Zero-One Laws and Almost Sure Valuations of First-Order Logic in Semiring Semantics

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Reminder: Classical 0-1 Law

\[
\psi = \exists x_1 \exists x_2 \exists x_3 \exists x_4 \left( \bigwedge_{i \neq j} x_i \neq x_j \land Ex_1 x_2 \land Ex_2 x_3 \land Ex_3 x_4 \right)
\]

\[G(n,p)\]

\[n = 6\]
\[p = \frac{1}{2}\]
Reminder: Classical 0-1 Law

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\[ n = 6 \]
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\[ \not \models \psi \]
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**G(n,p)**

- \( n = 6 \)
- \( p = \frac{1}{2} \)

If \( n \to \infty \), the probability that \( \psi \) holds converges to either 0 or 1.
Reminder: Proof of Classical 0-1 Law

**Extension Axioms**

“Every configuration of $k$ elements can be extended in every consistent way to $k + 1$ elements” (expressible in FO)

1. Each extension axiom is **almost surely true**
2. Theory of all extension axioms is **$\omega$-categorical** $\leadsto$ Rado graph
3. **Compactness**: $\psi$ or $\neg\psi$ follows from finitely many axioms
Reminder: Proof of Classical 0-1 Law

**Extension Axioms**

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**0-1 law for FO**

If $n \to \infty$, the probability that $\psi$ holds converges to either 0 or 1
Boolean semantics
Boolean semantics

Semiring semantics
Semiring Semantics

**Idea:** Replace Boolean values by values from a semiring $K$

$(0 = \text{false}, \ j > 0 = \text{shades of true})$

$$K = (\{0 < \text{foot} < \text{car} < \text{plane}\}, \ \text{max}, \ \text{min}, \ 0, \ \Rightarrow)$$
Semiring Semantics

**Idea:** Replace Boolean values by values from a semiring $K$
($0 = \text{false}, j > 0 = \text{shades of true}$)

$$K = (\{0 < \text{false} < \text{car} < \text{plane}\}, \text{max}, \text{min}, 0, \to)$$

Boolean Model

\[
\exists x \exists y \exists z (Exy \land Eyz) =: \psi
\]

$K$-Interpretation $\pi$

Matthias Naaf (RWTH Aachen)
**Semiring Semantics**

**Idea:** Replace Boolean values by values from a semiring $K$
($0 = \text{false, } j > 0 = \text{shades of true}$)

$$K = (\{0 < \text{human} < \text{car} < \text{plane}\}, \text{max, min, 0, } \rightarrow)$$

Boolean Model

$$G \models \exists x \exists y \exists z (E_{xy} \land E_{yz}) =: \psi$$

$K$-Interpretation $\pi$

Matthias Naaf (RWTH Aachen)
**Semiring Semantics**

**Idea:** Replace Boolean values by values from a semiring $K$

$(0 = \text{false}, j > 0 = \text{shades of true})$

$K = (\{ 0 < \text{man} < \text{car} < \text{airplane} \}, \max, \min, 0, \rightarrow)$

**Boolean Model**

$G \models \exists x \exists y \exists z (Exy \land Eyz) =: \psi$

**$K$-Interpretation $\pi$**

$\pi[\psi] = \max_{a,b,c} \min(\pi(Eab), \pi(Ebc)) = \text{car}$

alternative use

joint use of information

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Random process:

1. Choose a $G(n, \frac{1}{2})$ random graph
Random $K$-Interpretations

Random process:

1. Choose a $G(n, 1/2)$ random graph
2. Map false literals to 0
3. Map true literals to random values $> 0$
Random \( K \)-Interpretations

**Random process:**

1. Choose a \( G(n, \frac{1}{2}) \) random graph
2. Map false literals to 0
3. Map true literals to random values > 0

⚠️ Consistency: one of \( \pi(E_{ab}), \pi(\neg E_{ab}) \) is 0
Questions

How does the partition of FO into almost surely true and almost surely false sentences generalize to semiring semantics?

Example:

$$\exists x \exists y \exists z \ (E_{xy} \land E_{yz})$$

$$\exists x \forall y \ E_{xy}$$
Questions

How does the partition of FO into almost surely true and almost surely false sentences generalize to semiring semantics?

1. **0-1 law:**
   When $n \to \infty$, does the probability that $\psi$ evaluates to $j \in K$ in a random semiring interpretation converge to either 0 or 1?

2. **Almost sure valuations (ASV):**
   Which values $j \in K$ may appear with probability 1? Does this depend on the semiring?

3. **Complexity:**
   How can we compute $\text{ASV}(\psi)$?
Example: Almost Sure Valuation

\[ \psi = \exists x \exists y \exists z \, (Exy \land Eyz) \]

\[ n = 2 \]

\[ \pi[\psi] = 0 \]

\[ n \to \infty \]
Example: Almost Sure Valuation

\[ \psi = \exists x \exists y \exists z (E_{xy} \land E_{yz}) \]

\[ n = 3 \]

\[ \pi[\psi] = \]

\[ 0 \]

\[ n \to \infty \]
Example: Almost Sure Valuation

\[ \psi = \exists x \exists y \exists z (Exy \land Eyz) \]

\( n = 4 \)

\( \pi[\psi] = \)

\( n \rightarrow \infty \)
Example: Almost Sure Valuation

\[ \psi = \exists x \exists y \exists z \ (E_{xy} \land E_{yz}) \]

\[ n = 5 \]

\[ \pi \left[ \psi \right] = \text{car} \]

\[ 0 \quad \text{people} \quad \text{cars} \quad \text{cars} \]

\[ n \to \infty \]
Example: Almost Sure Valuation

\[ \psi = \exists x \exists y \exists z (E_{xy} \land E_{yz}) \]

\[ n = 6 \]

\[ \pi[\psi] = \text{Airplane} \]

\[ 0, \text{Car, Car, Airplane, …} \]

\[ n \to \infty \]
Example: Almost Sure Valuation

\[ \psi = \exists x \, \exists y \, \exists z \, (E_{xy} \land E_{yz}) \]

\[ n = 6 \]

\[ \pi[\psi] = \]

\[ ASV(\psi) = \]

\[ n \to \infty \]
“Every configuration of $k$ elements can be extended in every consistent way to $k + 1$ elements”.

Extension Property

includes semiring values
Extension Property

“Every configuration of \( k \) elements can be extended in every consistent way to \( k + 1 \) elements”.

“\begin{quote}
The first-order 0-1 law looks sophisticated but follows from shallow computations\end{quote}” \quad \textit{Béla Bollobás}
Extension Property

1. “Every configuration of $k$ elements can be extended in every consistent way to $k + 1$ elements”.

“The first-order 0-1 law looks sophisticated but follows from shallow computations”

Béla Bollobás

2. Theory of extension axioms

3. Compactness
Classical Proof Revisited

**Extension Property**

1. "Every configuration of \( k \) elements can be extended in every consistent way to \( k + 1 \) elements".

"The first-order 0-1 law looks sophisticated but follows from shallow computations"  
*Béla Bollobás*

2. Theory of extension axioms

3. Compactness

Use polynomials \( f_\psi \) to describe \( \text{ASV}(\psi) \)
Example: Polynomial for ASV(ψ)

\[ \psi = \forall x \left( Exx \lor (\neg Exx \land \exists y Exy) \right) \]
Example: Polynomial for $\text{ASV}(\psi)$

$$\psi = \forall x \left( Exx \lor (\neg Exx \land \exists y Exy) \right)$$

1. $f_{\varphi_1} = Exy$
Example: Polynomial for ASV(ψ)

\[ \psi = \forall x (E_{xx} \lor (\neg E_{xx} \land \exists y E_{xy})) \]

1. \( f_{\varphi_1} = E_{xy} \)
2. \( f_{\varphi_2} = \) Max value occurs almost surely (extension property!)
Example: Polynomial for \( \text{ASV}(\psi) \)

\[
\psi = \forall x \left( Exx \lor (\neg Exx \land \exists y E xy) \right)
\]

\( f_{\varphi_1} = Exy \)

\( f_{\varphi_2} = \)

\( f_{\varphi_3} = \max (Exx, \min(\neg Exx, \rightarrow)) \)

Max value \( \rightarrow \) occurs almost surely (extension property!)
Example: Polynomial for ASV(ψ)

\[ \psi = \forall x \left( Exx \lor (\neg Exx \land \exists y Exy) \right) \]

\[ f_{\varphi_1} = Exy \]
\[ f_{\varphi_2} = \text{Max value occurs almost surely (extension property!)} \]
\[ f_{\varphi_3} = \max (Exx, \min(\neg Exx, \rightarrow)) \]

Consistency: \( Exx, \neg Exx \)
Example: Polynomial for \( \text{ASV}(\psi) \)

\[
\psi = \forall x \left( Exx \lor (\neg Exx \land \exists y Exy) \right)
\]

\[f_{\varphi_1} = Exy\]

\[f_{\varphi_2} = \text{airplane}\]

\[f_{\varphi_3} = \max (Exx, \min(\neg Exx, \text{airplane}))\]

\[f_{\varphi_4} = \min \left\{ \max (0, \text{airplane}), \max (0, \text{airplane}) \right\} = \text{human}\]

Max value \( \text{airplane} \) occurs almost surely (extension property!)

⚠️ Consistency: \( Exx, \neg Exx \)
Example: Polynomial for $\text{ASV}(\psi)$

\[
\psi = \forall x \left( E_{xx} \lor (\neg E_{xx} \land \exists y E_{xy}) \right)
\]

1. $f_{\varphi_1} = E_{xy}$
2. $f_{\varphi_2} = E_{xx}$
3. $f_{\varphi_3} = \max (E_{xx}, \min(\neg E_{xx}, \quad))$
4. $\min \left\{ \max (\ , \min(0 , \quad)), \quad \right\} = \text{ASV}(\psi)$

Max value $\rightarrow$ occurs almost surely (extension property!)

⚠️ Consistency: $E_{xx}, \neg E_{xx}$
Max-Min Semirings

- 0-1 law holds, with almost sure valuation $\text{ASV}(\psi) = f_{\psi}$
- Possible ASVs: $0$, $\infty$, $\times$, $\rightarrow$
- Computing $\text{ASV}(\psi)$ is $\text{PSPACE}$-complete
Results

Max-Min Semirings

- 0-1 law holds, with almost sure valuation $\text{ASV}(\psi) = f_\psi$
- Possible ASVs: 0, ✗, ✗, ✗
- Computing $\text{ASV}(\psi)$ is $PSPACE$-complete

also: (in)finite lattice semirings
Results

Max-Min Semirings
- 0-1 law holds, with almost sure valuation $\text{ASV}(\psi) = f_\psi$
- Possible ASVs: 0 ✗
- Computing $\text{ASV}(\psi)$ is PSPACE-complete

Corollary
0-1 law for Tropical semiring $(\mathbb{R}_+, \text{min}, +, \infty, 0)$
Max-Min Semirings

- 0-1 law holds, with almost sure valuation $ASV(\psi) = f_\psi$
- Possible ASVs: 0, 0
- Computing $ASV(\psi)$ is $PSPACE$-complete

Corollary

0-1 law for Tropical semiring $(\mathbb{R}_{\infty}^+, \min, +, \infty, 0)$

Natural Numbers

- 0-1 law holds (by modifying $f_\psi$)
- ASVs: 0 or unbounded (except for trivial cases)
Conclusion:

- 0-1 law generalizes to semiring semantics
- Tools: extension property + polynomials $f_\psi$

Outlook:

- more general random structures (probability depends on $n$)
- different logic: $\Sigma_1$ (0-1 law depends on prefix class)
- non-definability results?
Summary & Outlook

\[ \exists x \exists y \exists z (E_{xy} \land E_{yz}) \]

\[
\begin{align*}
\max & \quad (Exy \land Eyz) \\
\min & \quad (Exy \land Eyz)
\end{align*}
\]

Conclusion:
- 0-1 law generalizes to semiring semantics
- Tools: extension property + polynomials \( f_\psi \)

Outlook:
- more general random structures (probability depends on \( n \))
- different logic: \( \Sigma_1^1 \) (0-1 law depends on prefix class)
- non-definability results?

Thanks for your attention