Zero-One Laws and Almost Sure Valuations of First-Order Logic in Semiring Semantics

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Mathematical Foundations of Computer Science
RWTH AACHEN UNIVERSITY

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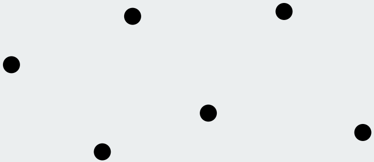
Reminder: Classical 0-1 Law

$$\psi = \exists x_1 \exists x_2 \exists x_3 \exists x_4 \Big(\bigwedge_{i \neq j} x_i \neq x_j \ \wedge \ E x_1 x_2 \wedge E x_2 x_3 \wedge E x_3 x_4 \Big)$$

G(n,p)

$n = 6$

$p = 1/2$



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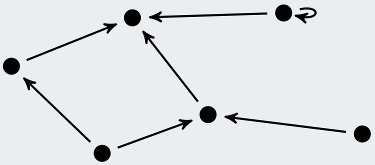

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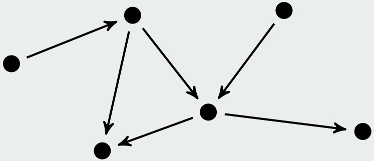

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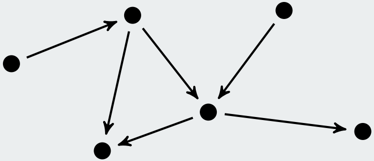

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0-1 law for FO

If $n \to \infty$, the probability that ψ holds converges to either 0 or 1

Reminder: Proof of Classical 0-1 Law

Extension Axioms

"Every configuration of k elements can be extended in every consistent way to $k+1$ elements" (expressible in FO)

1. Each extension axiom is almost surely true
2. Theory of all extension axioms is ω-categorical $\rightsquigarrow$ Rado graph
3. Compactness: ψ or $\neg\psi$ follows from finitely many axioms

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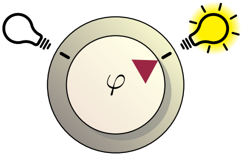
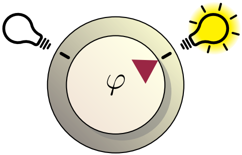


Boolean semantics

Semiring Semantics – An Analogy



Boolean semantics

Semiring semantics

Semiring Semantics

Idea: Replace Boolean values by values from a semiring K
($0 =$ false, $j > 0 =$ shades of true)

$K = (\{0 <$  $<$  $<$  $\}, \max, \min, 0,$  $)$

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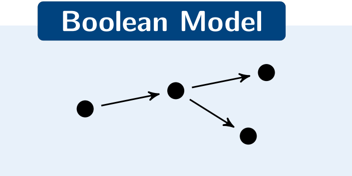
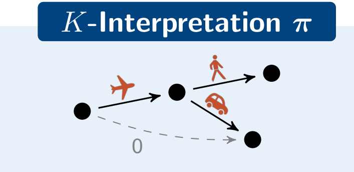


$\leadsto$



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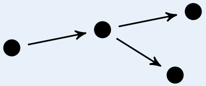
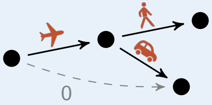
Boolean Model



$$G \models \exists x \exists y \exists z \, (Exy \wedge Eyz) =: \psi$$

$\rightsquigarrow$

K-Interpretation π



Semiring Semantics

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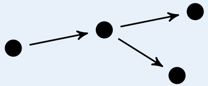
Boolean Model



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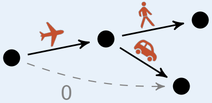
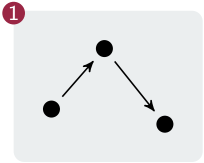


$$\pi[\![\psi]\!] = \max_{a,b,c} \min(\pi(Eab), \pi(Ebc)) = 🚗$$

alternative use — joint use of information

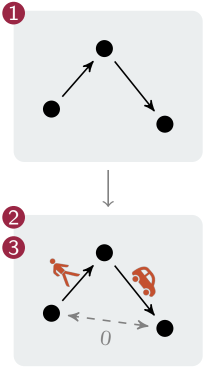
Random K-Interpretations



Random process:

1. Choose a $G(n, 1/2)$ random graph

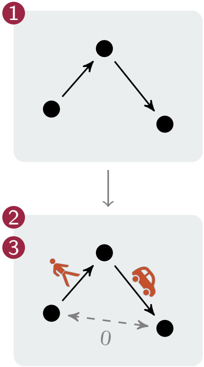
Random K-Interpretations



Random process:

1. Choose a $G(n, 1/2)$ random graph
2. Map false literals to 0
3. Map true literals to random values > 0

Random K-Interpretations



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! Consistency: one of $\pi(Eab)$, $\pi(\neg Eab)$ is 0

Questions

How does the partition of FO into almost surely true and almost surely false sentences generalize to semiring semantics?

Example:

$$\exists x\ \exists y\ \exists z\,(Exy \wedge Eyz)$$

$$\exists x\ \forall y\ Exy$$

Questions

How does the partition of FO into almost surely true and almost surely false sentences generalize to semiring semantics?

1. **0-1 law:**
   When $n \to \infty$, does the probability that ψ evaluates to $j \in K$ in a random semiring interpretation converge to either 0 or 1?

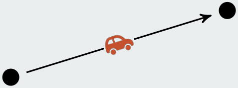
2. **Almost sure valuations (ASV):**
   Which values $j \in K$ may appear with probability 1?
   Does this depend on the semiring?

3. **Complexity:**
   How can we compute ASV(ψ)?

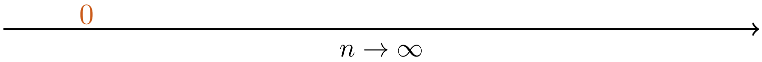
Example: Almost Sure Valuation

$$\psi = \exists x\, \exists y\, \exists z\, \big(Exy \wedge Eyz\big)$$

$n = 2$



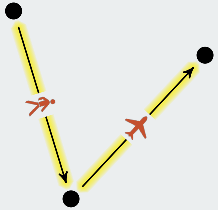
$\pi[\![\psi]\!] = 0$



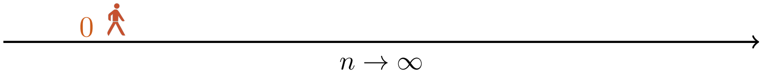
Example: Almost Sure Valuation

$$\psi = \exists x\ \exists y\ \exists z\ \big(Exy \wedge Eyz\big)$$

$n = 3$



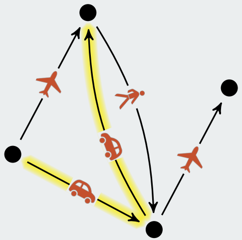
$\pi[\![\psi]\!] =$



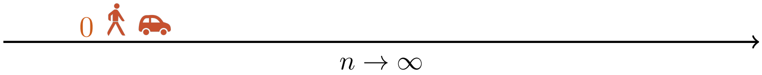
Example: Almost Sure Valuation

$$\psi = \exists x\ \exists y\ \exists z\ \big(Exy \wedge Eyz\big)$$

$n = 4$



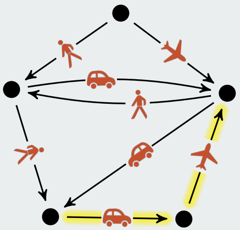
$\pi[\![\psi]\!] =$ 🚗



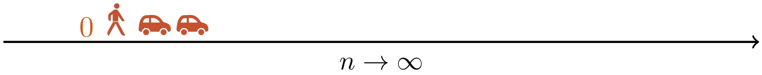
Example: Almost Sure Valuation

$$\psi = \exists x\ \exists y\ \exists z\ \big(Exy \wedge Eyz\big)$$

$n = 5$



$\pi[\![\psi]\!] =$



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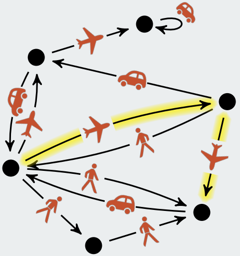

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$n \to \infty$

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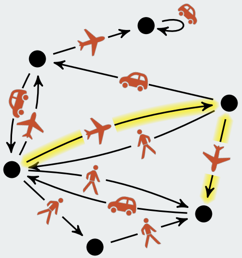

$\pi[\![\psi]\!] =$ 

0  $\dots$ ASV$(\psi) =$ 

$n \to \infty$

Classical Proof Revisited

Extension Property

includes semiring values

❶ "Every configuration of k elements can be extended in every consistent way to $k + 1$ elements".

✓

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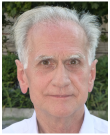

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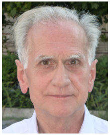

❷ Theory of extension axioms

❸ Compactness

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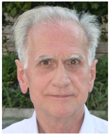

❷ ~~Theory of extension axioms~~

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?

Use polynomials f_ψ to describe $\mathrm{ASV}(\psi)$!

Example: Polynomial for ASV(ψ)

$$\psi = \forall x \left(Exx \lor (\neg Exx \land \exists y\, Exy) \right)$$

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❷

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Max value ✈ occurs almost surely (extension property!)

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1, 2, 3

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(1) Exy (2) $\exists y\, Exy$ (3) $Exx \lor (\neg Exx \land \exists y\, Exy)$

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(1) $\exists y\, Exy$ inner part Exy; (2) $\exists y\, Exy$; (3) $Exx \vee (\neg Exx \wedge \exists y\, Exy)$; (4) $\forall x (\dots)$

1. $f_{\varphi_1} = Exy$
2. $f_{\varphi_2} = \text{✈}$
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Results

Max-Min Semirings

- 0-1 law holds, with almost sure valuation $\text{ASV}(\psi) = f_\psi$
- Possible ASVs: 0 
- Computing $\text{ASV}(\psi)$ is PSPACE-complete

also: (in)finite lattice semirings

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0-1 law for Tropical semiring $(\mathbb{R}_+^\infty, \min, +, \infty, 0)$

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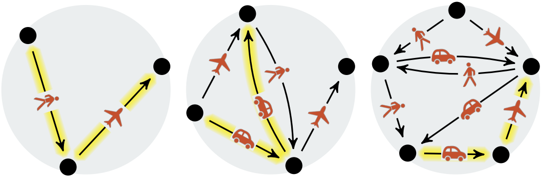
Natural Numbers

- 0-1 law holds (by modifying f_ψ)
- ASVs: 0 or unbounded (except for trivial cases)

Summary & Outlook

$\underbrace{\exists x \exists y \exists z}_{\max} \underbrace{(Exy \land Eyz)}_{\min}$



Conclusion:

- 0-1 law generalizes to semiring semantics

- Tools: extension property + polynomials f_ψ

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- more general random structures (probability depends on n)
- different logic: Σ_1^1 (0-1 law depends on prefix class)
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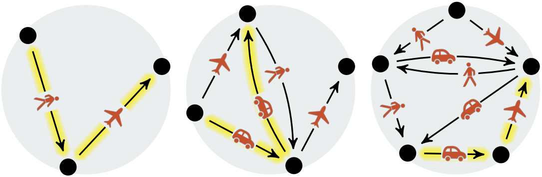

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Thanks for your attention