

RTN GAMES presents

Once Upon a Time in the West

(Determinacy, Definability, and Complexity of Path Games)

Actors: Ego



Alter



Story: Dietmar Berwanger, Erich Grädel, Stephan Kreutzer

Pictures: Eva Gajdos

Once upon a time in the west, . . .

. . . two players, Ego and Alter, set out for an infinite ride.



They were not exactly friends, and each of them would try to get the best



for him and leave nothing for the other.

They were not exactly friends, and each of them would try to get the best



for him and leave nothing for the other.

Consequently, they would never agree on the route to take . . .

... but common goals,
and other reasons, forgotten
in the mist of time,



... but common goals,
and other reasons, forgotten
in the mist of time,



forced them
to stay together.



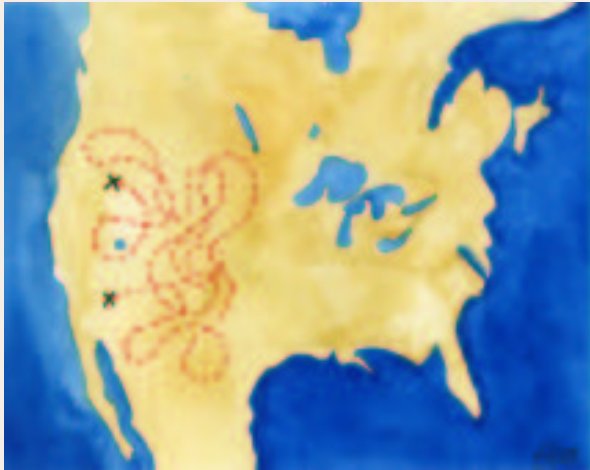
Rules

So they agreed on the following rules: Every morning, one of the two was to decide the day's route and destination. They would take turns in deciding the day's ride.

Rules

So they agreed on the following rules: Every morning, one of the two was to decide the day's route and destination. They would take turns in deciding the day's ride.

A player might choose a hard day's ride (as long as it is finite)



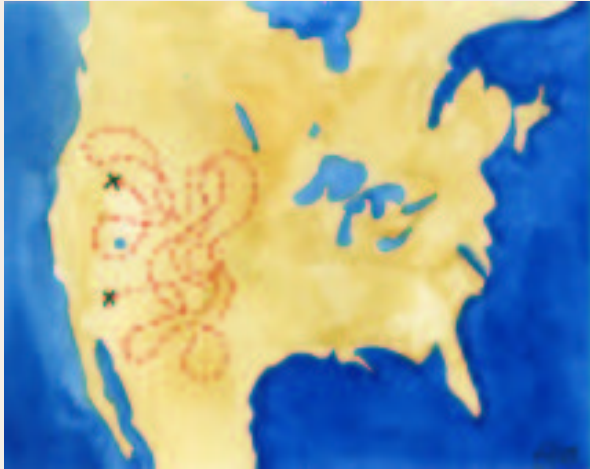
or take it easy



Rules

So they agreed on the following rules: Every morning, one of the two was to decide the day's route and destination. They would take turns in deciding the day's ride.

A player might choose a hard day's ride (as long as it is finite)



or take it easy



and make sure the day ends well



After ω days an infinite ride is completed and it is time for payoff!



Each player gets what he was able to collect during the ride. So his payoff depends on what they have seen, the infinite path they took!

(One of the reasons why they could never agree on the route.)

Path games with finite alternations

There were variants of this game, where after a certain number of days, one of the players would be eliminated (these things happened in the west),



and the other would complete the game alone (a very lonesome ride indeed)

Path games

Arena: (G, W) consisting of a game graph $G = (V, E, v_0)$ and a winning condition W : a set of infinite paths from v_0 through G

Players: Ego (E) and Alter (A)

Path games

Arena: (G, W) consisting of a game graph $G = (V, E, v_0)$ and a winning condition W : a set of infinite paths from v_0 through G

Players: Ego (E) and Alter (A)

Playing the game: The first player selects a finite path p_1 from v_0 ; the opponent extends p_1 to a path p_1q_1 ; then the first player prolongs this to $p_1q_1p_2$; and so on. All moves are non-empty and finite: $1 \leq |p_i|, |q_i| < \omega$.

Path games

Arena: (G, W) consisting of a game graph $G = (V, E, v_0)$ and a winning condition W : a set of infinite paths from v_0 through G

Players: Ego (E) and Alter (A)

Playing the game: The first player selects a finite path p_1 from v_0 ; the opponent extends p_1 to a path p_1q_1 ; then the first player prolongs this to $p_1q_1p_2$; and so on. All moves are non-empty and finite: $1 \leq |p_i|, |q_i| < \omega$.

Winning the game: After ω moves, an infinite path π is completed. Ego has won if $\pi \in W$, otherwise Alter has won.

Game prefixes

A game prefix $\gamma \in \{E, A\}^\omega$ indicates who begins and how many alternations are played. Obviously, $EE \equiv E$ and $AA \equiv A$. Hence, for any arena (G, W) we have the following games:

- $(EA)^\omega(G, W)$ and $(AE)^\omega(G, W)$:
path games with infinite alternations
- $(EA)^k E^\omega(G, W)$ and $A(EA)^k E^\omega(G, W)$:
games ending with infinite path extension by Ego
- $(AE)^k A^\omega(G, W)$ and $E(AE)^k A^\omega(G, W)$:
games where Alter chooses the final infinite lonesome ride

Comparing path games

$\mathcal{G} \succeq \mathcal{H}$ means that \mathcal{G} is better for Ego than \mathcal{H} :

Ego wins \mathcal{H} \implies Ego wins \mathcal{G}

Alter wins \mathcal{G} \implies Alter wins \mathcal{H}

$\mathcal{G} \equiv \mathcal{H}$ if $\mathcal{G} \succeq \mathcal{H}$ and $\mathcal{H} \succeq \mathcal{G}$

Comparing path games

$\mathcal{G} \succeq \mathcal{H}$ means that \mathcal{G} is better for Ego than \mathcal{H} :

Ego wins $\mathcal{H} \implies$ Ego wins \mathcal{G}

Alter wins $\mathcal{G} \implies$ Alter wins \mathcal{H}

$\mathcal{G} \equiv \mathcal{H}$ if $\mathcal{G} \succeq \mathcal{H}$ and $\mathcal{H} \succeq \mathcal{G}$

Problem: Do the game prefixes γ induce an infinite hierarchy (wrt. \succeq)?

Comparing path games

$\mathcal{G} \succeq \mathcal{H}$ means that \mathcal{G} is better for Ego than \mathcal{H} :

Ego wins $\mathcal{H} \implies$ Ego wins \mathcal{G}

Alter wins $\mathcal{G} \implies$ Alter wins \mathcal{H}

$\mathcal{G} \equiv \mathcal{H}$ if $\mathcal{G} \succeq \mathcal{H}$ and $\mathcal{H} \succeq \mathcal{G}$

Problem: Do the game prefixes γ induce an infinite hierarchy (wrt. \succeq)?

Obviously, $EAE^\omega(G, W) \succeq (EA)^k E^\omega(G, W)$.

Comparing path games

$\mathcal{G} \succeq \mathcal{H}$ means that \mathcal{G} is better for Ego than \mathcal{H} :

Ego wins $\mathcal{H} \implies$ Ego wins \mathcal{G}

Alter wins $\mathcal{G} \implies$ Alter wins \mathcal{H}

$\mathcal{G} \equiv \mathcal{H}$ if $\mathcal{G} \succeq \mathcal{H}$ and $\mathcal{H} \succeq \mathcal{G}$

Problem: Do the game prefixes γ induce an infinite hierarchy (wrt. \succeq)?

Obviously, $EAE^\omega(G, W) \succeq (EA)^k E^\omega(G, W)$.

But is $EAE^\omega(G, W)$ strictly better for Ego?

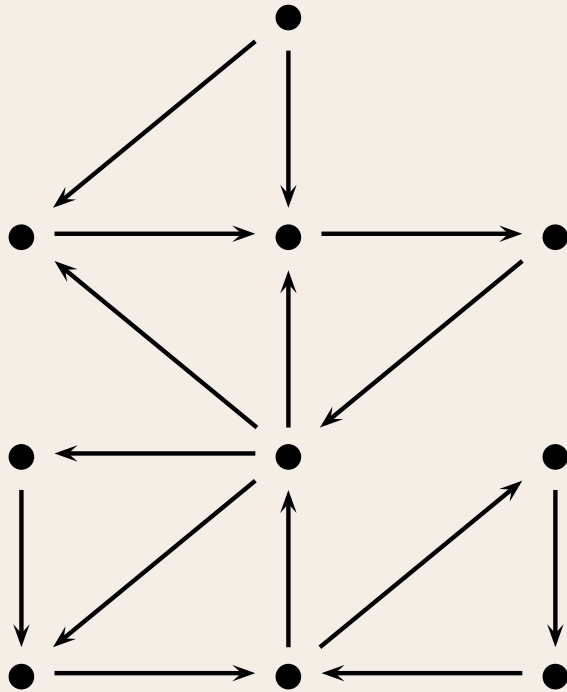
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

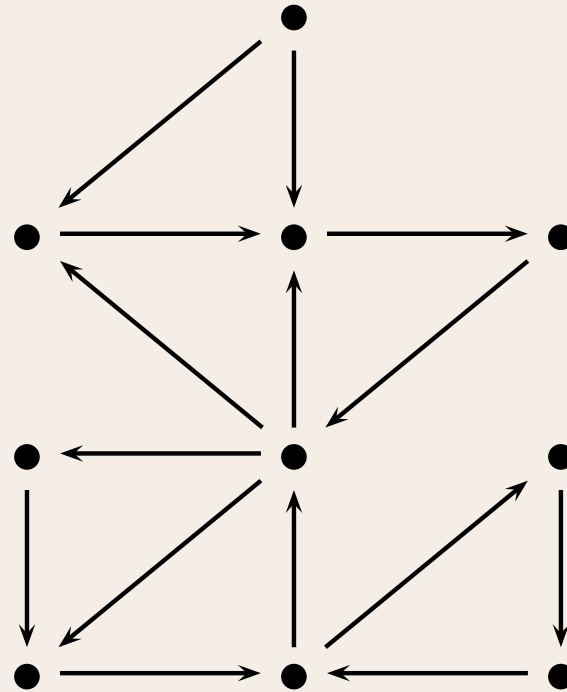
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



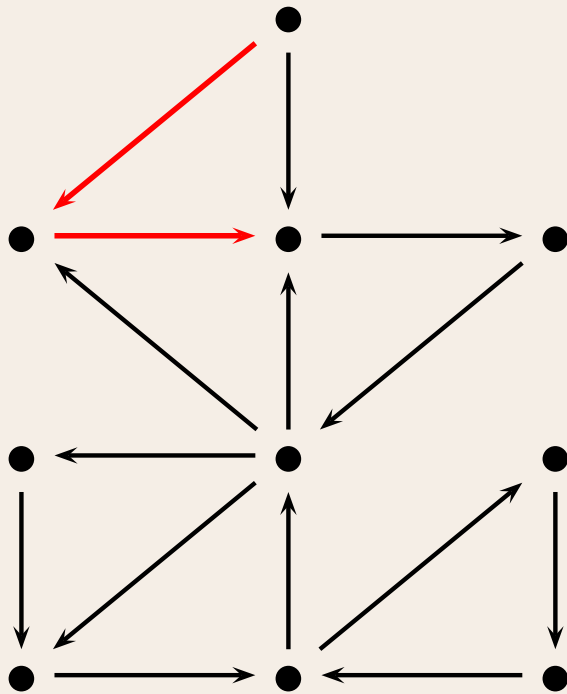
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

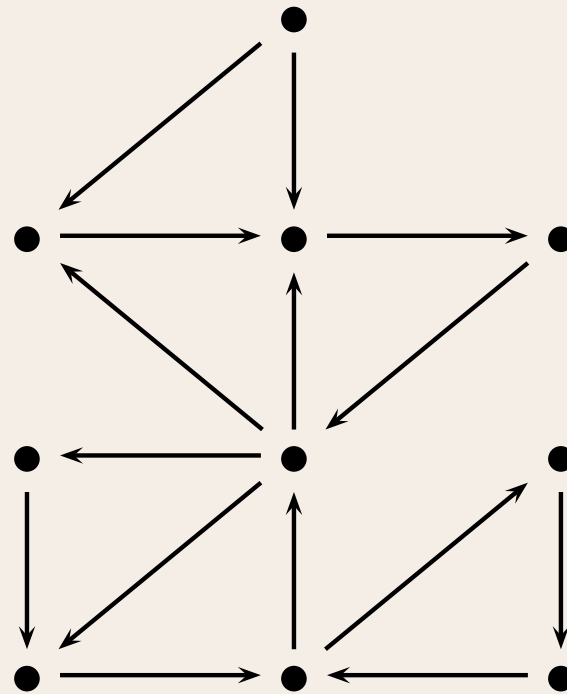
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



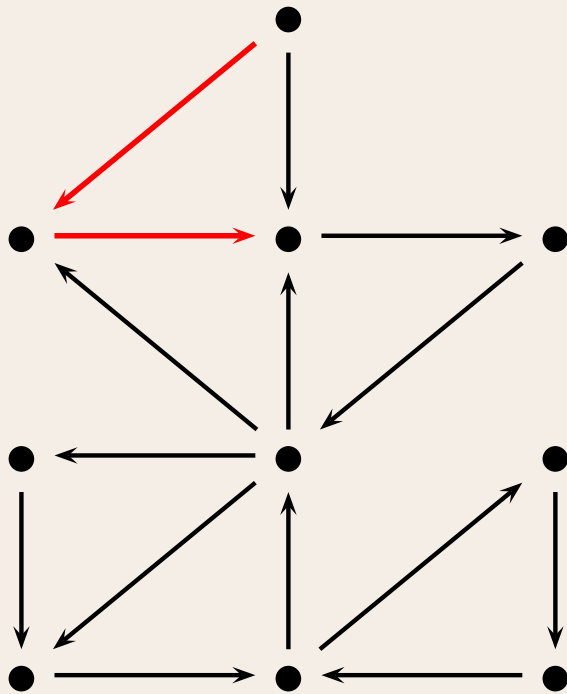
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

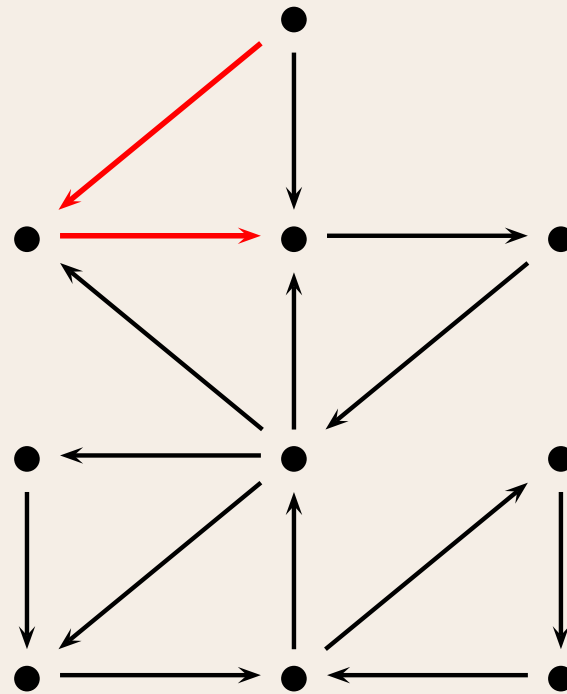
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



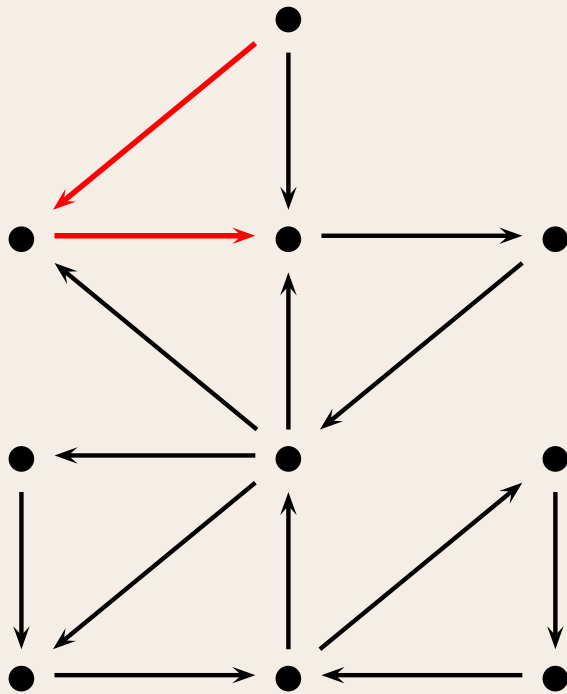
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

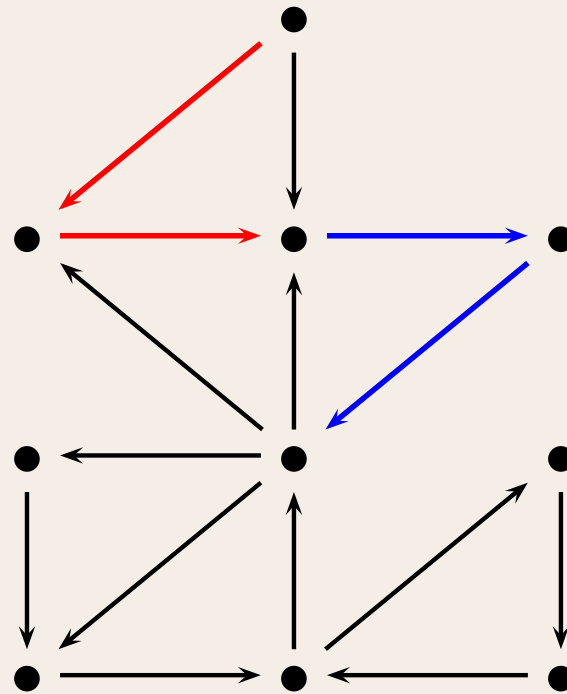
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



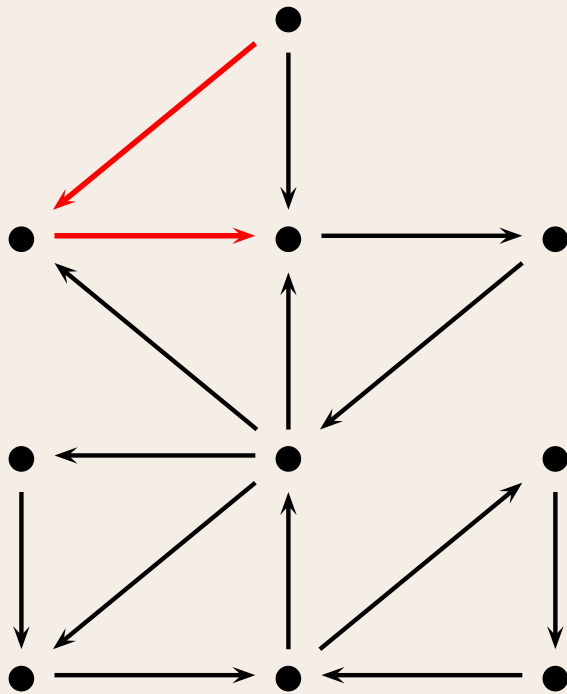
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

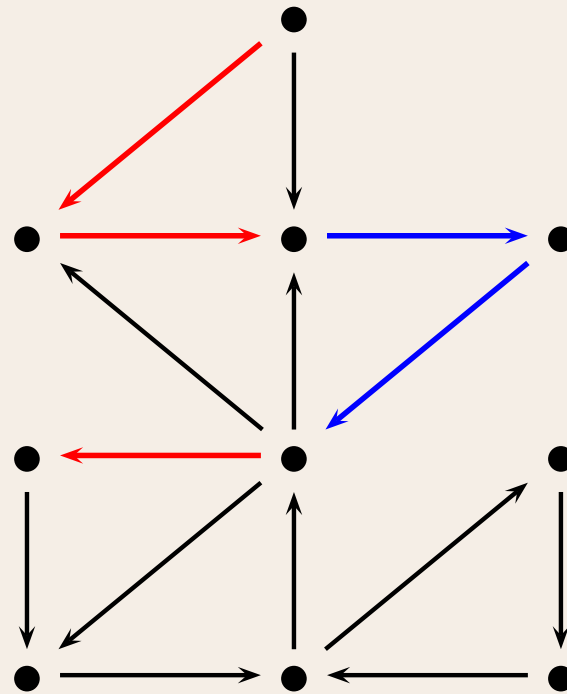
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



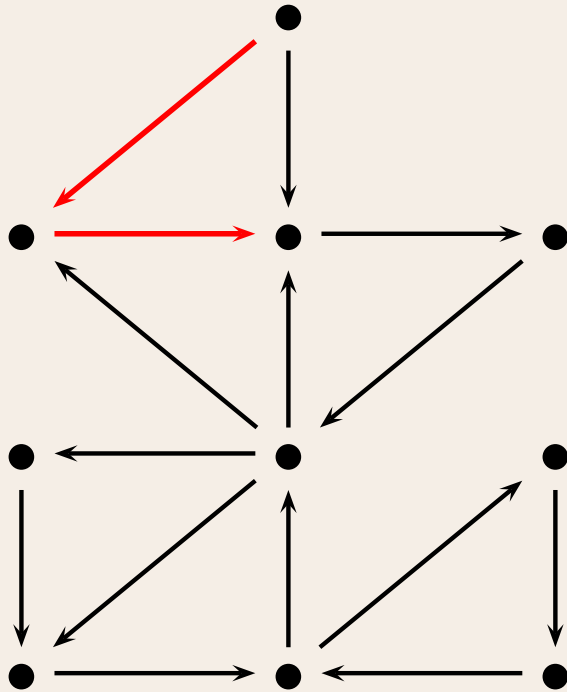
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

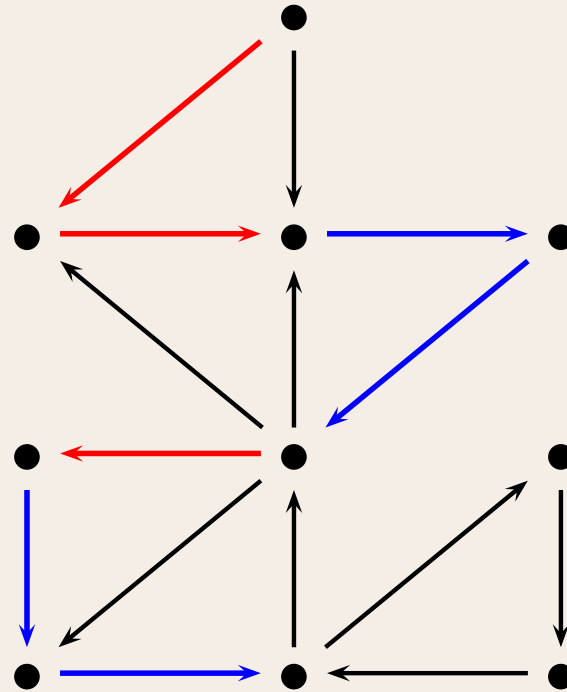
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



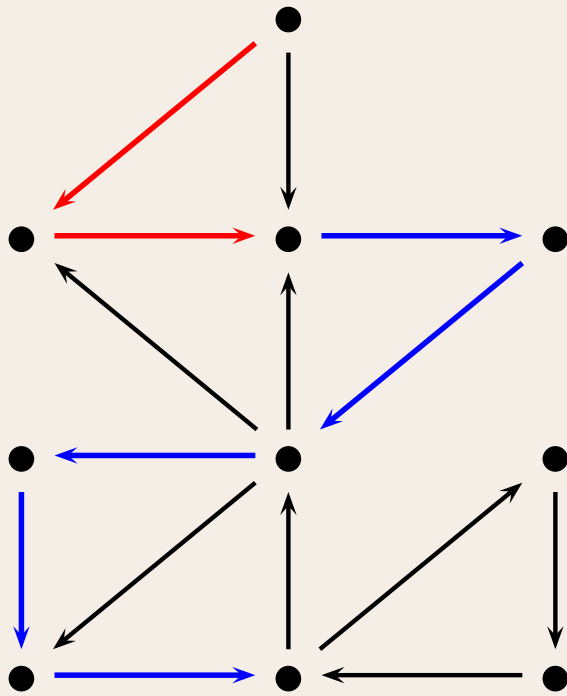
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

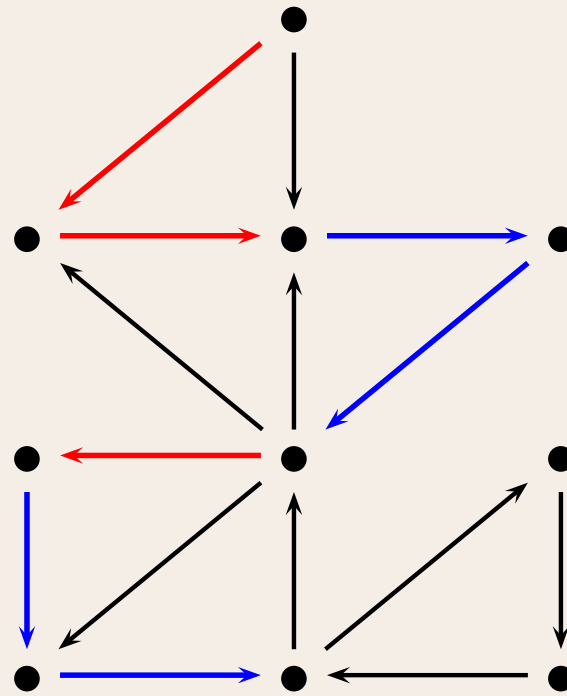
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



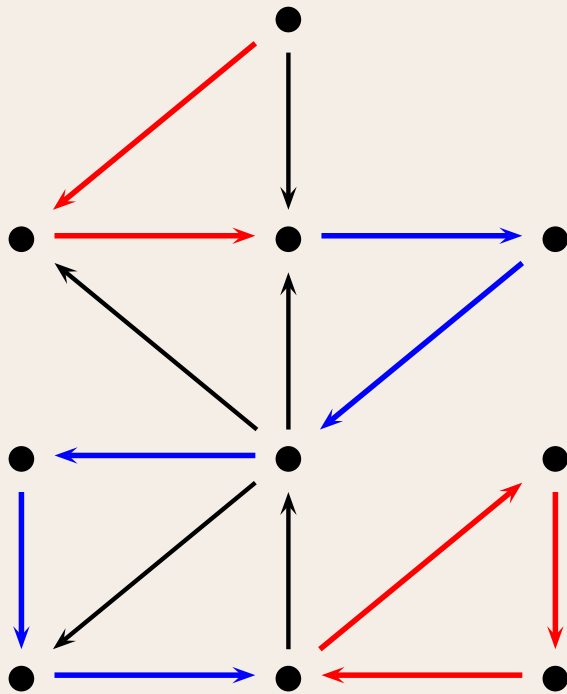
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

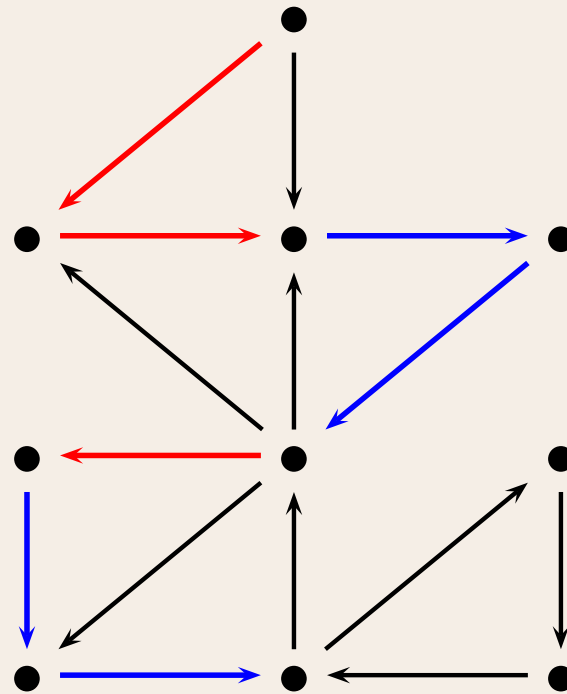
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



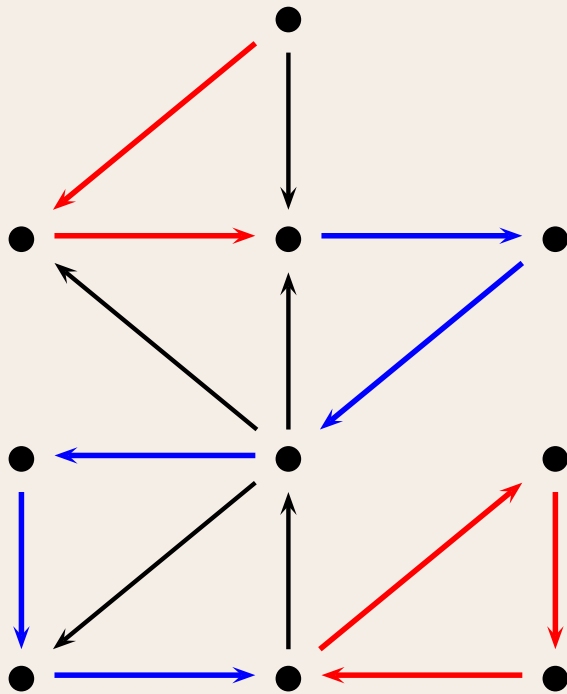
Game equivalence

Proposition. $EAE^\omega(G, W) \equiv (EA)^k E^\omega(G, W)$

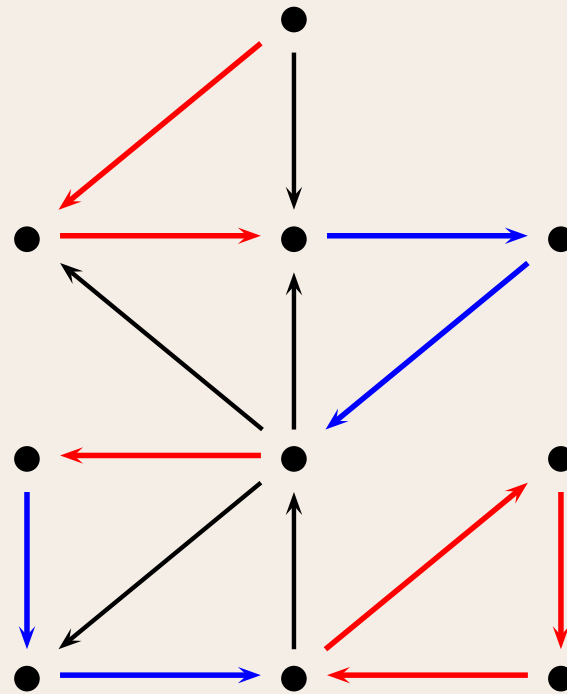
Proof. Ego wins $EAE^\omega(G, W) \implies$ Ego wins $(EA)^k E^\omega(G, W)$.

Let $k = 2$.

$EAE^\omega(G, W)$



$(EA)^2 E^\omega(G, W)$



Alternations: one, two, three, infinity

The hierarchy defined by the game prefixes collapses (Pistore/Vardi)

Theorem. For any game graph G and any winning condition W

$$E^\omega(G, W) \succeq EAE^\omega(G, W) \succeq AE^\omega(G, W)$$

$\mid \Upsilon$

$\mid \Upsilon$

$$(EA)^\omega(G, W) \succeq (AE)^\omega(G, W)$$

$\mid \Upsilon$

$\mid \Upsilon$

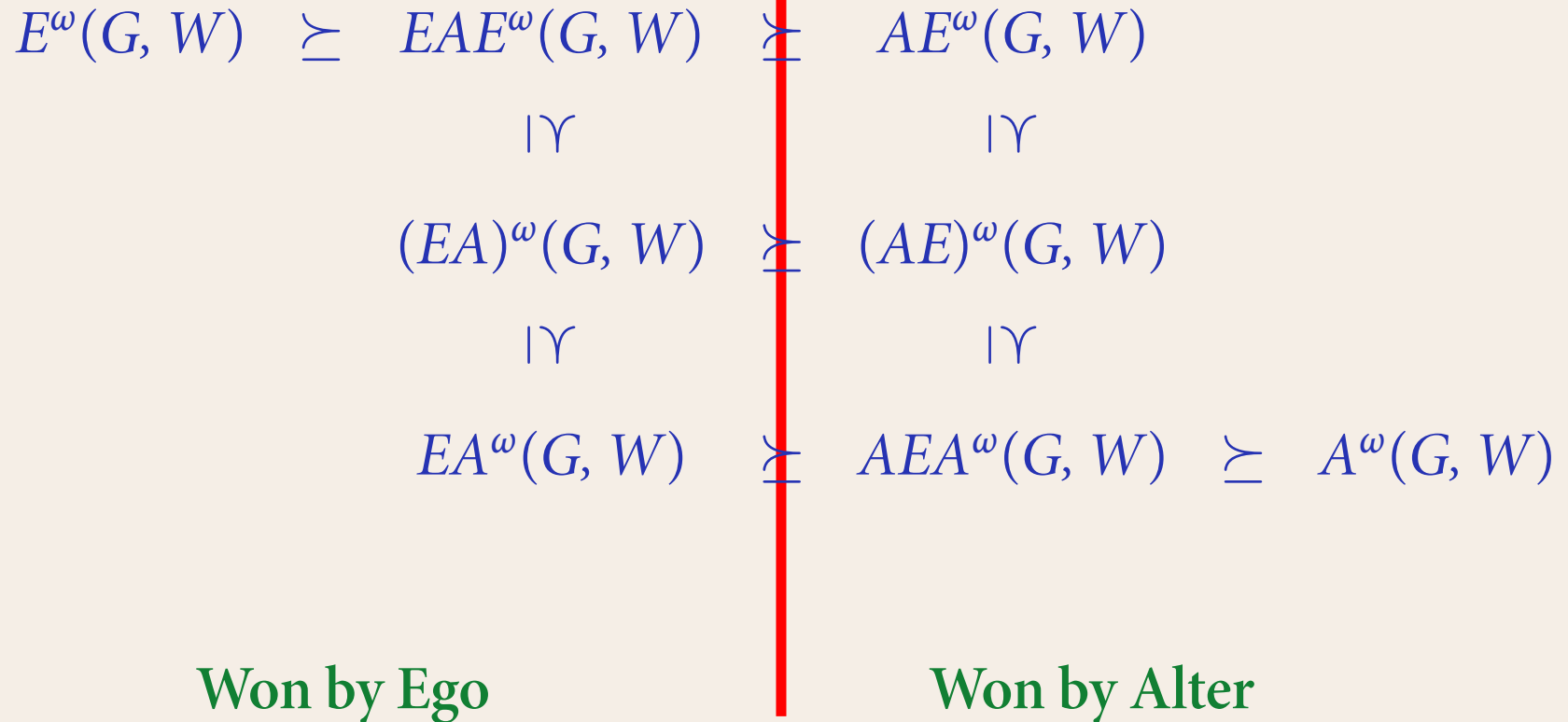
$$EA^\omega(G, W) \succeq AEA^\omega(G, W) \succeq A^\omega(G, W)$$

Every path game over (G, W) is equivalent to one of these eight games.

Remark. This holds for games with arbitrary payoff functions (that may take other values than 0 and 1). The games need not be determined.

Alternations: one, two, three, infinity

Start conditions: *Let's have a drink first.*



Alternations: one, two, three, infinity

Reachability conditions: *Some day, we'll have a drink.*

Guarantee conditions: *Every day, we have a drink.*

$$\begin{array}{ccccc} E^\omega(G, W) & \succeq & EAE^\omega(G, W) & \succeq & AE^\omega(G, W) \\ & & \text{I}\Upsilon & & \text{I}\Upsilon \\ (EA)^\omega(G, W) & \succeq & (AE)^\omega(G, W) & & \\ & & \text{I}\Upsilon & & \text{I}\Upsilon \\ EA^\omega(G, W) & \succeq & AEA^\omega(G, W) & \succeq & A^\omega(G, W) \end{array}$$

Alternations: one, two, three, infinity

Co-Büchi conditions:

Someday I will ride to the sunset and never come back.

$$E^\omega(G, W) \succeq EAE^\omega(G, W) \succeq AE^\omega(G, W)$$

$\downarrow \Upsilon$

$\downarrow \Upsilon$

Won by Ego

Won by Alter

$$(EA)^\omega(G, W) \succeq (AE)^\omega(G, W)$$

$\downarrow \Upsilon$

$\downarrow \Upsilon$

$$EA^\omega(G, W) \succeq AEA^\omega(G, W) \succeq A^\omega(G, W)$$

Alternations: one, two, three, infinity

Büchi conditions: *Again and again someone will play the harmonica.*

$$E^\omega(G, W) \succeq EAE^\omega(G, W) \succeq AE^\omega(G, W)$$

\Downarrow

\Downarrow

$$(EA)^\omega(G, W) \succeq (AE)^\omega(G, W)$$

\Downarrow

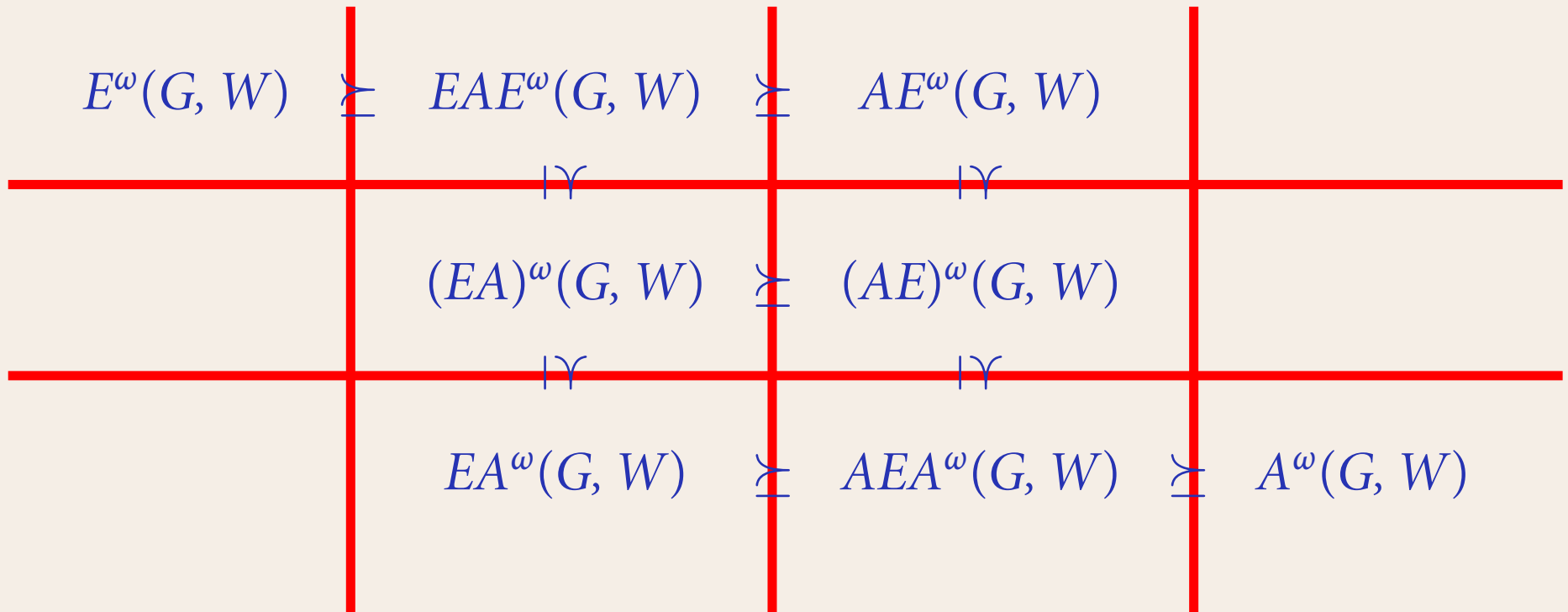
\Downarrow

Won by Ego

Won by Alter

$$EA^\omega(G, W) \succeq AEA^\omega(G, W) \succeq A^\omega(G, W)$$

Alternations: one, two, three, infinity



Path games in descriptive set theory

Banach-Mazur game $G^{**}(W)$:

(original version, see “*Mathematics from the Scottish Café*”): for a given winning condition $W \subseteq \mathbb{R}$, Ego first selects an interval $d_1 \subset \mathbb{R}$, then Alter chooses a subinterval $d_2 \subset d_1$, the Ego selects a further refinement $d_3 \subset d_2$, and so on . . .

Ego wins, iff $\bigcap_{n \in \omega} d_n$ contains an element of W .

Path games in descriptive set theory

Banach-Mazur game $G^{**}(W)$:

(original version, see “*Mathematics from the Scottish Café*”): for a given winning condition $W \subseteq \mathbb{R}$, Ego first selects an interval $d_1 \subset \mathbb{R}$, then Alter chooses a subinterval $d_2 \subset d_1$, the Ego selects a further refinement $d_3 \subset d_2$, and so on . . .

Ego wins, iff $\bigcap_{n \in \omega} d_n$ contains an element of W .

Banach-Mazur games are equivalent to path games on infinite trees:

$$G^{**}(W) \equiv (EA)^\omega(\mathcal{T}^\omega, W)$$

Path games in descriptive set theory

Banach-Mazur game $G^{**}(W)$:

(original version, see “*Mathematics from the Scottish Café*”): for a given winning condition $W \subseteq \mathbb{R}$, Ego first selects an interval $d_1 \subset \mathbb{R}$, then Alter chooses a subinterval $d_2 \subset d_1$, the Ego selects a further refinement $d_3 \subset d_2$, and so on . . .

Ego wins, iff $\bigcap_{n \in \omega} d_n$ contains an element of W .

Banach-Mazur games are equivalent to path games on infinite trees:

$$G^{**}(W) \equiv (EA)^\omega(\mathcal{T}^\omega, W)$$

Central issue in descriptive set theory: Characterise determined games by topological properties of the winning conditions.

Topology: Cantor space and Baire space

On B^ω , define a **topology** with **basic open sets** $O(x) := x \cdot B^\omega$, for $x \in B^*$.

- L is **open** $\iff L = W \cdot B^\omega$ (for some $W \subseteq B^*$)
- L is **closed** $\iff B^\omega - L$ is open $\iff L = [T]$
where $[T]$ is the set of infinite branches of a tree $T \subseteq B^*$.
- L is **nowhere dense** if the closure of L contains no non-empty open set
- L is **meager** if it is a countable union of nowhere dense sets.

For $B = \{0, 1\}$ this is **Cantor space**; for $B = \omega$ this is **Baire space**.

Examples:

- $O(x) = x \cdot B^\omega$ is **clopen** (both closed and open)
- in Cantor space, $1^*0\{0, 1\}^\omega$ is open, but not closed;
its complement $\{1^\omega\}$ is closed (and meager), but not open.

Borel sets

The class of Borel sets is the closure of the open sets under countable union and complementation.

Borel sets form a natural hierarchy of sets Σ_α^0 and Π_α^0 , for $1 \leq \alpha < \omega_1$.

The first levels of the Borel hierarchy:

- Σ_1^0 (or G) : the open sets
- Π_1^0 (or F) : the closed sets
- Σ_2^0 (or F_σ) : countable unions of closed sets
- Π_2^0 (or G_δ) : countable intersections of open sets

Determinacy of Banach-Mazur games

Theorem (Banach-Mazur) In the game $(EA)^\omega(T^\omega, W)$

- (1) Alter has a winning strategy $\iff W$ is meager.
- (2) Ego has a winning strategy \iff there exists a word $x \in \omega^*$ such that $(x \cdot \omega^\omega - W)$ is meager (W co-meager in a basic open set).

Determinacy of Banach-Mazur games

Theorem (Banach-Mazur) In the game $(EA)^\omega(T^\omega, W)$

- (1) Alter has a winning strategy $\iff W$ is meager.
- (2) Ego has a winning strategy \iff there exists a word $x \in \omega^*$ such that $(x \cdot \omega^\omega - W)$ is meager (W co-meager in a basic open set).

Corollary Path games with Borel winning conditions are determined.

(follows from the fact that Borel sets have the Baire property, i.e., their symmetric difference with some open set is meager)

Determinacy of Banach-Mazur games

Theorem (Banach-Mazur) In the game $(EA)^\omega(T^\omega, W)$

- (1) Alter has a winning strategy $\iff W$ is meager.
- (2) Ego has a winning strategy \iff there exists a word $x \in \omega^*$ such that $(x \cdot \omega^\omega - W)$ is meager (W co-meager in a basic open set).

Corollary Path games with Borel winning conditions are determined.

(follows from the fact that Borel sets have the Baire property, i.e., their symmetric difference with some open set is meager)

Remark. Standard winning conditions used in applications (e.g. all winning conditions in S1S) are in low levels of the Borel hierarchy.

Planning in nondeterministic domains

Planning domain: transition system $G := (V, (E_a)_{a \in A}, (P_b)_{b \in B})$

Planning goal: property of execution paths, specified by $\varphi \in \text{LTL}$

Plan: $\pi : V^* \rightarrow A$, assigns to each history an action

Execution tree: if the planning domain G is **deterministic**, then π defines a unique **execution path**. However, if G is **nondeterministic**, an action may have several outcomes, and a plan π then has not only one execution path, but an **execution tree** $\mathcal{T}_{G,\pi}$.

Planning in nondeterministic domains

Planning domain: transition system $G := (V, (E_a)_{a \in A}, (P_b)_{b \in B})$

Planning goal: property of execution paths, specified by $\varphi \in \text{LTL}$

Plan: $\pi : V^* \rightarrow A$, assigns to each history an action

Execution tree: if the planning domain G is **deterministic**, then π defines a unique **execution path**. However, if G is **nondeterministic**, an action may have several outcomes, and a plan π then has not only one execution path, but an **execution tree** $\mathcal{T}_{G,\pi}$.

It may be unrealistic to expect that **all** execution paths of a plan π satisfy the goal φ . On the other side, it is too optimistic to assume that a plan is good if just **one** execution path is consistent with φ .

Path games for planning

Pistore/Vardi study nondeterministic planning by means of path games: a plan π is good for the goal φ on domain G if Ego wins an associated game on the execution tree $\mathcal{T}_{G,\pi}$.

- **Weak planning**

There is a path in $\mathcal{T}_{G,\pi}$ that satisfies φ .

- **Strong planning**

Every path in $\mathcal{T}_{G,\pi}$ satisfies φ .

- **Strong cyclic planning**

Every partial execution of π can be extended to a successful path.

Path games for planning

Pistore/Vardi study nondeterministic planning by means of path games: a plan π is good for the goal φ on domain G if Ego wins an associated game on the execution tree $\mathcal{T}_{G,\pi}$.

- **Weak planning**

$$E^\omega(\mathcal{T}_{G,\pi}, \varphi)$$

There is a path in $\mathcal{T}_{G,\pi}$ that satisfies φ .

- **Strong planning**

$$A^\omega(\mathcal{T}_{G,\pi}, \varphi)$$

Every path in $\mathcal{T}_{G,\pi}$ satisfies φ .

- **Strong cyclic planning**

$$AE^\omega(\mathcal{T}_{G,\pi}, \varphi)$$

Every partial execution of π can be extended to a successful path.

Path games for planning

Pistore/Vardi study nondeterministic planning by means of path games: a plan π is good for the goal φ on domain G if Ego wins an associated game on the execution tree $\mathcal{T}_{G,\pi}$.

- **Weak planning**

$$E^\omega(\mathcal{T}_{G,\pi}, \varphi)$$

There is a path in $\mathcal{T}_{G,\pi}$ that satisfies φ .

- **Strong planning**

$$A^\omega(\mathcal{T}_{G,\pi}, \varphi)$$

Every path in $\mathcal{T}_{G,\pi}$ satisfies φ .

- **Strong cyclic planning**

$$AE^\omega(\mathcal{T}_{G,\pi}, \varphi)$$

Every partial execution of π can be extended to a successful path.

Theorem. The planning problem for LTL-goals, described by path games, can be solved by automata-based methods and is 2EXPTIME-complete.

Positional determinacy

A strategy is **positional** or **memoryless** if it only depends on the current position, and not on the history of the play.

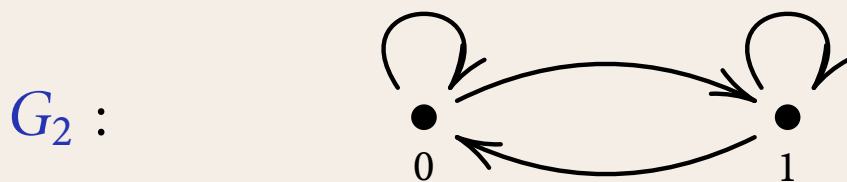
Proposition. If $W \in \Sigma_2^0$ (countable union of closed sets), and Ego has winning strategy for the game $(EA)^\omega(G, W)$, then he also has a positional winning strategy.

Positional determinacy

A strategy is **positional** or **memoryless** if it only depends on the current position, and not on the history of the play.

Proposition. If $W \in \Sigma_2^0$ (countable union of closed sets), and Ego has winning strategy for the game $(EA)^\omega(G, W)$, then he also has a positional winning strategy.

This is **not** always true for $W \in \Pi_2^0$:



$$W = \{\pi \in \{0, 1\}^\omega : (\forall m)(\exists n > m) |\{i < n : \pi(i) = 0\}| \geq n/2\}$$

(infinitely many initial segments of π have more zeros than ones)

Ego has a winning strategy for $EA^\omega(G_2, W)$, but no positional one.

Muller, parity, and S1S winning conditions

Game graph $\mathcal{G} = (V, E)$ with colouring $\lambda : V \rightarrow \{0, \dots, d-1\}$.

Logical winning conditions: given by formula φ in some logic on infinite paths, with predicates $\lambda(v) = i$ ($i < d$), such as

- **S1S:** monadic second-order logic on infinite paths;
- **LTL**, or equivalently, first-order logic $\text{FO}(<)$.

Muller, parity, and S1S winning conditions

Game graph $\mathcal{G} = (V, E)$ with colouring $\lambda : V \rightarrow \{0, \dots, d-1\}$.

Logical winning conditions: given by formula φ in some logic on infinite paths, with predicates $\lambda(v) = i$ ($i < d$), such as

- **S1S:** monadic second-order logic on infinite paths;
- **LTL**, or equivalently, first-order logic $\text{FO}(<)$.

Muller condition: given by collection $\mathcal{F} \subseteq \mathcal{P}(\{0, \dots, d-1\})$.

An infinite play $\pi = v_0 v_1 v_2 \dots$ is won by Ego if

$$\text{Inf}(\pi) := \{c : (\forall i)(\exists j > i)\lambda(v_j) = c\} \in \mathcal{F}.$$

Muller, parity, and SIS winning conditions

Game graph $\mathcal{G} = (V, E)$ with colouring $\lambda : V \rightarrow \{0, \dots, d-1\}$.

Logical winning conditions: given by formula φ in some logic on infinite paths, with predicates $\lambda(v) = i$ ($i < d$), such as

- **SIS:** monadic second-order logic on infinite paths;
- **LTL**, or equivalently, first-order logic $\text{FO}(<)$.

Muller condition: given by collection $\mathcal{F} \subseteq \mathcal{P}(\{0, \dots, d-1\})$.

An infinite play $\pi = v_0 v_1 v_2 \dots$ is won by Ego if

$$\text{Inf}(\pi) := \{c : (\forall i)(\exists j > i)\lambda(v_j) = c\} \in \mathcal{F}.$$

Parity condition: Ego wins if the least colour seen infinitely often is even.

Positional determinacy of classical graph games

For the more common, single step games on graphs

- **parity games** are positionally determined
- Positional strategies do **not** suffice for **Muller games**.

Example:

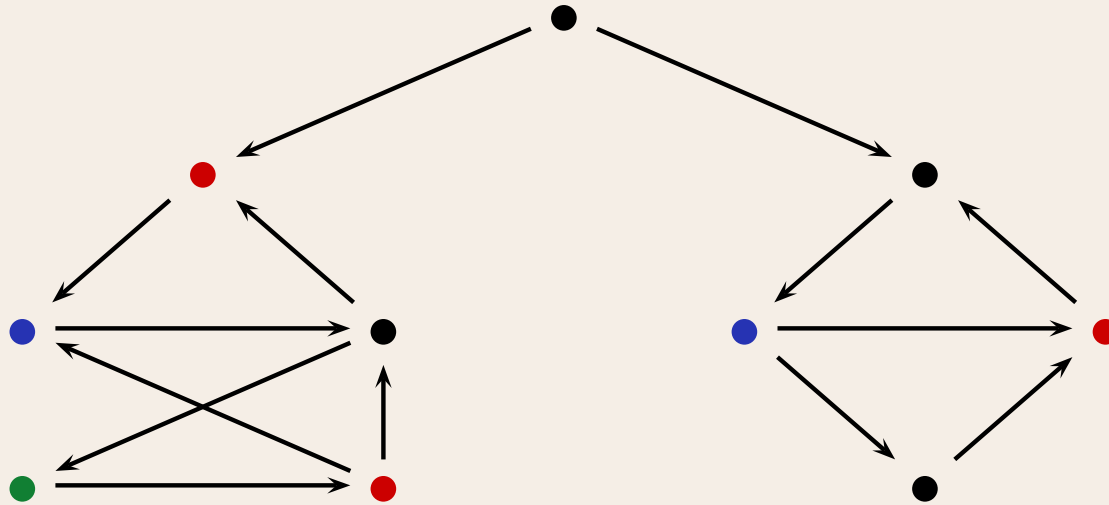


winning condition: all positions must occur infinitely often

Positional Determinacy of Muller Path Games

Proposition. Muller path games $(EA)^\omega(G, \mathcal{F})$ are positionally determined.

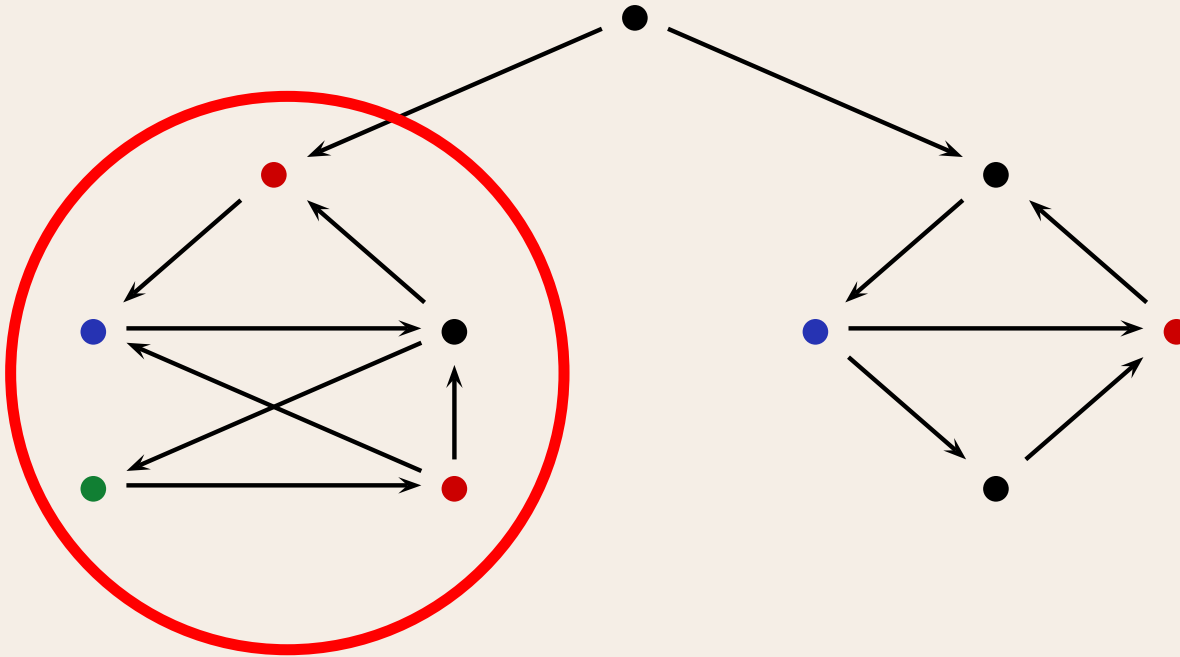
Proof. Decompose G into its strongly connected components



Positional Determinacy of Muller Path Games

Proposition. Muller path games $(EA)^\omega(G, \mathcal{F})$ are positionally determined.

Proof. Decompose G into its strongly connected components



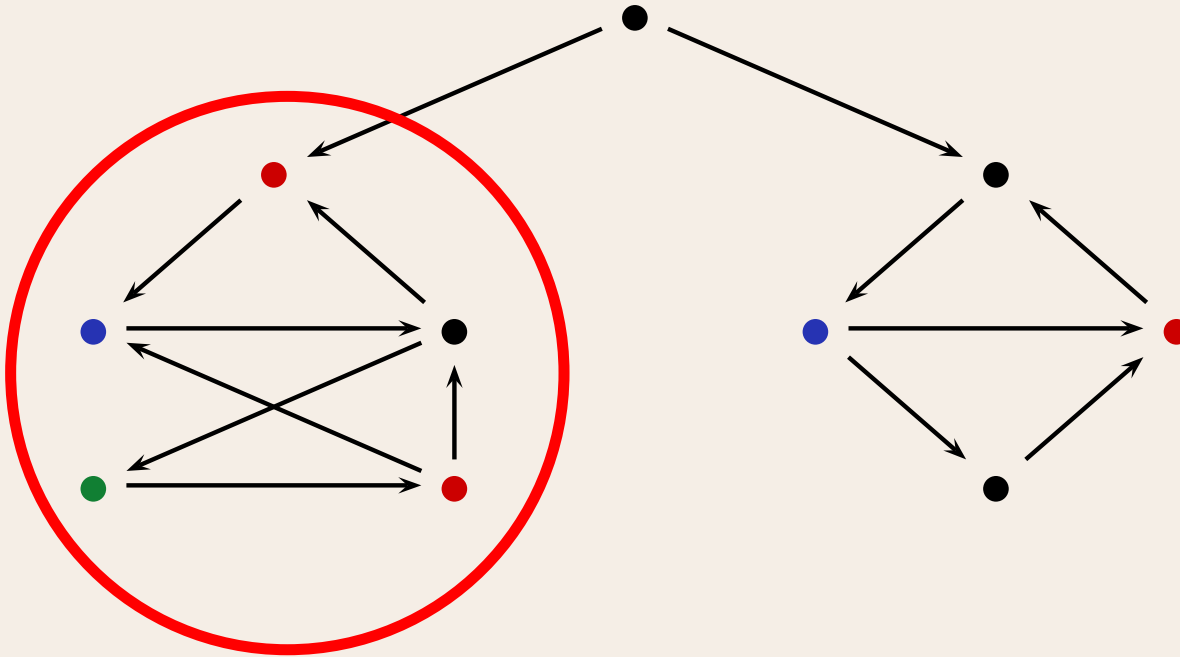
Ego wins iff there is a leaf component C such that $\lambda(C) \in \mathcal{F}$.

But then he also wins with a positional strategy.

Positional Determinacy of Muller Path Games

Proposition. Muller path games $(EA)^\omega(G, \mathcal{F})$ are positionally determined.

Proof. Decompose G into its strongly connected components



Ego wins iff there is a leaf component C such that $\lambda(C) \in \mathcal{F}$.

But then he also wins with a positional strategy.

Corollary. Muller path games can be solved in time $O(|G| \cdot |\mathcal{F}|)$.

Path games with S1S winning conditions

Theorem.

- All S1S path games $\gamma(G, \varphi)$ are determined via **finite memory strategies**.

Path games with S1S winning conditions

Theorem.

- All S1S path games $\gamma(G, \varphi)$ are determined via **finite memory strategies**.
- For every $\varphi \in \text{S1S}$, the games $(EA)^\omega(G, \varphi)$ and $(AE)^\omega(G, \varphi)$ are **positionally determined**.

Path games with S1S winning conditions

Theorem.

- All S1S path games $\gamma(G, \varphi)$ are determined via **finite memory strategies**.
- For every $\varphi \in \text{S1S}$, the games $(EA)^\omega(G, \varphi)$ and $(AE)^\omega(G, \varphi)$ are **positionally determined**.
- For **future** conditions $\varphi \in \text{S1S}$, all path games $\gamma(G, \varphi)$ are **positionally determined**.

future conditions: invariant under changes of finite initial segments

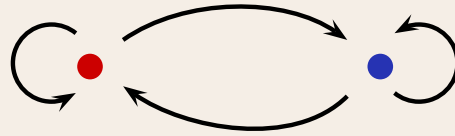
Path games with S1S winning conditions

Theorem.

- All S1S path games $\gamma(G, \varphi)$ are determined via **finite memory strategies**.
- For every $\varphi \in \text{S1S}$, the games $(EA)^\omega(G, \varphi)$ and $(AE)^\omega(G, \varphi)$ are **positionally determined**.
- For **future** conditions $\varphi \in \text{S1S}$, all path games $\gamma(G, \varphi)$ are **positionally determined**.
future conditions: invariant under changes of finite initial segments
- For any game prefix γ with **finite alternations** there exist games $\gamma(G, \varphi)$ with $\varphi \in \text{S1S}$ that do **not** admit positional winning strategies.

Games that are not positionally determined

Example. $\varphi :=$ “the number of red nodes is odd (and finite)”



For $\gamma \in \{E^\omega, AE^\omega, EAE^\omega\}$, the game $\gamma(G, \varphi)$ (starting from the blue node) is not positionally determined.

Definability of path games

Question: How relates the logical complexity of defining the winner of a path game to the logical complexity of the winning condition ?

Definability of path games

Question: How relates the logical complexity of defining the winner of a path game to the logical complexity of the winning condition ?

From a game prefix γ and a formula φ (e.g. from LTL or S1S) defining a winning condition, we obtain a game formula $\gamma.\varphi$, with

$$G \models \gamma.\varphi \quad \iff \quad \text{Ego wins the path game } \gamma(G, \varphi).$$

Definability of path games

Question: How relates the logical complexity of defining the winner of a path game to the logical complexity of the winning condition ?

From a game prefix γ and a formula φ (e.g. from LTL or S1S) defining a winning condition, we obtain a game formula $\gamma.\varphi$, with

$$G \models \gamma.\varphi \iff \text{Ego wins the path game } \gamma(G, \varphi).$$

This maps a logic L on paths to a logic $\gamma.L := \{\gamma.\varphi : \varphi \in L\}$ on graphs.

Definability of path games

Question: How relates the logical complexity of defining the winner of a path game to the logical complexity of the winning condition ?

From a game prefix γ and a formula φ (e.g. from LTL or S1S) defining a winning condition, we obtain a game formula $\gamma.\varphi$, with

$$G \models \gamma.\varphi \iff \text{Ego wins the path game } \gamma(G, \varphi).$$

This maps a logic L on paths to a logic $\gamma.L := \{\gamma.\varphi : \varphi \in L\}$ on graphs.

Question: Expressive power of γ .LTL and γ .S1S, compared to common logics on (game) graphs, like μ -calculus, CTL*, FO, and MSO ?

Definability theorem for path games

Theorem. For any game prefix γ

$$(1) \quad \gamma. \text{S1S} \leq L_\mu$$

$$(2) \quad \gamma. \text{LTL} \equiv \gamma. \text{FO} \leq \text{CTL}^*$$

That is, the winner of any path game with S1S resp. LTL winning condition is definable in the modal μ -calculus resp. CTL*.

Simplification via bisimulation invariance

It suffices to prove, that on **trees**,

$$(1) \quad \gamma. S1S \leq MSO$$

$$(2) \quad \gamma. FO \leq MPL$$

MPL is **monadic path logic**: MSO on trees with second-order quantification restricted to (finite or infinite) paths.

Simplification via bisimulation invariance

It suffices to prove, that on **trees**,

$$(1) \quad \gamma. \text{S1S} \leq \text{MSO}$$

$$(2) \quad \gamma. \text{FO} \leq \text{MPL}$$

MPL is **monadic path logic**: MSO on trees with second-order quantification restricted to (finite or infinite) paths.

- winning a (path) game is **invariant under bisimulation**
- bisimulation-invariant MSO $\equiv L_\mu$ (Janin/Walukiewicz)
- bisimulation-invariant MPL $\equiv \text{CTL}^*$ (Hafer/Thomas)
(Moller/Rabinovitch)

The simple case: games with finite alternations

Ego wins $EAE^\omega(\mathcal{T}, \varphi) \iff \mathcal{T} \models \psi$, where

$$\psi := (\exists X . X \text{ finite path})(\forall Y . X \subseteq Y \wedge Y \text{ finite path}) \\ (\exists Z . Y \subseteq Z \wedge Z \text{ infinite path} \wedge \varphi|_Z)$$

$\varphi|_Z := \varphi$ relativized to the path Z

- $\varphi \in \text{S1S} \implies \psi \in \text{MSO}$
- $\varphi \in \text{LTL} \implies \psi \in \text{MPL}$

The slightly harder case: games with infinite alternations

Winning strategy of Ego for $(EA)^\omega(\mathcal{T}, \varphi)$ on tree $\mathcal{T} = (V, E)$:
described by set $X \subseteq V$ with

- X is non-empty
- for all $x \in X$ and $y > x$ there is a $z > y$ with $z \in X$
- every path through \mathcal{T} hitting X infinitely often satisfies φ

The slightly harder case: games with infinite alternations

Winning strategy of Ego for $(EA)^\omega(\mathcal{T}, \varphi)$ on tree $\mathcal{T} = (V, E)$:
described by set $X \subseteq V$ with

- X is non-empty
- for all $x \in X$ and $y > x$ there is a $z > y$ with $z \in X$
- every path through \mathcal{T} hitting X infinitely often satisfies φ

Obviously, this can be formalised in MSO, if $\varphi \in \text{S1S}$.

The slightly harder case: games with infinite alternations

Winning strategy of Ego for $(EA)^\omega(\mathcal{T}, \varphi)$ on tree $\mathcal{T} = (V, E)$:
described by set $X \subseteq V$ with

- X is non-empty
- for all $x \in X$ and $y > x$ there is a $z > y$ with $z \in X$
- every path through \mathcal{T} hitting X infinitely often satisfies φ

Obviously, this can be formalised in **MSO**, if $\varphi \in \mathbf{S1S}$.

For $\varphi \in \mathbf{FO}$, we have to formalise in **MPL**.

In fact we can even formalise in **FO** !

Normal form for FO:

on infinite paths, every first-order formula is equivalent to

$$\bigvee_i \left(\exists x (\forall y \geq x) \varphi_i \wedge \forall x (\exists y \geq x) \vartheta_i \right)$$

where φ_i and ϑ_i contain only **bounded quantifiers** ($Qz \leq y$).

In terms of **LTL**: Every LTL-formula is equivalent to a disjunction of formulae $(FG\varphi \wedge GF\vartheta)$, where φ and ϑ are **past**-formulae.

Use this to show that on trees,

$$(EA)^\omega. \text{FO} \leq \text{FO} \quad \text{and} \quad (AE)^\omega. \text{FO} \leq \text{FO}.$$

Conclusion

- Path games are a natural kind of games on graphs.
- They arose in the wild west, but also in descriptive set theory and for planning in non-deterministic domains.
- The hierarchy defined by game prefixes collapses to eight different games.
- Many path games with SIS winning conditions (in particular all Muller games) are positionally determined.
- The winner of LTL path games can be defined in CTL*.

The End

