Banach-Mazur Games on Graphs

Erich Grädel

(joint work with Dietmar Berwanger, Stephan Kreutzer, and Simon Leßenich)

Infinite Games on Graphs

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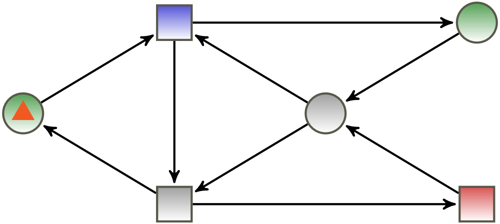

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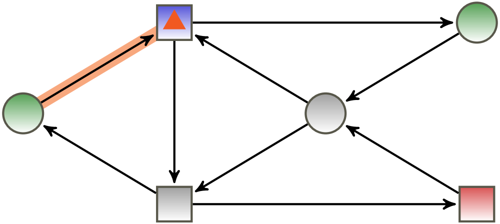

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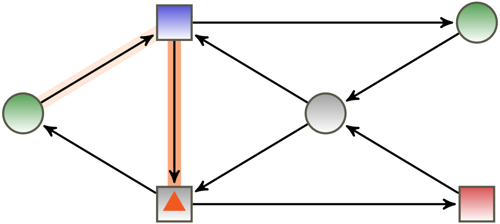

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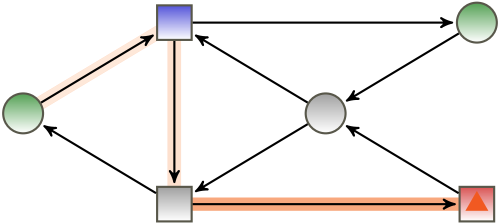

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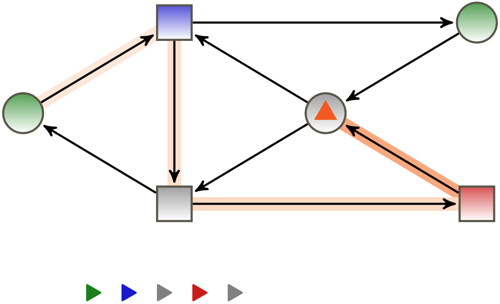

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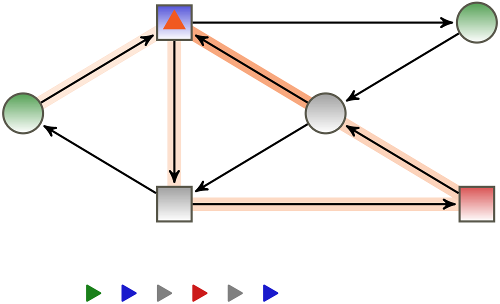

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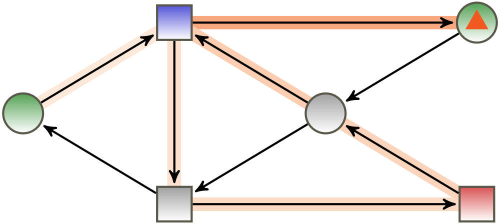

Objectives of the players are given by properties of infinite paths

The classical model of graph games

Arena: $\mathcal{G} = (V, V_0, V_1, E, \Omega)$, $\quad V = V_0 \cup V_1$, $\quad E \subseteq V \times V$

$\Omega : V \to C$ assigns to each position a priority or colour.

Player 0 moves from positions $v \in V_0$, Player 1 moves from $v \in V_1$

In each move the token is taken along an edge to its next position.

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Player 0 wins π, if $\Omega(\pi) \in \mathrm{Win}$, otherwise Player 1 wins.

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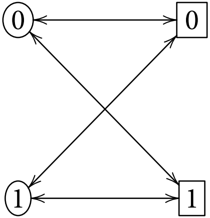
Strategy for Player σ: partial function that assigns moves to initial segments of plays that end in a position in V_σ:

$f : V^* V_\sigma \to V \qquad \text{with} \quad f(v_0 \dots v_n) \in v_n E$

Classical Theory of Gale-Stewart Games

Consider graph games where the arena is the complete bipartite graph K_{22}



or, equivalently, the infinite binary tree.

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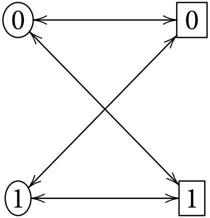

or, equivalently, the infinite binary tree.
Abstract winning condition $\text{Win} \subseteq \{0,1\}^\omega$

This amounts to a game where the players alternatingly select bits $a_i \in \{0,1\}$ in an infinite number of moves and thus produce an infinite word
$a_0 a_1 a_2 a_3 \cdots \in \{0,1\}^\omega$.

Banach-Mazur games

There is another, in fact older, variant of such infinite games where, in each move, the player selects not just a bit, but an arbitrary finite word $w_i \in \{0,1\}^*$, again producing an infinite word $w_0 w_1 w_2 w_3 \cdots \in \{0,1\}^\omega$.

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These games play an important role in topology.

Original version of Banach-Mazur games $G^{**}(W)$ (see *"The Scottish Book. Mathematics from the Scottish Café"*, Birkhäuser 1981, Problem 43):

For a given winning condition $W \subseteq \mathbb{R}$, Player 0 first selects an interval $d_1 \subset \mathbb{R}$, then Player 1 chooses a subinterval $d_2 \subset d_1$, then Player 0 selects a further refinement $d_3 \subset d_2$, and so on ...

Player 0 wins if, and only if, $\bigcap_{n\in\omega} d_n$ contains an element of W.

Banach-Mazur games on topological spaces

Such a game can be played on any topological space X. Let $\mathcal{V}$ be a collection of subsets of X such that each $V \in \mathcal{V}$ contains a non-empty open subset of X, an each non-empty open subset of X contains a $V \in \mathcal{V}$.

Banach-Mazur game on $(X, \mathcal{V})$ with winning condition $W \subseteq X$:

Players take turns to choose sets $V_0 \supset V_1 \supset V_2 \supset \dots$ in $\mathcal{V}$
Player 0 wins if $\bigcap_{n<\omega} V_n \cap W \neq \emptyset$.

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Banach-Mazur games on graphs are a special case in this general topological setting.

Banach-Mazur games on graphs

Arena: a directed graph $G = (V, E)$, with initial position v_0

Winning condition: $\mathrm{Win} \subseteq \mathrm{Paths}(G, v_0)$ of infinite paths from v_0 through G

Playing the game: Player selects a finite path x_0 from v_0; the opponent extends x_0 to a path $x_0 x_1$; then the first player prolongs this to $x_0 x_1 x_2$; and so on. All moves are non-empty and finite: $1 \leq |x_i| < \omega$.

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$\mathrm{Paths}(G, v_0)$ is a topological space, whose basic open sets are $\mathcal{O}(x)$, the set of infinite prololongations of a finite path from v_0.

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When a player prolongs the finite path x played so far to a new path xy, she reduces the set of possible outcomes from $\mathcal{O}(x)$ to $\mathcal{O}(xy)$, and Player 0 wins an infinite play $x_0 x_1 \dots$ if, and only if $\bigcap_{n<\omega} \mathcal{O}(x_0 \dots x_{n-1}) \cap \text{Win} \neq \emptyset$.

Determinacy and Topology

On $\mathrm{Paths}(G, v_0)$, the basic open sets are $\mathcal{O}(x)$, where x is a finite path from v_0 and $\mathcal{O}(x)$ is the set of its infinite prolongations.

A set is open if it is a union of basic open sets, and it is dense if its intersection with every basic open set is non-empty.

Lemma. For any strategy g of Player 1 in a Banach-Mazur game on G, v_0, the set $\mathrm{Plays}(g)$ of plays that are consistent with g is a countable intersection of dense open sets.

$\mathrm{Plays}(g) = \bigcap_{n \in \omega} \mathrm{Plays}_n(g)$ where Plays_n is the set of plays that may arise if Player 1 plays according to g in her first n moves.

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$\mathrm{Plays}_n(g)$ is clearly open. It is also dense because every finite path x can be used by Player 0 as the opening move, so there is a prolongation of x in $\mathrm{Plays}_n(g)$. Hence $\mathcal{O}(x) \cap \mathrm{Plays}_n(g) \neq \emptyset$.

Determinacy and Topology (2)

Topologically, dense open sets are large sets, and so are all countable intersections of such.

Hence, by any strategy, Player 1 can exclude only a topologically small set of plays. Hence, she can only have a winning strategy if the winning set Win of Player 0 is small, and her own winning set $\text{Paths}(G, v_0) \setminus \text{Win}$ is topologically large.

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On the other side, for any topologically small set Y Player 1 has a strategy g to exclude Y. Small sets have the form $Y = \bigcup_{n \in \omega} Y_n$ such that each Y_n is nowhere dense, i.e. its complement contains a dense open set.

In her n-th move, she prolongs the current path to a new path x_n such that $\mathcal{O}(x_n) \cap Y_n = \emptyset$. Then $\text{Plays}(g) \cap Y = \emptyset$.

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Topologically small sets are also called meagre and topologicaly large sets are co-meagre.

Determinacy of Banach-Mazur games

For Player 0, the situation is different: in the opening move, she chooses a finite path x, so that $\text{Plays}(f) \subseteq \mathcal{O}(x)$. But then the remaining play has switched roles, and it follows that, for any strategy f of Player 0, $\text{Plays}(f)$ is large inside $\mathcal{O}(x)$ (i.e. $\mathcal{O}(x) \setminus \text{Plays}(f)$ is topologically small.

Theorem (Banach-Mazur)

In a Banach-Mazur game $\text{BM}(G, v_0, \text{Win})$

(1) Player 1 has a winning strategy $\iff$ Win is meagre.

(2) Player 0 has a winning strategy $\iff$
there exists a finite path x from v_0 such that $\mathcal{O}(x) \setminus W$ is meagre
(i.e. W co-meagre in some basic open set).

A nondetermined Banach-Mazur game

Ultrafilters: A set $U \subseteq \mathcal{P}(\omega)$ is an ultrafilter if it is closed under intersection, contains with any set also all its supersets, does not contain the empty set, and if, for all $x \subseteq \omega$, either $x \in U$ or $\omega \setminus x \in U$.

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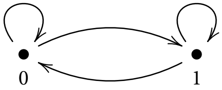

Player 0 wins $x_0 x_1 x_2 \dots \in \{0,1\}^\omega \iff \{n : x_n = 1\} \in U$

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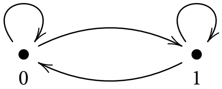

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This game amounts to the following:

Players 0 and 1 choose, in a strictly alternating fashion, an infinite sequence $a_0 < a_1 < a_2 < a_3 < \dots$ of natural numbers. Player 0 wins if, and only if, the set $[0, a_0) \cup [a_1, a_2) \cup [a_3, a_4) \cup \dots$ is in U.

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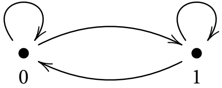

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Theorem. If U is a free ultrafilter, then $G(U)$ is not determined.

Playing the ultrafilter game

Assume: Player 0 has a winning strategy $f : (a_0 < a_1 < \cdots < a_{2n-1}) \quad \mapsto \quad a_{2n}$

Two plays against f: Against a_0 chosen according to f, Player 1 selects an arbitrary $a_1 > a_0$ and proceeds as follows:

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Play 2: a_0

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Play 2: $a_0 \quad < \quad a_2 \quad < \quad a_3 \quad < \quad a_4 \quad < \quad a_5$

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Play 2: $a_0 < a_2 < a_3 < a_4 < a_5 <$

In both plays, Player 0 moves according to her winning strategy f. Hence

$$X_1 := [0, a_0) \cup [a_1, a_2) \cup [a_3, a_4) \cup \cdots \in U$$

$$X_2 := [0, a_0) \cup [a_2, a_3) \cup [a_4, a_5) \cup \cdots \in U$$

Hence $X_1 \cap X_2 \in U$. But this is impossible since $X_1 \cap X_2 = [0, a_0)$ is finite.

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Hence $X_1 \cap X_2 \in U$. But this is impossible since $X_1 \cap X_2 = [0, a_0)$ is finite.

By a similar argument, Player 1 cannot have a winning strategy.

Borel determinacy

Banach-Mazur games with Borel winning conditions are determined.

This follows from the Banach-Mazur Theorem, by the fact that Borel sets have the Baire property, i.e., their symmetric difference with some open set is meagre.

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Remark. Standard winning conditions used in computer science applications (e.g. all ω-regular winning conditions) are in low levels of the Borel hierarchy.

An application: Characterisation of fairness

In verification one often wants to exclude unfair runs from consideration and verify the specification only for the fair ones.

Fairness: Anything that is infinitely often possible, should indeed happen infinitely often.

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Definition. (Varacca, Völzer) A set F of infinite runs through a transition system (G, v) is a fairness property if F is topologically large (i.e. co-meagre) in the set $\mathrm{Paths}(G, v)$ of all possible runs.

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By the Theorem of Banach-Mazur this means that the first player has a winning strategy in the Banach-Mazur game (G, v, F).

Game-theoretic view of fairness:
The scheduler has a strategy to ensure a fair run.

Strategies and positional strategies

A decomposition invariant strategy is of the form $f : V^* \to V^*$. It only depends on the finite path constructed so far, not how it has been built up.

Example. "Do the same what your opponent did in his last move" is not a decomposition invariant strategy.

Proposition. A player who has a winning strategy for a Banach-Mazur game also has one that is decomposition invariant.

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A positional strategy is of the form $f : V \to V^*$. It only depends on the current position, not on the history of the play.

A game is positionally determined if one of the players has a positional winning strategy. A winning condition guarantees positional determinacy if all games with that winning condition are positionally determined.

Importance of positional determinacy

To investigate positional determinacy (or finite-memory determinacy) is a fundamental first step in the analysis of an infinite determined game.

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The positional determinacy of parity games immediately implies that winning regions can be decided in NP $\cap$ Co-NP.
It is conjectured by many that parity games can be solved in polynomial time.
Although this is still open, all known approaches towards an efficient algorithmic solution make use of positional determinacy.

Positional winning strategy for one player

Proposition. If $W \in \Sigma_2^0$ (countable union of closed sets), and Player 0 has a winning strategy for the Banach-Mazur game (G, v, W), then she also has a positional winning strategy.

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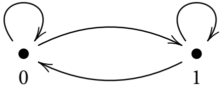
This is not always true for $W \in \mathbf{\Pi_2^0}$:

G_2 :



$$W = \{\pi \in \{0,1\}^\omega : (\forall m)(\exists n > m)\, |\{i < n : \pi(i) = 0\}| \geq n/2\}$$

(infinitely many initial segments of π have more zeros than ones)

Player 0 has a winning strategy for $(G_2, 0, W)$, but no positional one.

ω-regular winning conditions

Consider game graphs $\mathcal{G} = (V, E)$ with colouring $\Omega : V \to C$ for some finite set C of colours.

ω-regular winning conditions are given either by formula in S1S, monadic second-order logic on infinite paths, with predicates $P_c := \{v : \Omega(v) = c\}$ (for $c \in C$), or by finite automata.

Question. What kind of winning strategies are required for Banach-Mazur games with ω-regular winning conditions?

Muller games

An important special case of ω-regular conditions:

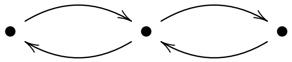
Muller conditions: given by a pair $(\mathcal{F}_0, \mathcal{F}_1)$ such that $\mathcal{F}_0 \subseteq \mathcal{P}(C))$ and $\mathcal{F}_1 = \mathcal{P}(C) \setminus \mathcal{F}_0$.

An infinite play $\pi = v_0 v_1 v_2 \dots$ is won by Player σ if

$$\mathrm{Inf}(\pi) := \{c : (\forall i)(\exists j > i)\Omega(v_j) = c\} \in \mathcal{F}_\sigma.$$

For classical graph games with Muller conditions, positional strategies do not suffice.

Example:



winning condition: all positions must occur infinitely often

Positional Determinacy of Muller Games

Proposition. Muller conditions guarantee positional determinacy for Banach-Mazur games.

Proof. For finite graphs: decompose G into its strongly connected components

Player 0 wins iff there is a leaf component $H \subseteq G$ such that $\Omega(H) \in \mathcal{F}_0$.
But then she also wins with a positional strategy.

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Banach-Mazur games with Muller conditions are positionally determined also on infinite graphs.

Games with ω-regular winning conditions

Theorem. (Büchi-Landweber) Classical graph games with ω-regular winning conditions are determined via finite-memory strategies that can be effectively computed and realised by finite automata.

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Reduction to Muller and parity games. Every game G with ω-regular winning condition W can be transformed into an equivalent game with a game graph $G \times M$ and a Muller or even parity winning condition.

Simplify the winning condition at the cost of enlarging the game graph

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But we can do better!

Eliminating the finite memory

Theorem. Banach-Mazur games that are determined via finite-memory strategies are in fact positionally determined.

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Corollary. For Banach-Mazur games, ω-regular winning conditions guarantee positional determinacy.

Eliminating the finite memory

Assume that Player 0 wins (G, v_0) with a finite memory strategy $f : V \times M \to \text{FinPaths}(G)$ with opening move $x_0 = f(m_0, v_0)$.

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Construct paths $y_1 < y_2 < \dots y_n \in \text{FinPaths}(G, v)$:

- $y_1 := f(v, m_1)$
- $y_{i+1} := y_i \cdot f(v_i, \text{memory}(m_i, y_i))$ (where $v_i = \text{end}(y_i)$)

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Claim. Every play π consistent with g is also consistent with f.

Construct a decomposition $x \cdot g(v) = x \cdot y \cdot f(v', m') \cdot z$.

If $\text{memory}(m_0, x) = m_i$, put $y = y_i$ and $v' = v_i = \text{end}(y_i)$.
Then $\text{memory}(m_0, x y_i) = \text{memory}(m_i, y_i) := m'$.

Positional determinacy and fairness

From the positional determinacy of the Banach-Mazur games with ω-regular winning conditions, it follows that, on finite graphs, Player 0 has a winning strategy if, and only if, the winning condition is probabilistically large (under certain well-behaved measures).

Corollary (Varacca, Völzer) Any ω-regular fairness property has probability 1 under randomised scheduling.

As a further consequence, one can use results about finite Markov chains for checking whether a finite system is fairly correct with respect to LTL or ω-regular specification.

Corollary. Banach-Mazur games on finite graphs with LTL-winning condition can be solved in Pspace.

Beyond ω-regular winning conditions

There are many winning conditions that are not ω-regular, but still guarantee positional determinacy for Banach-Mazur games.

Example. Given a colouring $\Omega : V \to \omega$, require that some colour $n \in \omega$ occurs infinitely often.

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For classical graph games, an infinitary Muller condition gurantees positional determinacy if, and only if, it reduces to a parity condition on some $\alpha \leq \omega$. (Grädel, Walukiewicz)

For Banach-Mazur games, this is clearly not necessary (as shown by finitary Muller conditions) but is it sufficient?

From classical graph games to Banach-Mazur games

Theorem. If $W \subseteq C^\omega$ guarantees finite memory determinacy for classical graph games, then W guarantees positional determinacy for Banach-Mazur games.

For each Banach-Mazur game G, one can construct a classical graph game G' with the same winning condition, such that any finite memory strategy for G' translates into a finite-memory strategy for G. The colourings of plays that are consistent with the two strategies are the same.

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Corollary. The parity condition on ω guarantees positional determinacy for Banach-Mazur games.

(since based on infinitely many colours, this is not an ω-regular condition)

Prefix-independent winning conditions

A set $W \subseteq C^\omega$ is prefix-independent if for all $u, v \in C^*, \alpha \in C^\omega$:

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Proposition.

If a prefix-independent condition W guarantees positional determinacy

- on all strongly connected graphs, and
- on all (infinite, non-terminating) acyclic ones,

then it does so on all graphs.

Infinitary Muller conditions

A set $\mathcal{F} \subseteq \mathcal{P}(C)$ is finitely based if for each infinite $X \in \mathcal{F}$ there is a finite $Y \subseteq X$ such that all Z with $Y \subseteq Z \subseteq X$ belong to $\mathcal{F}$.

Example. The parity condition on ω is finitely based. Each infinite set X has the finite basis $\{\min X\}$.

Conjecture. A Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees positional determinacy of Banach-Mazur games if, and only if, both $\mathcal{F}_0$ and $\mathcal{F}_1$ are finitely based.

It is easy to see that this condition is necessary, and that it is sufficient for strongly connected graphs. It remains to show that is sufficient also for infinite acyclic graphs.

Goal. Find a more general characterisation, for all prefix-independent conditions that guarantee positional determinacy.

Beyond positional determinacy

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Consider the Banach-Mazur game on the completely connected graph with vertex set $\omega = \{0, 1, 2, 3, \dots\}$ where Player 0 wins if all nodes are seen infinitely often.

Clearly, Player 0 wins, but not with a positional strategy.

Nevertheless Player 0 can win using very simple strategies based either on counting or on storing the maximal vertex seen so far:

Take the path $0, 1, 2, \dots, n$

- in your n-th move,
- after a path of length n has been played
- when the maximal node seen so far is $n - 1$.

Simple strategies

Besides the positional strategies and the finite-memory strategies (which anyway provide no additional power for Banach-Mazur games) there are other classes of simple strategies that deserve to be investigated.

- Strategies based on finite appearance records FAR
- Move-counting and length-counting strategies.

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- Move-counting and length-counting strategies.

Let $\mathcal{S}$ be a class of strategies, and $\mathcal{W}$ be a class of winning conditions $W \subseteq C^\omega$. $\mathcal{W}$ guarantees determinacy via $\mathcal{S}$, if every Banach-Mazur game with a winning condition in $\mathcal{W}$ is determined, and the winner has a winning strategy in $\mathcal{S}$.

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We have seen that ω-regular winning conditions guarantee determinacy via positional strategies. Investigate winning conditions that do not guarantee positional determinacy, but determinacy via FAR-strategies and/or counting strategies.

Move-counting strategies

Move-counting strategies have the form $g\colon V \times \omega \to \mathrm{FinPaths}(G)$

- The moves of the strategy's player are counted
- i-th move: $g(v, i)$

Theorem. All Muller conditions $(\mathcal{F}_0, \mathcal{F}_1)$ such that either $\mathcal{F}_0$ or $\mathcal{F}_1$ is countable guarantee determinacy via move-counting strategies.

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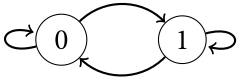
However, move-counting strategies fail for some simple non-prefix-independent winning conditions:

Example



Winning condition: infinitely many initial segments have more 0 than 1.

But this game is determined by a length-counting strategy.

Length-counting strategies

Length-counting strategies have the form $h\colon V \times \omega \to \mathrm{FinPaths}(G)$.
After prefix $\pi \cdot v$, the next move is given by $h(v, |\pi|)$.

Theorem. If $\mathcal{W}$ guarantees determinacy via move-counting strategies, then also via length-counting strategies.

Proof. Given a move-counting strategy $g : V \times \omega \to \mathrm{FinPaths}(G)$, define the length-counting strategy

$$h(v, i) := g(v, 0) \cdot g(\_, 1) \cdot \dots \cdot g(\_, i)$$

Every play consistent with h is also consistent with g.

Finite appearance records (FAR)

FAR are infinite memory structures, motivated by latest appearance records. Defined by Grädel and Kaiser (2007) in the setting of classical graph games with infinitary Muller conditions.

Let $G = (V, E, \Omega : V \to C)$, for an infinite set C of colours.
Memory states: $(C \uplus \Sigma)^d$, for some finite set Σ and $d \in \mathbb{N}$
Updates: at state v, update state $(m_1, \dots m_d)$ to a new tuple, composed of old entries, letters from Σ, and the current colour $\Omega(v)$.
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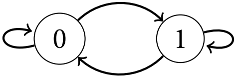
Theorem. If an infinitary Muller condition $(\mathcal{F}_0, \mathcal{F}_1)$ guarantees determinacy via move-counting strategies, then also via one-dimensional FAR strategies.

In particular, this holds when $\mathcal{F}_0$ or $\mathcal{F}_1$ is countable.

FAR strategies versus counting strategies

The classes of winning conditions determined via counting and via FAR strategies are incomparable.

Example.



Winning condition: For every n, the sequence 1^n must be seen infinitely often.

Player 0 wins with a move counting strategy but not with an FAR-strategy.

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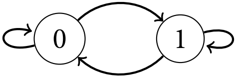

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More complicated examples determined by FAR strategies but not by length-counting strategies.

A game won by an FAR-stratgey but not by a length counting strategy

Vertex set: $\omega \cup (\omega \times \omega)$

Edges: $n \to (n, i)$ and $(n, i) \to n + 1$ for all $n, i \in \omega$

Plays: $0 \cdot (0, n_0) \cdot 1 \cdot (1, n_1) \cdot 2 \cdot (2, n_2) \dots$

Winnning condition: $n_0 n_1 n_2 \dots$ can be split into finite segments of form $m_0 \dots m_k$ such that $m_0 = \prod_{j=1}^{k} p_j^{m_j}$.

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Player 1 can win, with an FAR-strategy, by destroying in her first move any given sequence of prime exponents.

She cannnot win with a length counting strategy g. Indeed, node n can only be reached by paths of length $2n$, hence g assigns to each n a unique move $g(n, 2n)$.

Player 0 wins against g by playing from $(n-1)$ or $(n-2, k)$ to $(n-1, N) \cdot n$ such that $g(n, 2n)$ correspond to the prime factorization of N.

Summary

- Banach-Mazur games are a natural kind of games on graphs.
- Mathematically and algorithmically, Banach-Mazur games tend to be simpler than the common single-step graph games.
- Finite memory determinacy collapses to positional determinacy.
- All Banach-Mazur games with ω-regular winning conditions are positionally determined. Muller games can be solved efficiently.
- Prefix-independent winning conditions guarantee positional determinacy if they do so on strongly connected and infinite acyclic game graphs.

Summary

- Determinacy via move-counting strategies implies determinacy via length-counting strategies. Counting strategies and FAR-strategies are incomparable
- For infinitary Muller conditions
  - Finitely based + strongly connected: positional strategies
  - Countable $\mathcal{F}_\sigma$: move-counting strategies
  - Move-counting strategies imply one-dimensional FAR-strategies
- Banach-Mazur games have recently found applications for the characterisation of fairness and fair verification, for planning in nondeterministic domains, and for the semantics of timed automata.