Finite presentations of infinite structures: automata and interpretations

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(joint work with Achim Blumensath)

Logical definability versus computational complexity

Important issue in many fields:

- finite model theory
- databases
- verification
- complexity theory
- knowledge representation
- . . .

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Considerable efforts to extend methodology to relevant classes of infinite structures

- infinite databases: spatial databases, constraint databases, . . .
- verification for systems with infinite state spaces
- model theory of finitely presented structures

Computational model theory

extends approach and methods of finite model theory to suitable classes of infinite structures

- finite presentations of infinite structures
- complexity of model checking problems
- capturing complexity classes
- model theoretic constructions
- games

Computational model theory

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Effective semantics (for relevant logic L): Given $\psi \in L$ and a (presentation of) $\mathfrak{A} \in \mathcal{D}$ it should be decidable whether $\mathfrak{A} \models \psi$. That is, model checking of L on $\mathcal{D}$ must be effective.

Other possibly relevant conditions (depending on context):

Closure: For $\mathfrak{A} \in \mathcal{D}$ and $\psi(\bar{x}) \in L$, also the expanded structure $(\mathfrak{A}, \psi^{\mathfrak{A}})$ is in $\mathcal{D}$.

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Effective query evaluation: Given a presentation of $\mathfrak{A} \in \mathcal{D}$ and a formula $\psi(\bar{x}) \in L$ one can effectively compute a presentation of $(\mathfrak{A}, \psi^{\mathfrak{A}})$.

Note: contrary to finite structures, query evaluation does not necessarily reduce to model checking.

Outline of this talk

- survey on different classes of finitely presented structures
- structures presented by interpretations
- structures presented by automata
- automatic groups
- algorithmic problems for automatic structures
- characterizing automatic structures by interpretations

Finitely presentable structures

- **recursive structures**
- **tree-interpretable structures**
  - context-free graphs
  - HR-equational and VR-equational graphs
  - prefix-recognizable graphs
- **tree constructible structures**
- **automatic structures, automatic groups, ω-automatic structures**
- other classes with finite presentations
  - tree-automatic structures, rational structures
  - ground tree rewriting graphs
  - constraint databases
  - metafinite structures

Recursive structures

Countable structures $\mathfrak{A} = (A, f_1, \dots, f_m, R_1, \dots, R_n)$ with computable functions and decidable relations

Long tradition in model theory since 1960s

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Some work studying finite model theory issues for recursive structures

- failure of classical results (compactness, completeness, interpolation, Beth,. . .) on recursive structures. (Stolboushkin, Hirst–Harel)
- descriptive complexity (mostly on non-recursive levels) (Hirst–Harel)
- 0-1 laws (Hirst–Harel, G.–Malmström)

Interpretations

$\mathfrak{A}$ σ-structure, L logic, $\mathfrak{B}$ τ-structure

(k-dimensional) $L[\tau,\sigma]$-interpretation : sequence

$$I = \langle \underbrace{D(\overline{x})}_{\text{domain formula}}, \underbrace{E(\overline{x},\overline{y})}_{\text{equality formula}}, \underbrace{(\varphi_R(\overline{x}_1,\dots,\overline{x}_r))_{R\in\sigma}}_{\text{formulae defining relations}} \rangle$$

of $L[\tau]$-formulae (where $\overline{x}, \overline{u}, \overline{x}_i$ are k-tuples of variables)

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domain formula — equality formula — formulae defining relations

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I interprets $\mathfrak{A}$ in $\mathfrak{B}$ (in short $I(\mathfrak{B}) = \mathfrak{A}$) if I defines a copy of $\mathfrak{A}$ inside $\mathfrak{B}$.

$$h : \quad I(\mathfrak{B}) := \langle D^{\mathfrak{B}}, (\varphi_R^{\mathfrak{B}})_{R\in\sigma}\rangle / E^{\mathfrak{B}} \quad \xrightarrow{\sim} \quad \mathfrak{A}$$

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$\mathfrak{A} \leq_L \mathfrak{B}$: there exists L-interpretation of $\mathfrak{A}$ in $\mathfrak{B}$

Interpretation Lemma

$L[\tau, \sigma]$-interpretation $\quad I = \langle D(\overline{x}),\ E(\overline{x}, \overline{y}),\ (\ \varphi_R(\overline{x}_1, \ldots, \overline{x}_r)\ )_{R\in\sigma}\rangle$

- I maps τ-structures $\mathfrak{B}$ to σ-structures $I(\mathfrak{B})$
- in turn, I maps σ-formulae ψ to τ-formulae $I(\psi)$:
  - replace variables $x, y, \ldots$ by k-tuples $\overline{x}, \overline{y}, \ldots$
  - relativize quantifiers to $D(\overline{x})$
  - replace equalities $x = y$ by $E(\overline{x}, \overline{y})$
  - replace atoms $Rx_1 \ldots x_r$ (for $R \in \sigma$) by $\varphi_R(\overline{x}_1, \ldots, \overline{x}_r)$

Interpretation Lemma: $\quad I(\mathfrak{B}) \models \psi \quad \Longleftrightarrow \quad \mathfrak{B} \models I(\psi)$

Structures presented by interpretations

Interpretations provide general and powerful way for defining classes of finitely presentable structures with effective semantics

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Take structure $\mathfrak{B}$ with "nice" properties and study closure $\{\mathfrak{A} : \mathfrak{A} \leq_L \mathfrak{B}\}$ under L-interpretations for suitable L.

Finite presentations: by interpretations into $\mathfrak{B}$

Effective semantics: if L is closed under interpretations and L is effective on $\mathfrak{B}$, then L is effective on any $\mathfrak{A} \leq_L \mathfrak{B}$.
(Interpretation Lemma)

Tree interpretable structures

$\mathcal{T}^2 = (\{0,1\}^*, \sigma_0, \sigma_1)$ infinite binary tree

MSO: monadic second-order logic

A structure $\mathfrak{A}$ is tree-interpretable if $\mathfrak{A} \leq_{\text{MSO}} \mathcal{T}^2$:

(one-dimensional) MSO-interpretation of $\mathfrak{A}$ in the infinite binary tree.

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Tree-interpretable structures admit effective evaluation of MSO

- Rabin's Theorem: The MSO-theory of $\mathcal{T}^2$ is decidable
- Interpretation Lemma

Tree interpretable graphs

Tree-interpretable graphs generalize various classes of finitely presentable graphs that admit effective evaluation of MSO.

Context-free graphs: (Muller, Schupp)

configuration graphs of pushdown automata

HR-equational and VR-equational graphs: (Courcelle)

defined by graph grammars

Prefix-recognizable graphs: (Caucal)

$G = (V, (E_a)_{a \in A}$ where V regular language, and

$$E_a = \bigcup_{i=1}^{m} X_i(Y_i \times Z_i) = \bigcup_{i=1}^{m} \{(xy, xz) : x \in X_i, y \in Y_i, z \in Z_i\}$$

for regular languages X_i, Y_i, Z_i.

Tree interpretable structures

Theorem (Barthelmann, Blumensath, Caucal, Courcelle, Stirling)

For any graph G, the following are equivalent.

(1) G is tree-interpretable

(2) G is VR-equational

(3) G is prefix-recognizable

(4) G is the restriction to a regular set of the configuration graph of a pushdown automaton with ε-transitions.

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The classes of context-free graphs and HR-equational graphs are strictly contained in the class of tree-interpretable graphs.

Tree-like structures

More powerful domains than the tree-interpretable structures on which MSO is effective?

Tree constructions:

- **Unfolding** of a labeled graph G from a node v to the tree $\mathcal{T}(G, v)$.
- **Muchnik´s construction:** With relational structure $\mathfrak{A} = (A, R_1, \ldots, R_m)$, associate its iteration

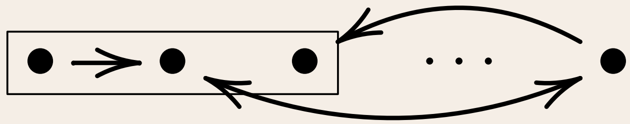
$$\mathfrak{A}^* := (A^*, R_1^*, \ldots, R_m^*, \text{son}, \text{clone})$$

with relations

$$R_i^* := \{(wa_1, \ldots, wa_r) : w \in A^*, (a_1, \ldots, a_r) \in R_i\}$$
$$\text{son} := \{(w, wa) : w \in A^*, a \in A\}$$
$$\text{clone} := \{waa : w \in A^*, a \in A\}$$

Muchnik's construction

$\mathfrak{A}$



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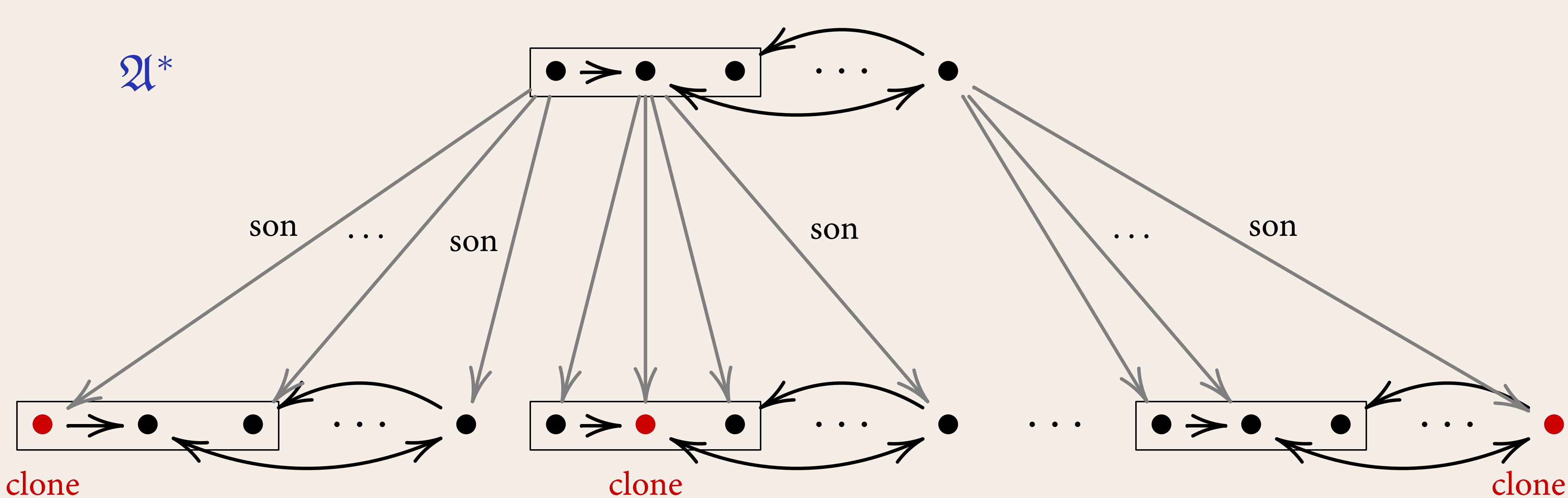

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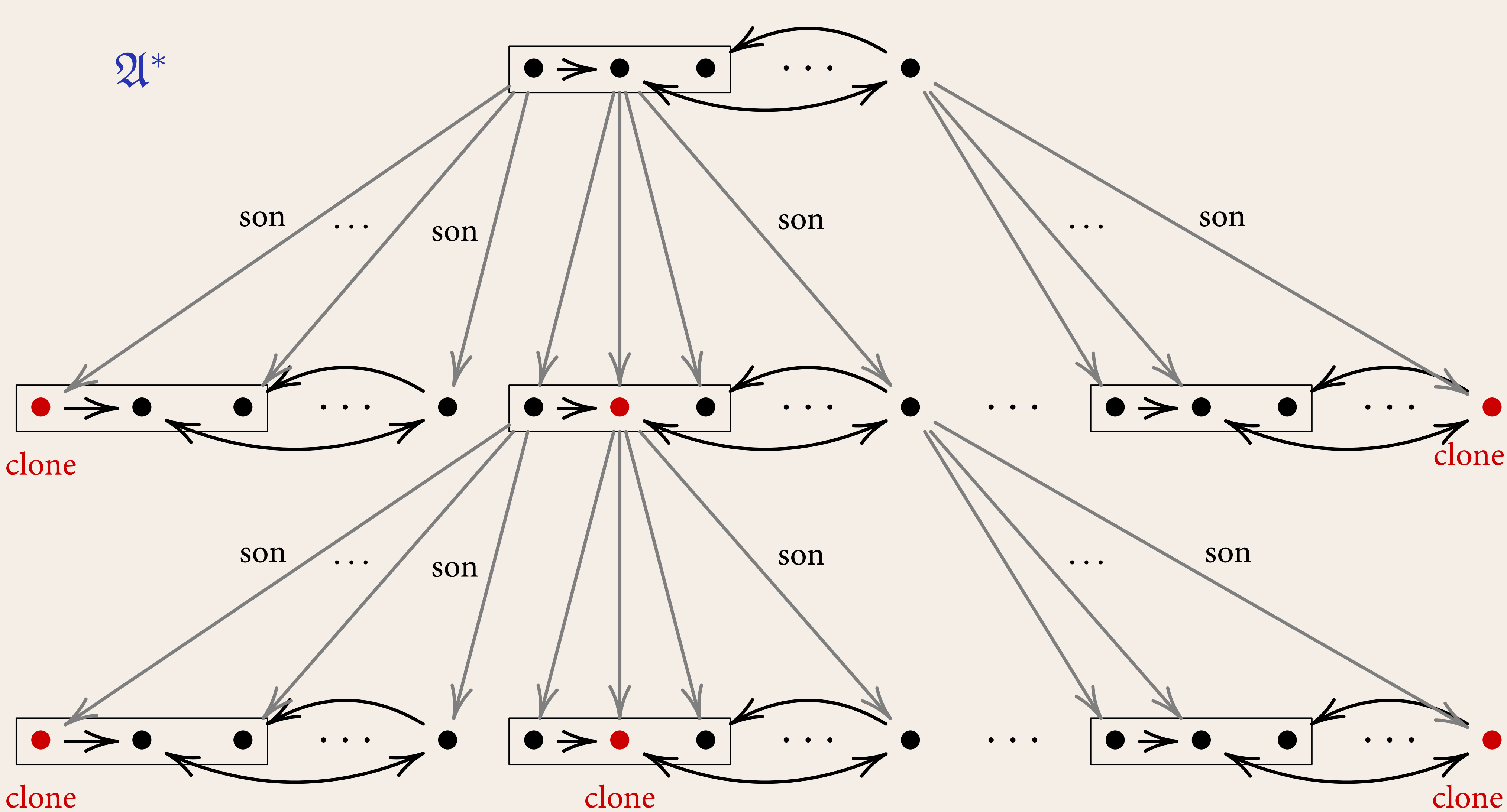

Tree constructible structures

Decidability

- If the MSO-theory of (G, v) is decidable, then so is the MSO-theory of its unfolding $\mathcal{T}(G, v)$ (Courcelle, Walukiewicz).
- If the MSO-theory of $\mathfrak{A}$ is decidable, then so is the MSO-theory of its iteration $\mathfrak{A}^*$ (Muchnik, Walukiewicz, Berwanger-Blumensath)

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Tree constructible structures: Closure of finite structures under MSO-interpretations and Muchnik's construction.

- MSO is effective on tree constructible structures
- There exist tree constructible structures that are not tree interpretable (Courcelle)

Automatic structures

$\mathfrak{A} = (A, R_1, \dots, R_s)$ is automatic if there exist a regular language $L_\delta \subseteq \Sigma^*$ and a surjective function $h : L_\delta \to A$ such that the relations

$$L_= := \{(u, v) : h(u) = h(v)\} \subseteq L_\delta \times L_\delta$$

$$L_{R_i} := \{(u_1, \dots, u_r) : \mathfrak{A} \models R_i h(u_1) \dots h(u_r)\} \subseteq L_\delta \times \dots \times L_\delta$$

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Automatic presentation of $\mathfrak{A}$: list of automata

$$\langle M_\delta, M_=, M_{R_1}, \dots, M_{R_s} \rangle$$

recognizing $L_\delta, L_=, L_{R_1}, \dots, L_{R_s}$.

(Khoussainov-Nerode, Blumensath, Blumensath-G.)

Synchronous automata

Automaton M, recognizing a relation $R \subseteq \underbrace{\Sigma^* \times \cdots \times \Sigma^*}_{r}$:

works on alphabet $\Gamma := (\Sigma \cup \{\Box\})^r - \{\Box\}^r$

$$(u_1, \ldots, u_r) \in R \qquad \Longleftrightarrow$$

$$M \text{ accepts} \quad \underbrace{\begin{matrix} u_{11}u_{12} & \ldots & u_{1j}\Box\Box\Box\ldots\Box \\ u_{21}u_{22}\ldots & \ldots & u_{2k}\Box\ldots\Box \\ \vdots & \vdots & \vdots \\ u_{i1}u_{i_2} & \ldots & u_{i\ell} \\ \vdots & \vdots & \vdots \\ u_{r1}u_{r2} & \ldots & u_{rj}\Box\Box\Box\ldots\Box \end{matrix}}_{\ell=\max\{|u_i|:i=1,\ldots,r\}} \in \Gamma^*$$

Examples of automatic structures

- $(\mathbb{N}, +)$ is automatic

  – $L_\delta = \{0,1\}^*1 \cup \{0\}$

  – $h(w_0 \dots w_{n-1}) = \sum_{i<n} w_i 2^i$ (h injective)

  – L_+ recognised by automaton M_+

  scans
  $$\begin{array}{l} u_0 u_1 \dots \quad \dots u_m \square \dots\dots \square \\ v_0 v_1 \dots \qquad\qquad \dots v_{n-1} \square \\ w_0 w_1 \dots \qquad\qquad\qquad \dots w_m \end{array}$$

  remembering carry bit c_i for $u_0 \dots u_{i-1} + v_0 \dots v_{i-1}$

  checks whether $w_i = u_i + v_i + c_i \pmod 2$

- every finite structure is automatic

- the configuration graphs of Turing machines are automatic

Examples of automatic structures

- $(\mathbb{N}, +, |_m)$ is automatic

$$x \,|_m\, y \qquad :\Longleftrightarrow \qquad x \text{ is a power of } m \text{ dividing } y$$

use m-ary representation of numbers

$$L_{|_m} = \left\{ \begin{matrix} u \\ v \end{matrix} : \begin{matrix} u \\ v \end{matrix} = \begin{matrix} 0 \dots & 0\,1\,\square \dots & \dots \square \\ 0 \dots & 0 v_r v_{r+1} & \dots v_n \end{matrix} \right\}$$

- $\text{Tree}(m) = (\{0, \dots, m-1\}^*, \sigma_0, \dots, \sigma_{m-1}, \leq, \text{el})$ is automatic

  - $\sigma_i : \quad u \mapsto ui$
  - $u \leq v : \quad \exists w \quad uw = v$
  - $\text{el}(u, v) : \quad |u| = |v|$

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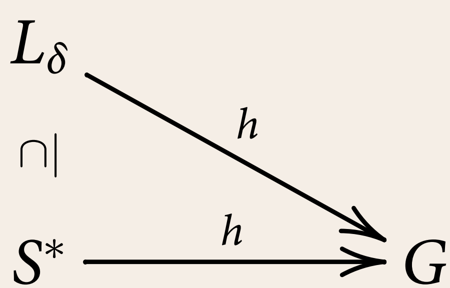
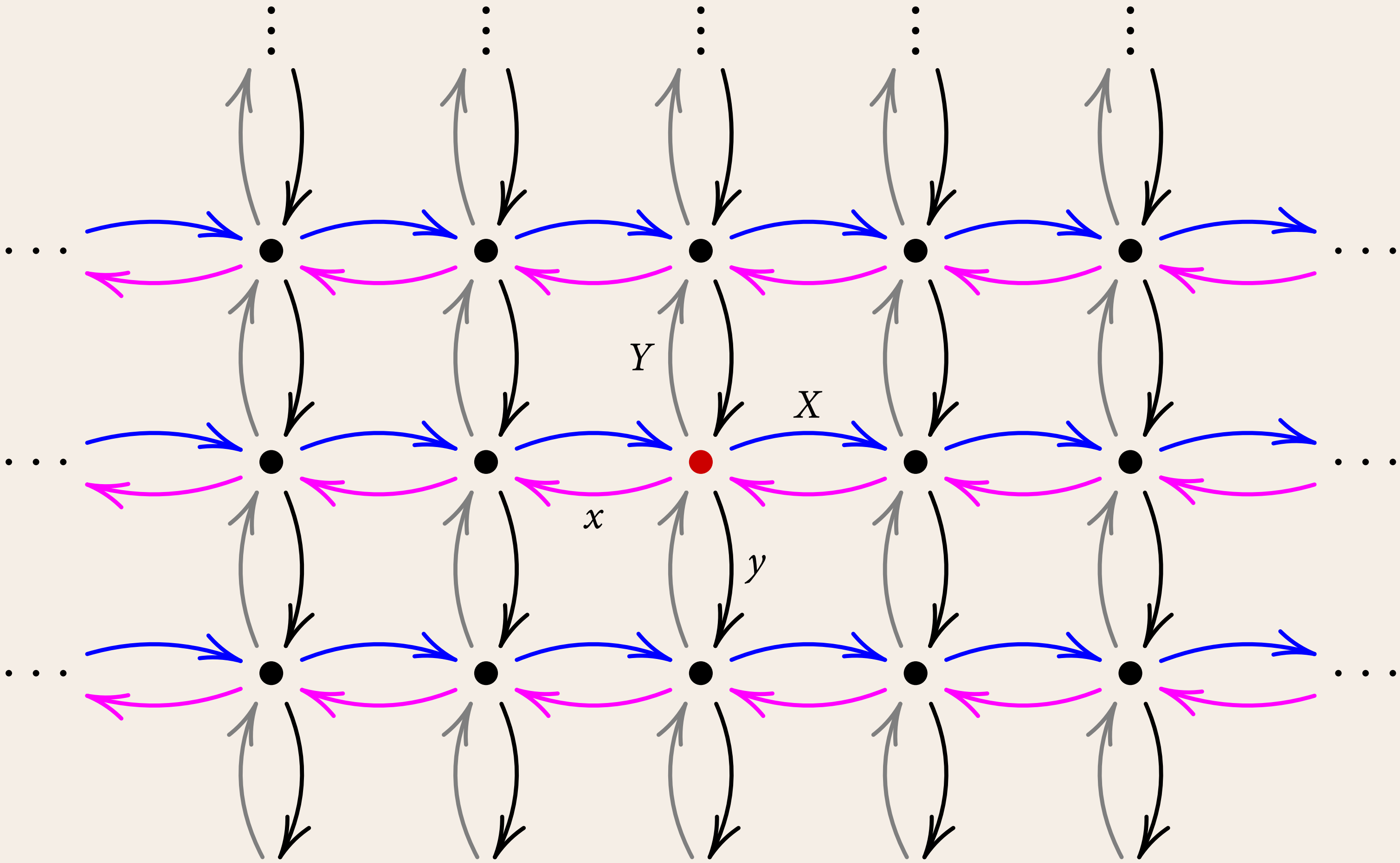
$(G, \cdot)$ is an **automatic group** if there is a finite set $S \subseteq G$ of semigroup generators, such that $\Gamma(G, S)$ is an **automatic structure** with presentation $\langle L_\delta, h, \dots \rangle$ where $L_\delta \subseteq S^*$ and



Automatic groups: Example

$(\mathbb{Z} \times \mathbb{Z}, +)$, with $S = \{X, Y, x, y\}$, $x = X^{-1}, y = Y^{-1}$



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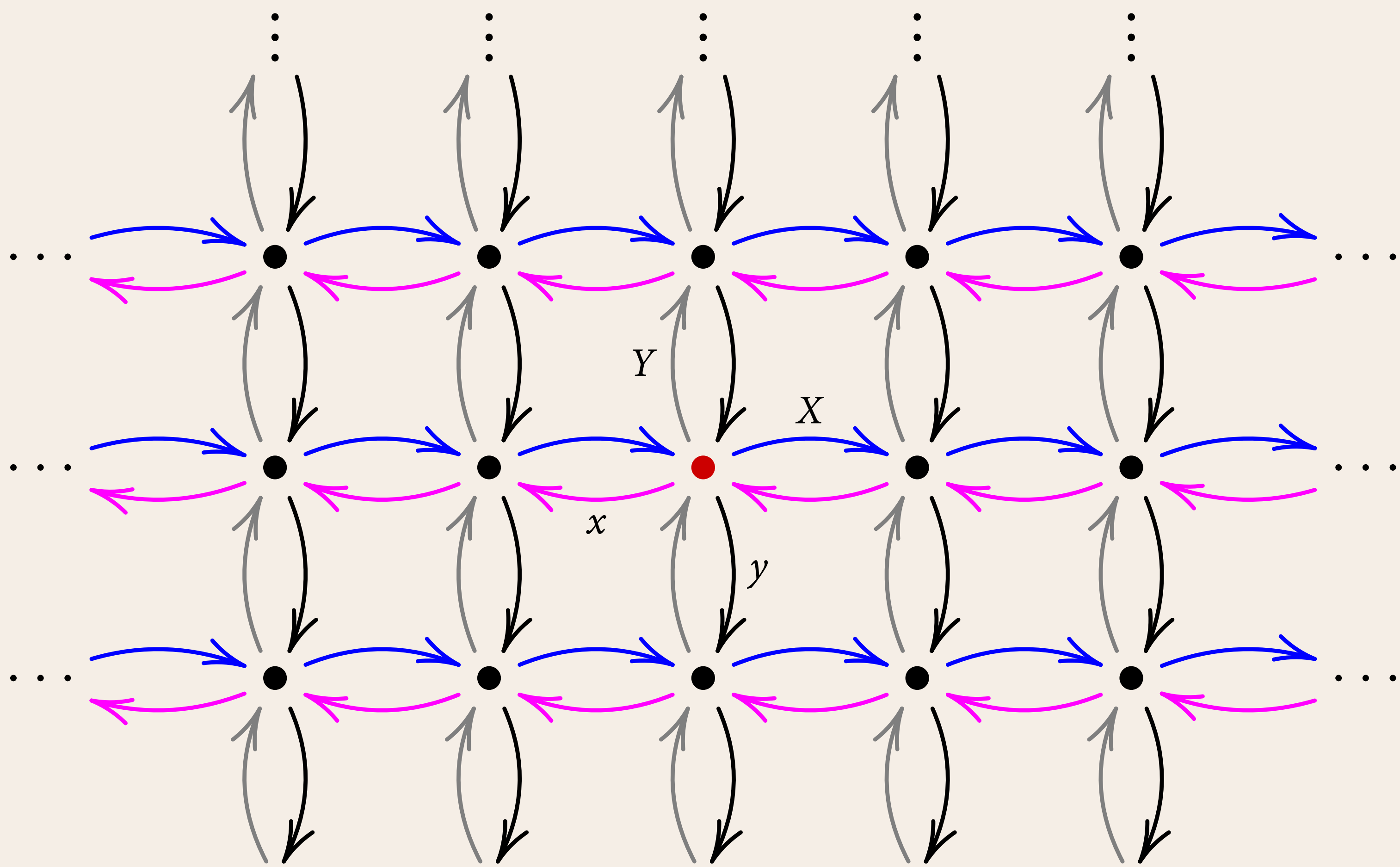

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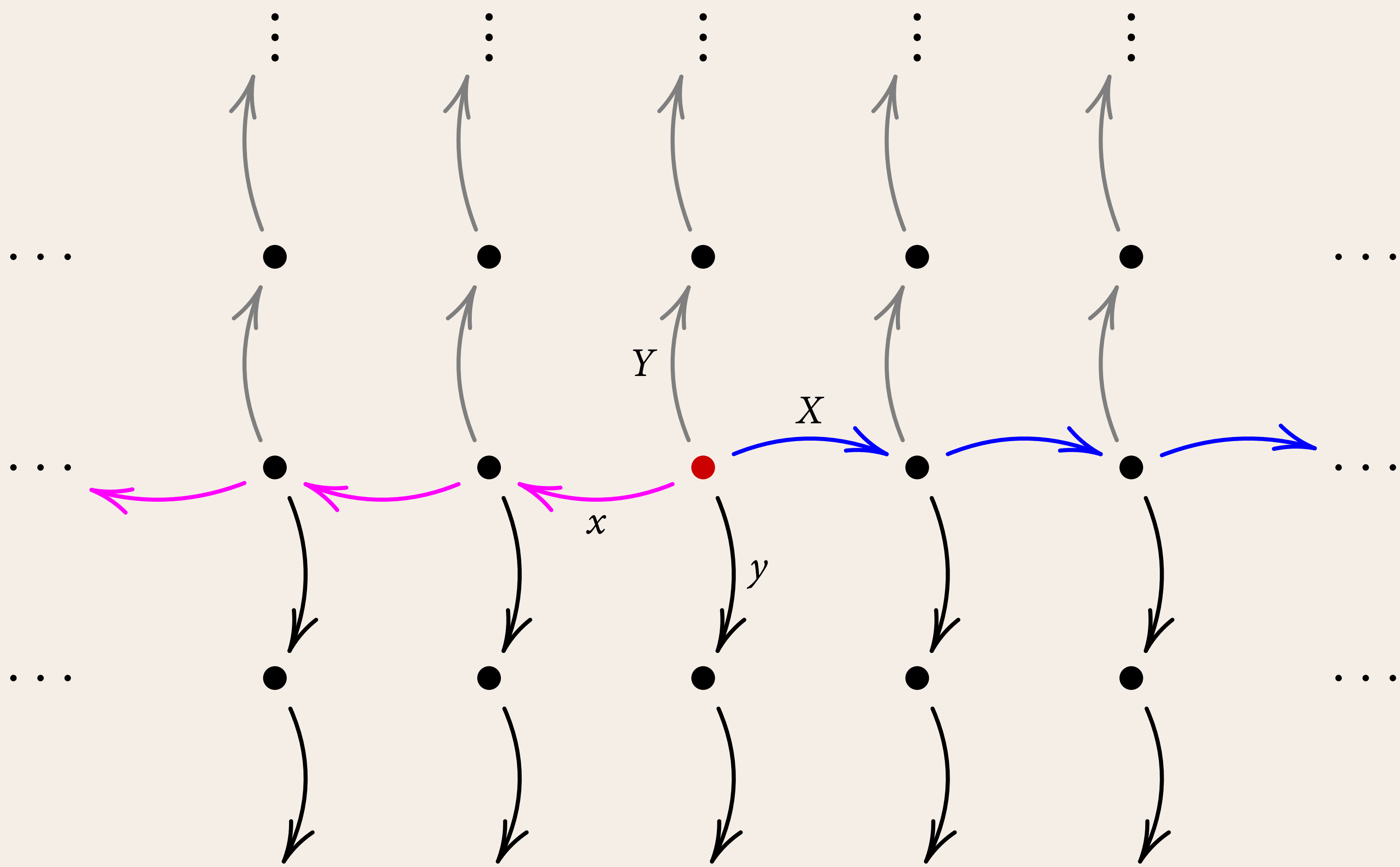

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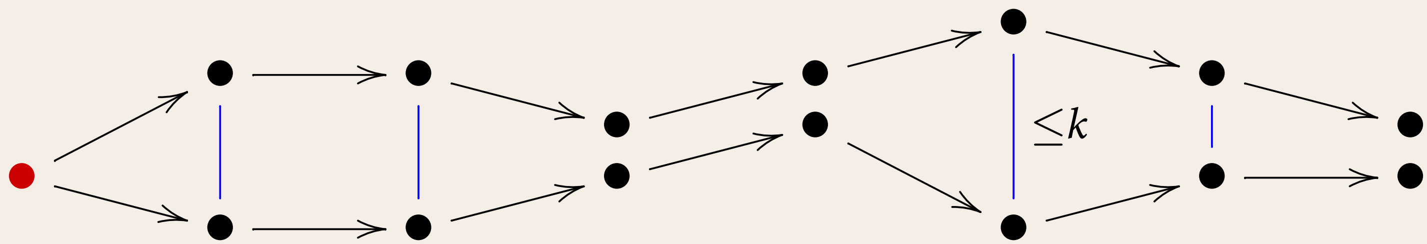
The k-fellow traveller property

$(G, \cdot)$ automatic group, with automatic presentation $h : L_\delta \to G$.

For all $u, v \in L_\delta$, if $\mathrm{dist}(u, v) \leq 1$ in $\Gamma(G, S)$, then
$\mathrm{dist}(u_1 \dots u_i, v_1 \dots v_i) \leq k$ for all $i \leq \max(|u|, |v|)$.



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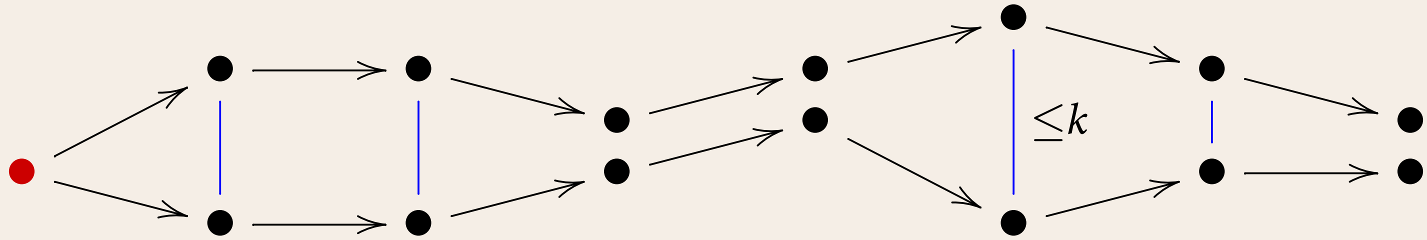

Proposition. $(G, \cdot)$ is automatic $\iff$ for some S and k, there is a regular language $L_\delta \subseteq S^*$ such that the canonical map $h : L_\delta \to G$ is surjective and satisfies the k-fellow traveller property.

Automatic groups versus automatic Caley graphs

By definition, if $(G, \cdot)$ is an automatic group, then for some S, the Cayley graph $\Gamma(G, S)$ is an automatic graph.

The converse is not true!

Counterexample: (Senizergues) The Heisenberg group H is the group of affine transformations of $\mathbb{Z}^3$ generated by $S = \{\alpha, \beta, \gamma\}$.

$$\alpha : (x, y, z) \mapsto (x + 1, y, z + y)$$

$$\beta : (x, y, z) \mapsto (x + 1, y + 1, z)$$

$$\gamma : (x, y, z) \mapsto (x, y, z + 1)$$

- Obviously, $\Gamma(H, S)$ is first-order interpretable in $(\mathbb{N}, +)$. Hence, $\Gamma(H, S)$ is an automatic graph.
- But H is not an automatic group (Epstein et al.).

ω-automatic structures

$\mathfrak{A} = (A, R_1, \dots, R_s)$ is ω-automatic if there exist a ω-regular language $L_\delta \subseteq \Sigma^\omega$ and a surjective function $h : L_\delta \to A$ such that the relations

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are ω-regular, i.e. recognizable by synchronous Büchi automata.

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- every automatic structure is ω-automatic
- $(\mathbb{R}, +)$ and $(\mathbb{R}, +, \leq, |_m, 1)$ are ω-automatic

$$x \mid_m y \quad :\Longleftrightarrow \quad \exists k, r \in \mathbb{Z} : x = m^k, \quad y = r \cdot x$$

- ω-Tree$(m) = (\{0, \ldots, m-1\}^{\leq\omega}, \sigma_0, \ldots, \sigma_{m-1}, \leq, \mathrm{el})$ is ω-automatic

First-order logic on ω-automatic structures

$\mathrm{FO}(\exists^\omega)$: FO + "$\exists$ infinitely many x such that …"

Theorem. Given $\varphi(\bar{x}) \in \mathrm{FO}(\exists^\omega)$ and an ω-automatic structure $\mathfrak{A}$, one can effectively compute an automatic presentation of $(\mathfrak{A}, \varphi^{\mathfrak{A}})$.

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Regular and ω-regular relations are closed under

- first-order operations: classical automata theory
- the quantifier $\exists^\omega$:
  for regular relations, this follows from the Pumping Lemma
  for ω-regular relations, more complicated arguments needed

Corollary.
The $\mathrm{FO}(\exists^\omega)$-theory of every ω-automatic structure is decidable.

Query evaluation and model checking

Query evaluation: (for a logic L on (ω)-automatic structures)

Given: $\varphi(\overline{x}) \in L$, and an automatic presentation of $\mathfrak{A}$

Compute: a presentation of $\varphi^{\mathfrak{A}} := \{\overline{a} : \mathfrak{A} \models \varphi(\overline{a})\}$

(compatible with presentation of $\mathfrak{A}$)

Model checking:

Given: $\varphi(\overline{x}) \in L$, a presentation of $\mathfrak{A}$ and $\overline{a}$

Decide: $\mathfrak{A} \models \varphi(\overline{a})$?

Query evaluation and model checking

Query evaluation: (for a logic L on (ω)-automatic structures)

Given: $\varphi(\overline{x}) \in L$, and an automatic presentation of $\mathfrak{A}$

Compute: a presentation of $\varphi^{\mathfrak{A}} := \{\overline{a} : \mathfrak{A} \models \varphi(\overline{a})\}$
(compatible with presentation of $\mathfrak{A}$)

Model checking:

Given: $\varphi(\overline{x}) \in L$, a presentation of $\mathfrak{A}$ and $\overline{a}$

Decide: $\mathfrak{A} \models \varphi(\overline{a})$?

structure complexity: For fixed φ, determine complexity of $\mathfrak{A} \mapsto \varphi^{\mathfrak{A}}$
in terms of size of deterministic automata representing $\mathfrak{A}$.

expression complexity: For fixed $\mathfrak{A}$, determine complexity of
$\mathfrak{A} \mapsto \varphi^{\mathfrak{A}}$ or $\mathrm{Th}_L(\mathfrak{A})$ in terms of length of φ

combined complexity: both inputs variable

More powerful logics than FO

Query evaluation and model checking are undecidable for

- transitive closure logics
- fixed point logics (μ-calculus, LFP, . . .)
- FO + counting
- monadic second-order logic

on certain fixed automatic structures.

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on certain fixed automatic structures.

- define multiplication in $(\mathbb{N}, +)$
- configuration graphs of Turing machines are automatic.

Any logic that is strong enough for REACHABILITY can express the halting problem.

Complexity

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Simple fragments of (relational) FO:

	structure complexity	expression complexity
Model Checking		
quantifier-free	LOGSPACE	ALOGTIME
existential	NP	PSPACE
Query-Evaluation		
quantifier-free	LOGSPACE	PSPACE
existential	PSPACE	EXPSPACE

Automatic structures with functions

Model checking complexity of quantifier-free formulae with functions:

structure complexity: NLOGSPACE-complete

expression complexity: PTIME-complete and solvable in time $O(|\varphi|^2)$

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Corollary. The word problem of any automatic group can be solved in quadratic time.

$(G, \cdot)$ automatic group, generated by $\{s_1, \ldots, s_m\}$

$\Rightarrow \quad G' := (G, e, g \mapsto gs_1, \ldots, g \mapsto gs_m)$ is automatic structure

Word problem for $(G, \cdot)$ described by term equations over G'.

The isomorphism problem

Theorem.

The isomorphism problem for automatic structures is undecidable.

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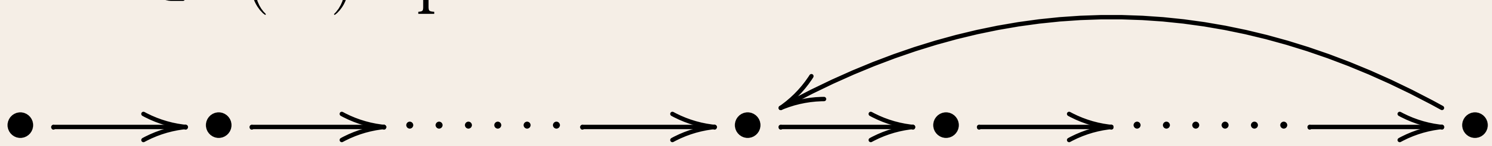
Proof. Every deterministic TM M can be effectively translated into TM M' whose configuration graph $C(M')$ contains

- ω many copies of $(\mathbb{N}, \mathrm{succ})$
- for each $x \in L(M)$ a path



Hence, $$L(M) = \varnothing \iff \underbrace{C(M')}_{\text{automatic}} \cong \underbrace{\omega \cdot (\mathbb{N}, \mathrm{succ})}_{\text{automatic}}$$

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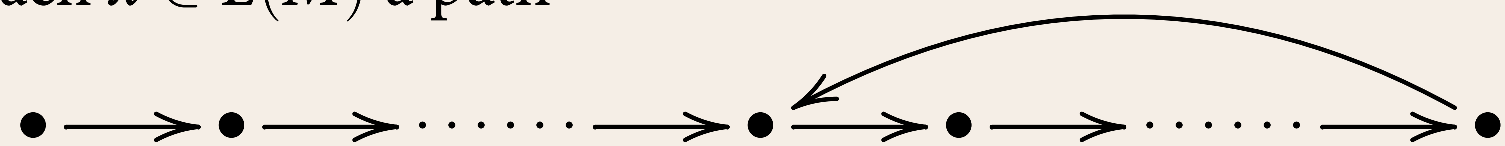

Hence, $$L(M) = \varnothing \iff \underbrace{C(M')}_{\text{automatic}} \cong \underbrace{\omega \cdot (\mathbb{N}, \mathrm{succ})}_{\text{automatic}}$$

Theorem.

The connectivity problem for automatic graphs is undecidable.

Automatic structures and interpretations

$\mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B}$: $\mathfrak{A}$ is first-order interpretable in $\mathfrak{B}$

Automatic structures and ω-automatic structures are closed under FO-interpretations:

$$\mathfrak{B} \text{ is } (\omega)\text{-automatic}, \quad \mathfrak{A} \leq_{\mathrm{FO}} \mathfrak{B} \quad \Longrightarrow \quad \mathfrak{A} \text{ is } (\omega)\text{-automatic}$$

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In particular, the (ω)-automatic structures are closed under

- expansion by definable relations
- factorisation by definable congruences
- substructures with definable universe
- finite powers

Note: They are not closed under taking arbitrary substructures

Model theoretic characterisation of automatic structures

Theorem. The following are equivalent:

(1) $\mathfrak{A}$ is automatic

(2) $\mathfrak{A} \leq_{\mathrm{FO}} (\mathbb{N}, +, |_m)$ for some (and hence all) $m \geq 2$

(3) $\mathfrak{A} \leq_{\mathrm{FO}} \mathrm{Tree}(m)$ for some (and hence all) $m \geq 2$

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(2) $\mathfrak{A} \leq_{\mathrm{FO}} (\mathbb{R}, +, \leq, |_m, 1)$ for some (and hence all) $m \geq 2$

(3) $\mathfrak{A} \leq_{\mathrm{FO}} \omega\text{-}\mathrm{Tree}(m)$ for some (and hence all) $m \geq 2$

Characterising automatic groups

Theorem. $(G, \cdot)$ is an automatic group

$$\Longleftrightarrow$$

there is finite set $S \subseteq G$ of semigroup generators such that Caley graph $\Gamma(G, S)$ is FO-definable in $\mathrm{Tree}(S)$.

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There exist first-order formulae $D(x), E(x, y), \varphi_1(x, y), \ldots, \varphi_m(x, y)$ of vocabulary $\{s_1, \ldots, s_m, \leq, \mathrm{el}\}$ such that

$$\langle D^{\mathrm{Tree}(S)}, \varphi_1^{\mathrm{Tree}(S)}, \ldots, \varphi_m^{\mathrm{Tree}(S)} \rangle / E^{\mathrm{Tree}(S)} \quad = \quad \Gamma(G, S)$$

(equality rather than isomorphism)

Structures that are not automatic

How to prove that a structure $\mathfrak{A}$ is not automatic ?

1) $\mathfrak{A}$ not countable: $(\mathbb{R}, +, , \cdot)$

2) $\mathrm{Th}(\mathfrak{A})$ undecidable: $(\mathbb{N}, +, , \cdot)$

3) Growth rates

Take automatic presentation of $\mathfrak{A}$ with bijective $h : L_\delta \longrightarrow A$

For any set $a_1, a_2, \dots$ of definable elements in $\mathfrak{A}$ (ordered by $|h^{-1}(a_i)|$), and any finite set F of definable functions on $\mathfrak{A}$, let

$$G_1 := \{a_1\}, \qquad G_{n+1} := G_n \cup \{a_{n+1}\} \cup \bigcup_{f \in F} f(G_n \times \cdots \times G_n)$$

Theorem. $|G_n| = 2^{O(n)}$ for all n

Elements of G_n are represented by words of length $O(n)$

Application

Corollary. $(\mathbb{N}, \cdot)$ is not automatic.

For $a_1, a_2, \ldots$ enumeration of primes, $f(x, y) = x \cdot y$

$$|G_n| = 2^{\Omega(n^2)}$$

But: $(\mathbb{N}. \cdot)$ is tree automatic