

# Decidable Fragments of First-Order and Fixed-Point Logic

From prefix-vocabulary classes to guarded logics

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# The classical decision problem

part of Hilbert's formalist programme for the foundations of mathematics

in modern terminology:

find an algorithm that solves the satisfiability problem for first-order logic

**D. Hilbert and W. Ackermann:**

“Das Entscheidungsproblem ist gelöst, wenn man ein Verfahren kennt, das bei einem vorgelegten logischen Ausdruck durch endlich viele Operationen die Entscheidung über die Allgemeingültigkeit bzw. Erfüllbarkeit erlaubt. (...) Das Entscheidungsproblem muss als das Hauptproblem der mathematischen Logik bezeichnet werden.”

Grundzüge der theoretischen Logik (1928), pp 73ff

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Similar statements by many other logicians, including **Bernays, Schönfinkel, Herbrand, von Neumann, . . .**

# Early results: decision procedures

## Decision algorithms for fragments of first-order logic

- the monadic predicate calculus (Löwenheim 1915, Kalmár 1929)
- the prefix class  $\exists^*\forall^*$  (Bernays–Schönfinkel 1928, Ramsey 1932)
- the  $\exists^*\forall\exists^*$ -prefix class (Ackermann 1928)
- the  $\exists^*\forall^2\exists^*$ -prefix class, without equality (Gödel, Kalmár, Schütte 1932-1934)

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These are often called the classical solvable cases of the decision problem.

# Early results: reduction classes

A fragment  $X \subseteq \text{FO}$  is a **reduction class** if there is a computable function  $f : \text{FO} \rightarrow X$  such that

$$\psi \text{ satisfiable} \iff f(\psi) \text{ satisfiable}$$

## Early reduction classes:

- relational sentences without  $=$  (the pure predicate calculus)
- dyadic sentences (only predicates of arity 2)
- $\forall^*$ -sentences, or relational  $\forall^* \exists^*$ -sentences (Skolem normal form)
- only one binary predicate (Kalmár 1936)
- relational  $\forall^3 \exists^*$ -sentences (Gödel 1933)

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After the negative solution, the decision problem did not disappear, but became a **classification problem**.

# The decision problem as a classification problem

- For which classes  $X \subseteq \text{FO}$  is  $\text{Sat}(X)$  decidable, which are reduction classes ?
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What kind of classes  $X \subseteq \text{FO}$  should be considered?

There are continuum many such classes, one cannot study all of them.

A direction is needed.

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In view of the early results, most attention (in logic) was given to classes determined by **quantifier prefix** and **vocabulary**, i.e. the number and arity of relation and function symbols.



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**Example:**  $[\exists^* \forall \exists^*, (\omega, 1), all]_ =$

sentences  $\exists x_1 \dots \exists x_m \forall y \exists z_1 \dots \exists z_n \varphi$  where  $\varphi$  is quantifier-free and

- contains at most one binary predicate, and no predicates of arity  $\geq 3$ ,
- may contain any number of monadic predicates,
- may contain any number of function symbols of any arity,
- may contain equality.

# The complete classification: undecidable cases

## A: Pure predicate logic (without functions, without =)

- (1)  $[\forall\exists\forall, (\omega, 1), (0)]$  (Kahr 1962)
- (2)  $[\forall^3\exists, (\omega, 1), (0)]$  (Surányi 1959)
- (3)  $[\forall^*\exists, (0, 1), (0)]$  (**Kalmár**-Surányi 1950)
- (4)  $[\forall\exists\forall^*, (0, 1), (0)]$  (Denton 1963)
- (5)  $[\forall\exists\forall\exists^*, (0, 1), (0)]$  (Gurevich 1966)
- (6)  $[\forall^3\exists^*, (0, 1), (0)]$  (**Kalmár**-Surányi 1947)
- (7)  $[\forall\exists^*\forall, (0, 1), (0)]$  (Kostyrko-Genenz 1964)
- (8)  $[\exists^*\forall\exists\forall, (0, 1), (0)]$  (Surányi 1959)
- (9)  $[\exists^*\forall^3\exists, (0, 1), (0)]$  (Surányi 1959)

# The complete classification: undecidable cases

## B: Classes with functions or equality

(10)	$[\forall, (0), (2)]_=_$	(Gurevich 1976)
(11)	$[\forall, (0), (0, 1)]_=_$	(Gurevich 1976)
(12)	$[\forall^2, (0, 1), (1)]$	(Gurevich 1969)
(13)	$[\forall^2, (1), (0, 1)]$	(Gurevich 1969)
(14)	$[\forall^2 \exists, (\omega, 1), (0)]_=_$	(Goldfarb 1984)
(15)	$[\exists^* \forall^2 \exists, (0, 1), (0)]_=_$	(Goldfarb 1984)
(16)	$[\forall^2 \exists^*, (0, 1), (0)]_=_$	(Goldfarb 1984)



# The complete classification: decidable cases

(Exclude the trivial classes: finite prefix and finite relational vocabulary)

## A: Classes with the finite model property

- (1)  $[\exists^*\forall^*, all, (0)]_ =$  (Bernays, Schönfinkel 1928)
- (2)  $[\exists^*\forall^2\exists^*, all, (0)]$  (Gödel 1932, Kalmár 1933, Schütte 1934)
- (3)  $[all, (\omega), (\omega)]$  (Löb 1967, Gurevich 1969)
- (4)  $[\exists^*\forall\exists^*, all, all]$  (Gurevich 1973)
- (5)  $[\exists^*, all, all]_ =$  (Gurevich 1976)

## B: Classes with infinity axioms

- (6)  $[all, (\omega), (1)]_ =$  (Rabin 1969)
- (7)  $[\exists^*\forall\exists^*, all, (1)]_ =$  (Shelah 1977)

# Why prefix-vocabulary classes?

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**Classifiability Theorem** (Gurevich) Let  $P$  be a property of prefix vocabulary classes that is closed under subsets and under union:

$$X \in P, Y \subseteq X \implies Y \in P$$

$$X, Y \in P \implies X \cup Y \in P$$

Then there is a finite list **MIN** of prefix vocabulary classes such that

$$X \notin P \iff \exists Y \in \text{MIN} : Y \subseteq X$$

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Then there is a finite list **MIN** of prefix vocabulary classes such that

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Take  $P = \{X : \text{Sat}(X) \text{ decidable}\}$ .

$\implies$  there is a finite list of minimal undecidable prefix-vocabulary classes.

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Modern applications of logic in computer science often lead to quite different classes. For instance:

modal logics, two-variable logics, guarded logics.

On the other side the restriction to first-order logic is often too restrictive. Consider instead (fragments of) **fixed point logics**

# ML: propositional modal logic

Transition systems = Kripke structures = labeled graphs

$$\mathcal{K} = ( \underbrace{V}_{\substack{\text{states} \\ \text{elements}}} , \underbrace{(E_a)_{a \in A}}_{\substack{\text{actions} \\ \text{binary relations}}} , \underbrace{(P_i)_{i \in I}}_{\substack{\text{atomic propositions} \\ \text{unary relations}}} )$$

**Syntax of ML:**  $\psi ::= P_i \mid \neg P_i \mid \psi \wedge \psi \mid \psi \vee \psi \mid \langle a \rangle \psi \mid [a] \psi$

Example:  $P_1 \vee \langle a \rangle (P_2 \wedge [b] P_1)$

**Semantics:**  $\llbracket \psi \rrbracket^{\mathcal{K}} = \{v : \mathcal{K}, v \models \psi\} = \{v : \psi \text{ holds at state } v \text{ in } \mathcal{K}\}.$

$$\mathcal{K}, v \models \begin{array}{l} \langle a \rangle \psi \\ [a] \psi \end{array} \iff \mathcal{K}, w \models \psi \text{ for } \begin{array}{l} \text{some} \\ \text{all} \end{array} w \text{ with } (v, w) \in E_a$$

# ML as a fragment of first-order logic

There is standard translation, taking every  $\psi \in \text{ML}$  to a formula  $\psi^*(x) \in \text{FO}$  such that

$$\mathcal{K}, v \models \psi \iff \mathcal{K} \models \psi^*(v)$$

$$P_i \longmapsto P_i x$$

$$\langle a \rangle \psi \longmapsto \exists y (E_a x y \wedge \psi^*(y))$$

$$[a] \psi \longmapsto \forall y (E_a x y \rightarrow \psi^*(y))$$

# Properties of modal logic

## Good algorithmic properties:

- Sat(ML) is decidable and PSPACE-complete
- efficient model checking: time  $O(|\psi| \cdot \|\mathcal{K}\|)$
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## Interesting model-theoretic properties:

- invariance under bisimulation:

$$\mathcal{K}, v \models \psi, \quad \mathcal{K}, v \sim \mathcal{K}', v' \quad \Longrightarrow \quad \mathcal{K}', v' \models \psi$$

in fact ML is the bisimulation-invariant fragment of first-order logic

- ML has the tree model property
- ML has the finite model property
- ...

# Limitations of propositional modal logic

## ML has very limited expressive power

- no equality
- no unbounded quantification
- no possibility to define new transition relations, not even on the quantifier-free level
- no counting mechanism
- **no recursion mechanism:** reachability problems, game problems, etc. are not expressible

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**Note:** The absence of a recursion mechanism is also a limitation of FO

For numerous applications, in particular for reasoning about computation, reachability problems and game problems are essential.

⇒ **numerous logics that extend ML by recursion mechanisms**

# Computation tree logic CTL

ML + path quantification + temporal operators on paths

CTL is a very important logic for hardware verification.

Good balance between expressive power and algorithmic manageability

- $A P U Q \equiv (\forall \text{ paths}) (P \text{ until } Q)$



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- $A G A F P \equiv (\forall \text{ paths}) \text{ always } (\forall \text{ paths}) \text{ eventually } P$   
 $\equiv (\forall \text{ paths}) \text{ infinitely often } P$

# $L_\mu$ : modal $\mu$ -calculus

ML + least and greatest fixed points

For any definable **monotone relational operator**

$$F_\varphi : X \mapsto \{w : \varphi(X) \text{ true at } w\}$$

make also the least and the greatest fixed point of  $F_\varphi$  definable:

$$\mathcal{K}, w \models \mu X. \varphi \iff w \in \bigcap \{X : F_\varphi(X) = X\} \quad (\text{least fixed point})$$

$$\mathcal{K}, w \models \nu X. \varphi \iff w \in \bigcup \{X : F_\varphi(X) = X\} \quad (\text{greatest fixed point})$$

# $L_\mu$ : Examples

- $\mathcal{K}, w \models \nu X. \langle a \rangle X \iff$  there is an infinite  $a$ -path from  $w$  in  $\mathcal{K}$
- $\mathcal{K}, w \models \mu X. P \vee [a]X \iff$  every infinite  $a$ -path from  $w$  eventually hits  $P$

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- Two player game on graph  $\mathcal{G} = (V, E)$ .

Players alternate, moves along edges, who cannot move, loses.

Player 0 has winning strategy for game  $\mathcal{G}$  from position  $w$



$$\mathcal{G}, w \models \mu X. \diamond \square X$$

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$$\mathcal{G}, w \models \mu X . \Diamond \Box X$$

- $\mathcal{K}, w \models vX \mu Y . \Diamond((P \wedge X) \vee Y) \iff$

on some path from  $w$ ,  $P$  occurs infinitely often

# Importance of CTL and modal $\mu$ -calculus

- $ML \subseteq CTL \subseteq L_\mu$
- $\mu$ -calculus encompasses most of the popular logics used in hardware verification: LTL, CTL, CTL\*, PDL, . . . , and also many logics from other fields: game logic, description logics, etc.
- reasonably good algorithmic properties:
  - satisfiability problem decidable (EXPTIME-complete)
  - efficient model checking for CTL and other important fragments of  $\mu$ -calculus
  - automata-based algorithms
- nice model-theoretic properties:
  - finite model property
  - tree model property
- $L_\mu$  is the bisimulation-invariant fragment of MSO

# Summery: properties of modal logics

The basic modal logic ML has **good algorithmic and model-theoretic properties**, but for most applications in computer science, it is **not expressive enough**



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There are numerous modal logics, that extend ML by recursion mechanisms and other features.

These extensions are practically important because they provide the necessary expressive power for applications, while **preserving the good algorithmic properties**.

Of particular importance for theoretical studies: **modal  $\mu$ -calculus** (modal fixed point logic)

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Of particular importance for theoretical studies: **modal  $\mu$ -calculus** (modal fixed point logic)

**Question.** (Vardi 96)

**Why have modal logics so robust decidability properties ?**

# Bounded variable logics

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we may re-use variables: “there is a path of length 17”  $\in \text{FO}^2$

$$\begin{aligned} \exists x \exists y (E_{xy} \wedge \\ \quad \exists x (E_{yx} \wedge \\ \quad \quad \exists y (E_{xy} \wedge \dots \\ \quad \quad \quad \dots \exists y E_{xy} ) \dots ))) \end{aligned}$$

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**Note:** instead of the number of variables, we can restrict the **width**

$\text{width}(\psi)$ : maximal number of free variables in subformulae of  $\psi$

**Proposition.**  $\text{FO}^k \equiv \{\psi \in \text{FO} : \text{width}(\psi) \leq k\}$

**Proposition.**  $\text{Sat}(\text{FO}^k)$  is undecidable for all  $k \geq 3$

indeed, even the  $\forall\exists\forall$ -prefix class is undecidable

**Question:** What about  $\text{FO}^2$  ?

# Decidability of $\text{FO}^2$

**Theorem** (Mortimer 75)  $\text{FO}^2$  has the finite model property.

Mortimer's proof implies: every satisfiable  $\psi \in \text{FO}^2$  has a model with at most  $2^{2^{O(|\psi|)}}$  elements.

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**Theorem** (Grädel, Kolaitis, Vardi 97)

Every satisfiable  $\psi \in \text{FO}^2$  has a model with at most  $2^{O(|\psi|)}$  elements.

**Corollary**  $\text{Sat}(\text{FO}^2)$  is  $\text{NEXPTIME}$ -complete



# Modal logic and two-variable logic

ML is a fragment of FO<sup>2</sup>.

The standard translation, taking  $\psi \in \text{ML}$  to  $\psi^*(x) \in \text{FO}$  does not need more than two variables:

$$\begin{array}{lll} P_i & \longmapsto & P_i x \\ \langle a \rangle \psi & \longmapsto & \exists y (E_a x y \wedge \psi^*(y)) \\ [a] \psi & \longmapsto & \forall y (E_a x y \rightarrow \psi^*(y)) \end{array}$$

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Traditionally the embedding into two-variable logic has been used as an explanation for the good algorithmic properties of modal logics.

Is this explanation convincing?

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The standard translation, taking  $\psi \in \text{ML}$  to  $\psi^*(x) \in \text{FO}$  does not need more than two variables:

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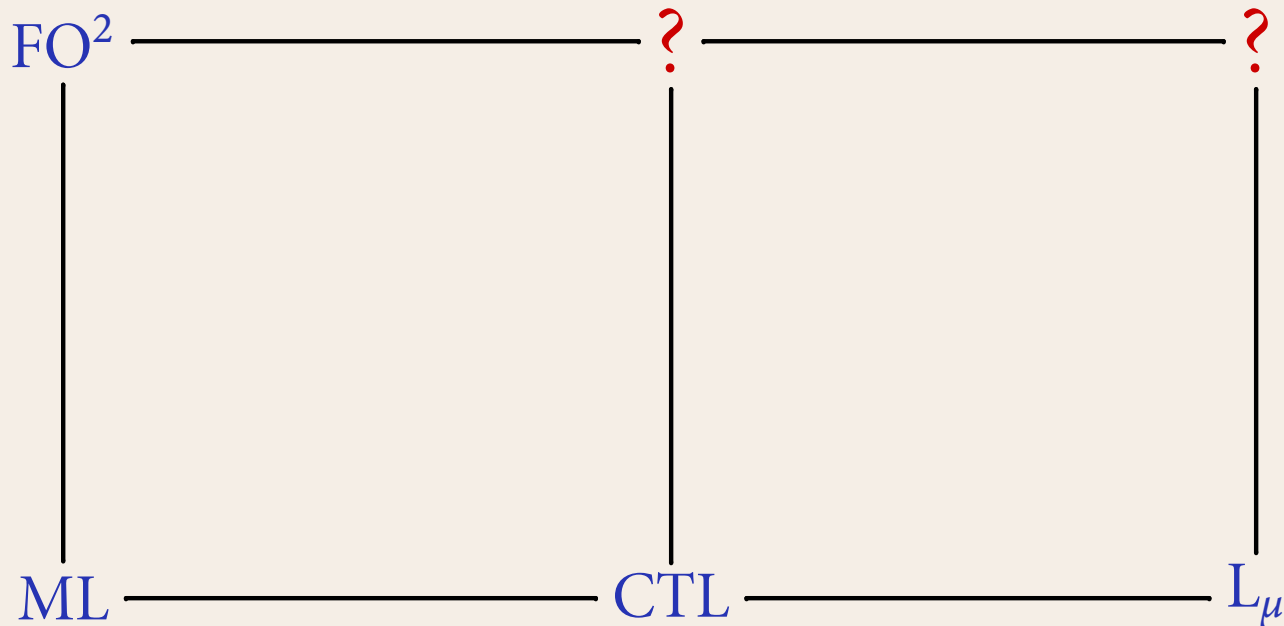
Traditionally the embedding into two-variable logic has been used as an explanation for the good algorithmic properties of modal logics.

Is this explanation convincing?

**Answer: No !!!**

# Stronger two-variable logics

**Challenge:** Extend  $\text{FO}^2$  in a similar way, as  $\text{CTL}$  and  $\mu$ -calculus extend propositional modal logic. Are the resulting logics still decidable?



# Stronger two-variable logics

$TC^2 = FO^2 +$  transitive closures

$(TC \varphi)(x, y)$

$CL^2 =$  least common natural extension of CTL and  $FO^2$

$FO^2 + \langle \beta(x, y) \rangle^* \varphi + [\beta(x, y)]^* \varphi$

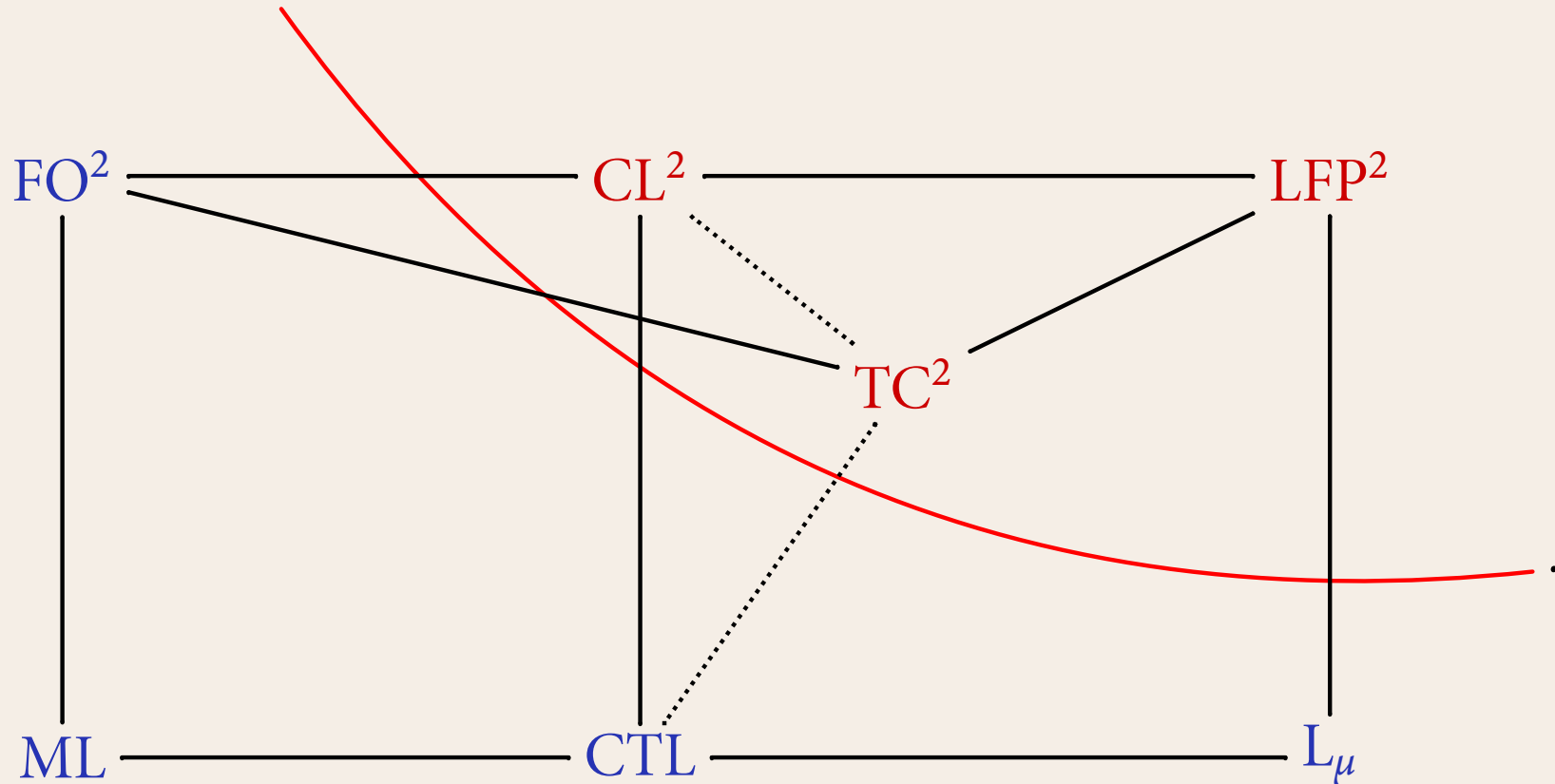
$LFP^2 = FO^2$  monadic least and greatest fixed points

$[lfp \ Tx. \varphi(T, x)](x)$

**Theorem.** (Grädel, Otto, Rosen)

$Sat(TC^2)$ ,  $Sat(CL^2)$ , and  $Sat(LFP^2)$  are highly undecidable ( $\Sigma_1^1$ -hard).

# Two-variable logics



Two-variable logics do **not** have the robust decidability properties of modal logics.

Is there another explanation?

# The guarded fragment of first-order logic (GF)

Fragment of first-order logic with only **guarded quantification**

$$\exists \bar{y}(\alpha(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{y})) \quad \forall \bar{y}(\alpha(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))$$

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Generalizes modal quantification:  $\text{ML} \subseteq \text{GF} \subseteq \text{FO}$

$$\langle a \rangle \varphi \equiv \exists y(E_a xy \wedge \varphi(y)) \quad [a] \varphi \equiv \forall y(E_a xy \rightarrow \varphi(y))$$



# Model-theoretic and algorithmic properties of GF

- Satisfiability for GF is **decidable** (Andréka, van Benthem, Németi)
- GF has **finite model property** (Andréka, Hodkinson, Németi; Grädel)
- GF has (generalized) **tree model property**:  
every satisfiable formula has model of small tree width (Grädel)
- Sat(GF) is 2EXPTIME-complete  
and EXPTIME-complete for formulae of bounded width (Grädel)
- Guarded logics have **small model checking games**:  
 $\|\mathcal{G}(\mathfrak{A}, \psi)\| = O(|\psi| \cdot \|\mathfrak{A}\|)$   
 $\implies$  **efficient game-based model checking algorithms**
- GF is invariant under **guarded bisimulation**

# A thesis – and a challenge

**Thesis:** (Andréka, van Benthem, Németi) The guarded nature of quantification in modal logics is the real reason for their good algorithmic and model-theoretic properties.

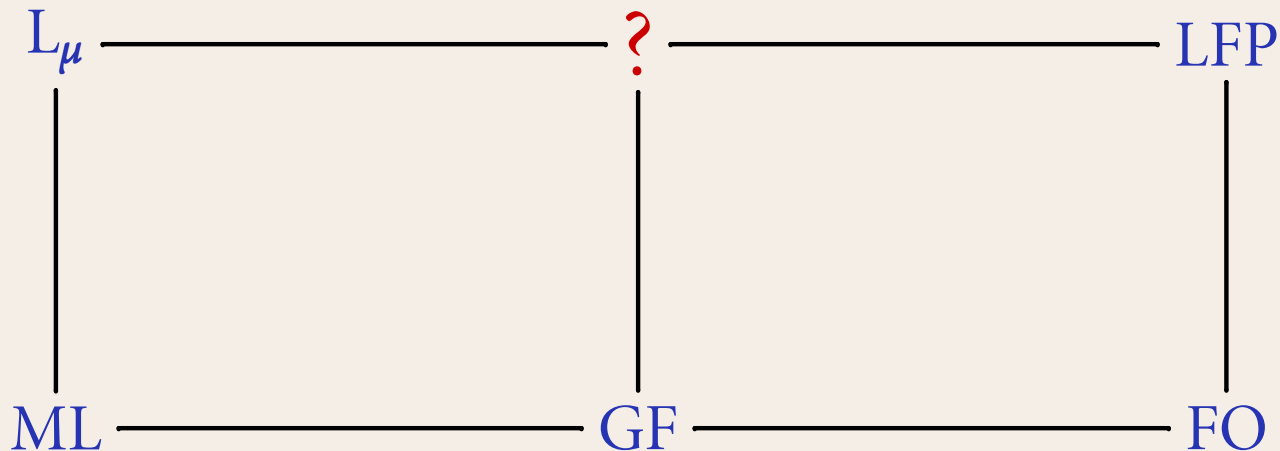
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Results on GF provide some evidence for this thesis.

Real test: algorithmic properties of stronger guarded logics than GF.

**Challenge:** Extend GF in a similar way, as  $\mu$ -calculus extends ML. Is the resulting **guarded fixed-point logic** still decidable ?



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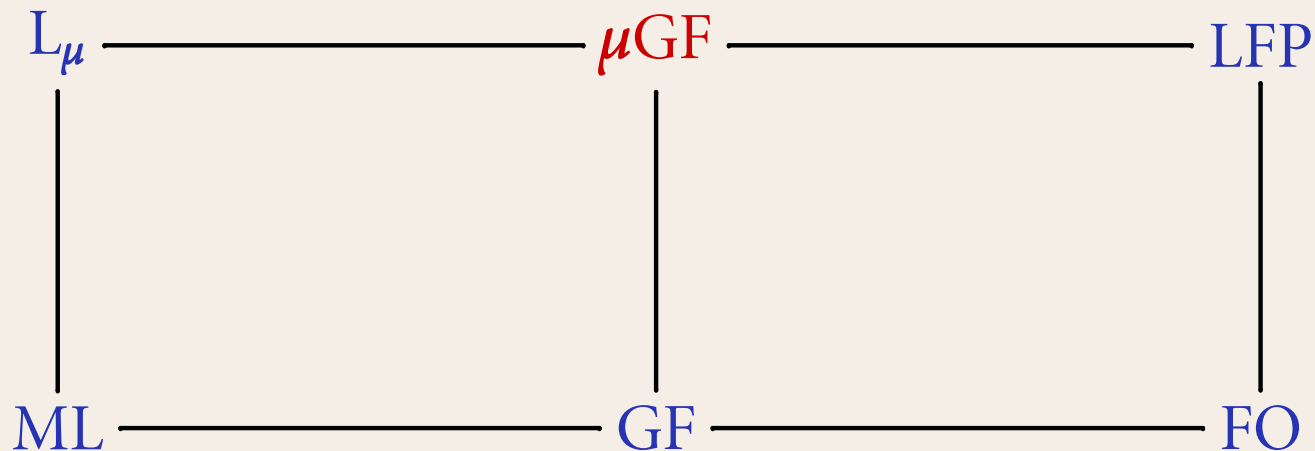
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build formulae  $[\mathbf{lfp} \ T\bar{x} . \psi](\bar{x})$  and  $[\mathbf{gfp} \ T\bar{x} . \psi](\bar{x})$

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On every structure  $\mathfrak{A}$ ,  $\psi(T, \bar{x})$  defines monotone operator

$$\psi^{\mathfrak{A}} : \mathcal{P}(A^k) \longrightarrow \mathcal{P}(A^k)$$

$$T \longmapsto \{\bar{a} : (\mathfrak{A}, T) \models \psi(T, \bar{a})\}$$

$\psi^{\mathfrak{A}}$  has a least fixed point **lfp**( $\psi^{\mathfrak{A}}$ ) and a greatest fixed point **gfp**( $\psi^{\mathfrak{A}}$ )

$$\mathfrak{A} \models [\text{lfp } T\bar{x}. \psi(T, \bar{x})](\bar{a}) \iff \bar{a} \in \text{lfp}(\psi^{\mathfrak{A}})$$

$$\mathfrak{A} \models [\text{gfp } T\bar{x}. \psi(T, \bar{x})](\bar{a}) \iff \bar{a} \in \text{gfp}(\psi^{\mathfrak{A}})$$

# Guarded fixed point logic

## Example.

$$\begin{aligned} \psi &:= \exists xy Fxy \wedge (\forall xy. Fxy) \exists z Fyz \wedge \\ &(\forall xy. Fxy) \underbrace{[\text{lfp } Wy. (\forall x. Fxy) Wx]}(y) \\ &\{y : \text{there is no infinite chain } y \leftarrow y_1 \leftarrow y_2 \leftarrow \dots \} \end{aligned}$$

- $\psi$  is satisfiable:  $(\omega, <) \models \psi$
- $\psi$  has no finite models

**Proposition.**  $\mu\text{GF}$  does not have the finite model property.



# Decidability of guarded fixed point logic

**Theorem.** (Grädel, Walukiewicz)  $\text{Sat}(\mu\text{GF})$  2-EXPTIME-complete, and EXPTIME-complete for formulae of bounded width.

same complexity bounds as for GF: no penalty for fixed points !

**Proof.** Relies on generalized tree model property:

Every satisfiable  $\psi \in \mu\text{GF}$  has a model of small tree width

Reduction of  $\text{Sat}(\mu\text{GF})$  to emptiness problem for certain automata (alternating two-way tree automata with parity acceptance condition).

# Final remarks

The world of guarded logics is much richer than shown in this talk:

- more liberal guardedness conditions: loosely guarded, clique guarded, action guarded, packed logics
- guarded fragments of other logics, e.g. GSO, Datalog LITE, ...

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**Conclusion.** Guarded logics provide a general and versatile family of logical systems that generalise, and to a large extent explain the good algorithmic and model-theoretic properties of modal logics.