order logic is orthogonal to P_{TIME} :

On words, MSO captures the regular languages, and not all P_{TIME} -languages are regular.

On graphs, MSO can express NP-complete properties, such as 3-colourability.

Fixed-point logic with counting

(FP + C): Two-sorted fixed-point logic with counting terms.

Two sorts of variables:

- x, y, z, \dots ranging over the domain of the given finite structure
- μ, ν, \dots ranging over natural numbers

On natural numbers, standard operations $+$, \cdot and $<$ are available, but variables must be explicitly restricted to take only polynomially bounded values.

Counting terms: For a formula $\varphi(x)$, the term $\#_x \varphi(x)$ denotes the number of elements a of the structure that satisfy $\varphi(a)$.

Least or inflationary fixed point operator defining formulae of the form

$$[\mathbf{fp} \ R \bar{x} \bar{\mu}_{\leq \bar{t}} \cdot \psi(R, \bar{x}, \bar{\mu})](\bar{y}, \bar{\nu}).$$

Fixed-point logic with counting is close to PTIME

Fixed-point logic with counting is powerful enough to **express fundamental algorithmic techniques** (such as the ellipsoid method) and captures PTIME on many interesting classes of finite structures, including

- linearly ordered structures (Immerman, Vardi)
- trees (Immerman, Lander) and structures of bounded tree-width (Grohe, Marino)
- planar graphs and graphs of bounded genus (Grohe)
- chordal line graphs (Grohe) and interval graphs (Laubner)
- all classes of graphs that exclude a minor (Grohe)

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(FP+C) is the logic of reference in this area!

(see survey by A. Dawar, SIGLOG-News, 2015)

Fixed-point logic with counting versus polynomial time

Theorem. $(FP + C) \not\subseteq P_{TIME}$ (Cai, Fürer, Immerman 1992)

Fixed-point logic with counting versus polynomial time

Theorem. $(FP + C) \not\subseteq P_{TIME}$ (Cai, Fürer, Immerman 1992)

The CFI-construction separating P_{TIME} from $(FP+C)$ is interesting and sophisticated, but originally seemed somewhat artificial.

However, Atserias, Bulatov, and Dawar proved that it very closely related to the fundamental problem of solving linear equation systems over finite Abelian groups, rings, and fields.

Hard problems for (FP+C)

There are two main sources of natural polynomial-time problems that are hard for logic, and not in (FP+C):

(1) Tractable cases of isomorphism problems

- Structures of colour class size q : coloured by an ordered set such that at most q elements get the same colour
- Multipedes (Blass and Gurevich)
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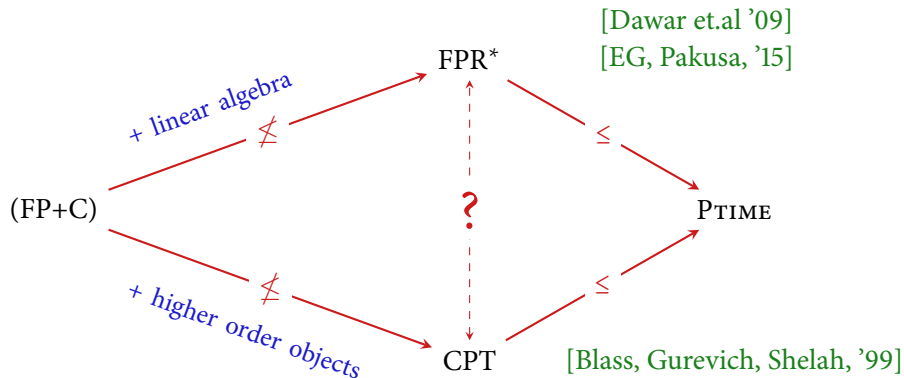
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(2) Problems from linear algebra and permutation group theory

- Solving linear equation systems
- Matrix rank over finite fields
- Permutation group membership problem

Candidates for a logic for PTIME



Fixed-point logic with rank

Rank logic FPR: Extend fixed-point logic by rank operators $\text{rk}_p \varphi$, to denote the rank (over the prime field \mathbb{F}_p) of the matrix defined by φ .

FPR has been proposed in 2009 by Dawar, Grohe, Holm, Laubner as a potential candidate for a logic for PTIME.

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Theorem. (EG, Pakusa, CSL 2015) Rank logic is dead, long live rank logic!

In its original form, FPR fails to capture PTIME, and must be replaced by a stronger variant, FPR^* , with a uniform rank operator, taking the prime as an additional input.

Open problem. Does FPR^* capture PTIME ?

Choiceless Polynomial Time

introduced by Blass, Gurevich, and Shelah in 1999

Idea. Model for efficient computation on abstract structures that preserves symmetries at every step of the computation. Disallow explicit choice, but permit essentially everything else, including fancy data structures and parallelism (explore all possible choices in parallel).

BGS-machines: model of abstract state machines that operate on hereditarily finite expansions $\text{HF}(\mathfrak{A})$ of finite structures \mathfrak{A}

$\text{HF}(\mathfrak{A})$ has universe consisting of

- **atoms:** the elements of \mathfrak{A}
- all **finite sets** of elements of $\text{HF}(\mathfrak{A})$ with set-theoretic operations such as $\emptyset, \in, \cup, \dots$ and a **cardinality operator**

BGS-logic

BGS-machines can be described and understood in logical terms

BGS-logic: terms $t(\bar{x})$ and formulae $\varphi(\bar{x})$ constructed via

- quantifier-free part of first-order logic
- basic set-theoretic operators $\emptyset, \in, \cup, \dots$
- a cardinality operator $x \mapsto |x|$ (as a von Neumann ordinal)
- comprehension terms of form $\{t(x) : x \in t' : \varphi(x)\}$
- an iteration operator $t(x)^*$

Evaluation: $[[t(x_1, \dots, x_n)]]^{\mathfrak{A}} : \text{HF}(\mathfrak{A})^n \rightarrow \text{HF}(\mathfrak{A})$,

For sentences of BGS-logic, we have $[[\varphi]]^{\mathfrak{A}} \in \{\text{false}, \text{true}\} = \{\emptyset, \{\emptyset\}\}$

Definition of choiceless polynomial time

For an iteration term $t(x)^*$, we get a sequence x_0, x_1, x_2, \dots of sets in $\text{HF}(\mathfrak{A})$ with $x_0 = \emptyset$ and $x_{i+1} := \llbracket t \rrbracket(x_i)$, and we let

$\llbracket t^* \rrbracket = x_\ell$ for the least ℓ with $x_{\ell+1} = x_\ell$ if it exists, and \emptyset otherwise

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A computation described in BGS-logic is a sequence of hereditarily finite sets.

Choiceless Polynomial time is the polynomial-time fragment of BGS-logic. It is the set of properties definable in BGS-logic such that

- all iterations have **polynomial length**
- only a **polynomial number of sets are activated**.

The power of choiceless polynomial time

CPT is a proper extension of (FP + C)

CPT can define any polynomial time property of small definable substructures X of the input structure \mathfrak{A} .

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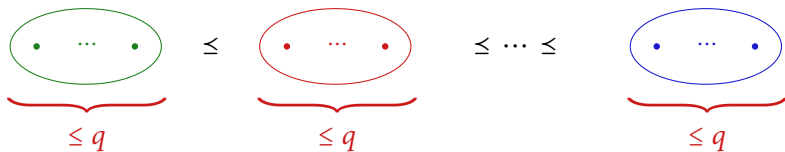
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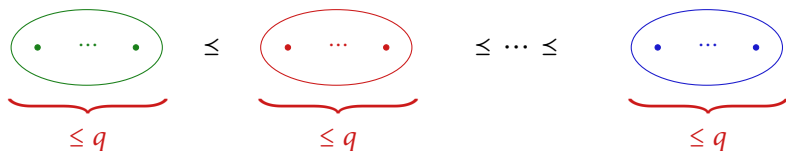
CPT (even without counting) can distinguish the CFI-graphs constructed from ordered graphs (so that the CFI-graphs themselves have a preorder)

CPT can solve certain systems of linear equations (with a preorder on the variables) that cannot be solved in (FP+C)

Structures of bounded colour class size



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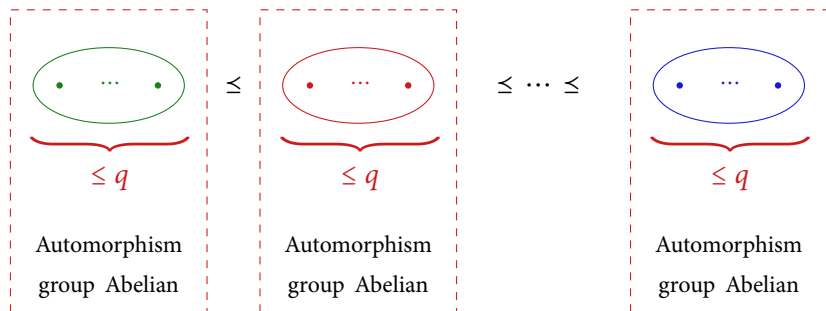


The isomorphism problem for q -bounded structures is solvable in PTIME, for any fixed $q \in \mathbb{N}$.

In general, such isomorphism problems are not (FP+C)-definable. CFI-graphs (over ordered input graphs) are 2-bounded structures.

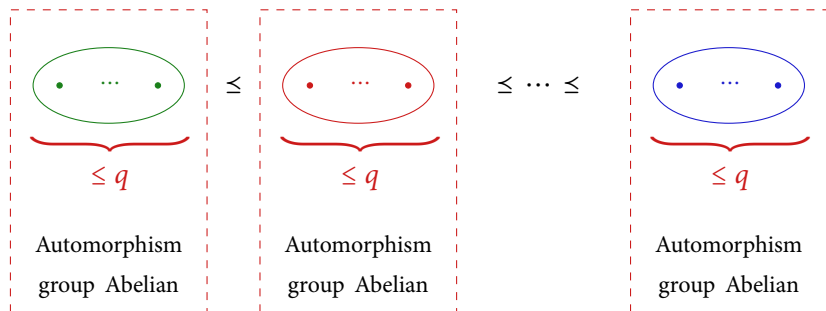
Challenge. Find CPT-canonization procedures for q -bounded structures.

Structures of bounded colour class size



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In particular this holds for 2-bounded structures, which also answers an open problem posed by Blass, Gurevich, and Shelah:

The isomorphism problem for multipedes is solvable in CPT.

Cyclic equation systems

Linear equation systems over an **ordered** set of variables can be solved in (FP+C). But classical solution algorithms require choice, and cannot be carried out in (FP+C) for **unordered sets of variables**.

Intermediate class: Cyclic equation systems (CES) over rings \mathbb{Z}_p^k

- pre-order \leq on variables
- \leq -equivalent variables are related by equations $x + a = y$
- fixing value of x in a solution fixes values of all variables in the same \leq -class

Theorem. Solvability of CES is definable in Choiceless Polynomial Time.

Problem. CPT cannot simultaneously fix one variable in each class since there would be exponentially many symmetric choices.

Solving CES with hyperterms

Hyperterms identify equivalent choices and encode equivalence class as hereditarily finite sets over the variables of the CES.

(Dawar, Richerby, Rossman)

Choiceless Polynomial Time is powerful enough to

- perform **arithmetic operations on hyper terms**
- translate CES into **ordered systems of hyper equations**
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Solving CES is an essential ingredient in the CPT-canonization procedure for q -bounded structures with Abelian colours.

For details, see **Phd-Thesis of Wied Pakusa (Aachen, 2015)**

Symmetric circuits

A circuit family $(C_n)_{n \in \mathbb{N}}$ decides a property of finite τ -structures if C_n takes as inputs the truth values of atomic τ -formulae of structures with universe $[n] = \{0, \dots, n-1\}$, and if it is **invariant under isomorphisms**.

Invariance: Any permutation of $[n]$ induces a permutation of the input gates of C_n . The result of the computation of C_n must be invariant under this.

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Translate any formula from FO or LFP into a circuit family $C = (C_n)_{n \in \mathbb{N}}$. Then this sequence is

p-uniform: The circuit C_n is polynomial-time computable in n

symmetric: Every permutation of $[n]$ induces an automorphism of C_n

Symmetric circuits are always invariant. The converse is not true.

Symmetric threshold circuits

For logics with counting it is natural to consider circuits with **threshold gates**.

The extension by threshold gates does not increase the power of polynomial-size circuits. But it can make a difference for restricted classes, such as bounded-depth circuits or symmetric circuits.

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Every formula in **(FP+C)** can be translated into a **p-uniform sequence of symmetric threshold circuits**.

Question. Can this also be done for Choiceless Polynomial Time?

Is there a circuit model for CPT?

Theorem (Anderson, Dawar)

p-uniform symmetric threshold circuits are equivalent to (FP+C).

Thus, translations from **Choiceless Polynomial Time** into equivalent sequences of **symmetric** threshold circuits cannot be **not p-uniform**.

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Thus, translations from **Choiceless Polynomial Time** into equivalent sequences of **symmetric** threshold circuits cannot be **not p-uniform**.

To put it differently, p-uniform translations from CPT into threshold circuits must **break symmetry** in some way. But how?

Challenge: Find a circuit model for CPT, based on a weaker notion of **symmetry**.

Anderson and Dawar suggest to require induced automorphisms of the circuits only for certain subgroups of the symmetric group on the input universe.

Choiceless polynomial time via interpretations

Idea: Replace the machinery of BGS-terms computing hereditarily finite sets by first-order interpretations.

Instead of a sequence of hereditarily finite sets, a computation then is a sequence of finite structures obtained by repeated application of a fixed first-order interpretation.

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Interpretations: A $\text{FO}[\tau, \sigma]$ -interpretation is a sequence

$$I = (\delta(\bar{x}), \varepsilon(\bar{x}, \bar{y}), (\varphi_R(\bar{x}_1, \dots, \bar{x}_{s(R)})_{R \in \sigma})$$

of $\text{FO}[\tau]$ -formulae. It maps a τ -structure \mathfrak{A} to a σ -structure

$$I(\mathfrak{A}) = (\delta^{\mathfrak{A}}, (\varphi_R^{\mathfrak{A}})_{R \in \sigma}) / \varepsilon^{\mathfrak{A}}$$

Notice that interpretations may change the size of the structures.

Computing by interpretations

Polynomial time interpretation Logic PIL^- : $\Pi = (I_{init}, I_{step}, \varphi_{halt}, \varphi_{out})$

- I_{init} is a $FO[\tau, \sigma]$ -interpretation defining from the input structure \mathfrak{A} an initial state $\mathfrak{A}_0 := I_{init}(\mathfrak{A})$
- I_{step} is a $FO[\sigma, \sigma]$ -interpretation defining from a state \mathfrak{A}_i the next state $\mathfrak{A}_{i+1} := I_{step}(\mathfrak{A}_i)$
- the run $\mathfrak{A}_0, \mathfrak{A}_1, \dots$ of Π in \mathfrak{A} terminates at the first state \mathfrak{A}_n with $\mathfrak{A}_n \models \varphi_{halt}$
- Π accepts \mathfrak{A} if the run terminates at state \mathfrak{A}_n with $\mathfrak{A}_n \models \varphi_{out}$

Explicit polynomial bounds on the length of the run and the size of all states are needed to get a polynomial-time variant.

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PIL^- is equivalent to CPT without counting.

PIL: Use $(\text{FO}+H)$ -interpretations, where H is the H\"artig quantifier.

Equivalence

Theorem $\text{CPT} \equiv \text{PIL}$ (EG, Kaiser, Pakusa, Schalthöfer, 2015)

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- **one-dimensional PIL** \equiv (FP+C)
- **one-dimensional PIL without counting** corresponds to $PFP \upharpoonright_{PTIME}$, which is equivalent to LFP if, and only if $PTIME = PSPACE$.
- **PIL without counting and without congruences** is equivalent to $while_{new} \upharpoonright_{PTIME}$.
- On structures of bounded colour-class size, **PIL without congruences** can be simulated by CPT-programs that access only sets of bounded rank. By Dawar, Richerby, and Rossman, this is too weak for the CFI-query.

Challenges for future research

A characterization of CPT without explicit polynomial bounds.

All known presentations of CPT need explicit polynomial bounds on length of iterations, number of active elements or size of interpreted structures. Find a presentation that guarantees this by construction, as in fixed-point logics.

Symmetric circuits for CPT

Find a circuit model for CPT. Understand better the symmetries inherent in CPT-computations.

CFI-graphs

Can the isomorphism problem for CFI-graphs constructed from **unordered** input graphs be solved in CPT ?

Actually this might be a candidate for separating CPT from PTIME

Challenges for future research

Constraint Satisfaction Problems (CSP)

Which CSPs are solvable in CPT?

The CSPs solvable in $(FP+C)$ are those with a property called bounded width. CPT can solve certain CSPs of unbounded width, such as cyclic equation systems (CES), which belong to the class of CSPs with Maltsev polymorphisms.

Can CPT solve **all** CSPs with Maltsev polymorphisms? This would imply that isomorphism of graphs with bounded colour class size is in CPT.

Challenges for future research

Choiceless Polynomial-Time versus Rank Logic

Besides CPT, logics with operators from linear algebra, such as the **rank logic** FPR^* , seem to be the most prominent candidates for a logic for PTIME.

The relationship between CPT and FPR^* is unclear but cyclic equation systems (CES) over rings might separate the two logics.

Conjecture. Solvability of CES over \mathbb{Z}_4 is definable in CPT but not in FPR^* .