Semiring Provenance for Büchi Games: Strategy Analysis with Absorptive Polynomials

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Traveling in Manhattan with minimal effort

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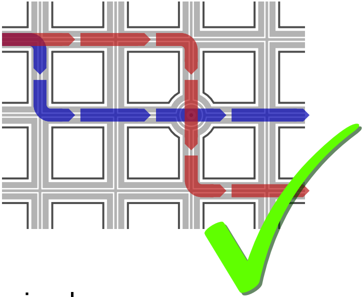

simple
(always go straight)

not positional
(no unique direction at roundabout)

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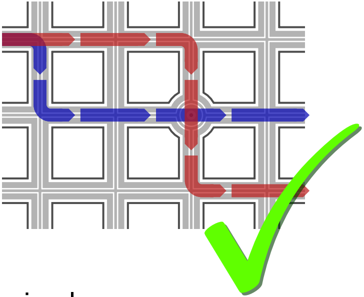

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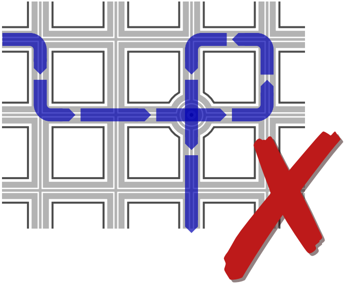
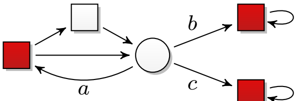


simple
(always go straight)

not minimal effort
(the loop is redundant)

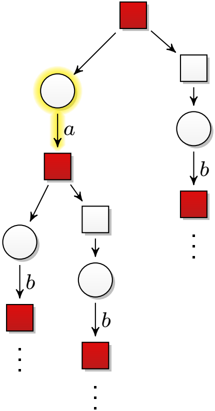
Winning Büchi games with minimal effort

Büchi Games



- infinite duration
- ◯ = Player 0 (we)
  □ = Player 1 (opponent)
- we win an infinite play if target nodes occur infinitely often
- positional determinacy

Strategies as Trees



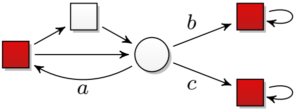
Strategies with minimal effort

> “Perfection is achieved, not when there is nothing more to add, but when there is nothing left to take away”
>
> *Antoine de Saint-Exupéry*

Idea

- positional = minimize the set of edges
- “Manhattan” = minimize the multi-set of edges

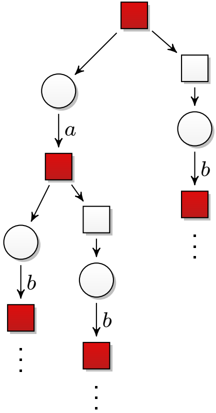
Strategies with minimal effort – Example



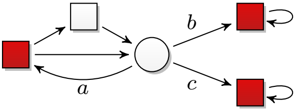
Strategy: tree ex.

Edge profile:

a	1
b	3
c	$-$



Strategies with minimal effort – Example

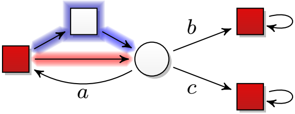


Strategy: tree ex. always a

Edge profile:

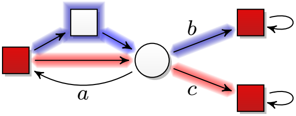
	tree ex.		always a
a	1	a	∞
b	3	b	–
c	–	c	–

Strategies with minimal effort – Example



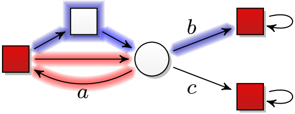
Strategy:	tree ex.	always a	always b
Edge profile: a	1	∞	–
b	3	–	2
c	–	–	–

Strategies with minimal effort – Example



Strategy:	tree ex.	always a	always b	b or c
Edge profile:	a 1 / b 3 / c –	a ∞ / b – / c –	a – / b 2 / c –	a – / b 1 / c 1

Strategies with minimal effort – Example

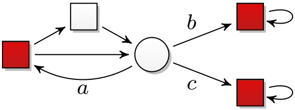


Strategy: tree ex. | always a | always b | b or c | b or a

Edge profile:

Strategy	tree ex.	always a	always b	b or c	b or a
a	1	∞	–	–	∞
b	3	–	2	1	∞
c	–	–	–	1	–

Strategies with minimal effort – Example

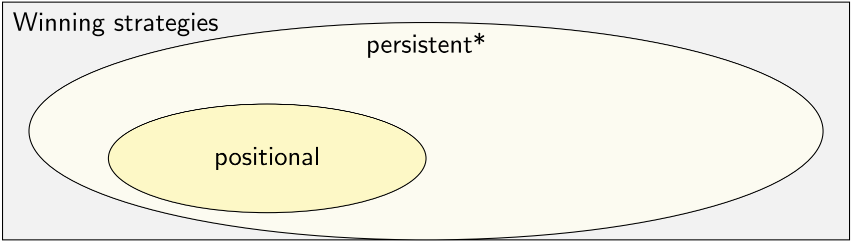


Strategy:	tree ex.	always a	always b	b or c	b or a
Edge profile:	a 1	a ∞	a –	a –	a ∞
	b 3	b –	b 2	b 1	b ∞
	c –	c –	c –	c 1	c –

absorbed absorbed

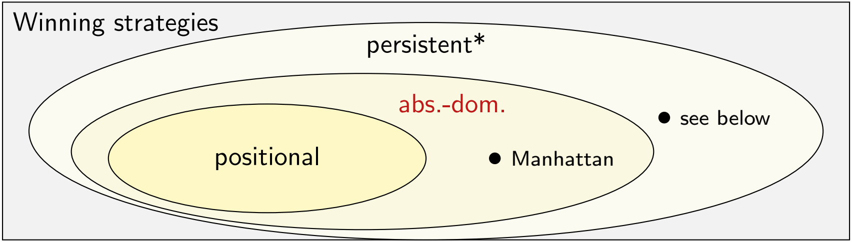
The remaining strategies are **absorption-dominant** ("Manhattan").

Relation to other Strategies



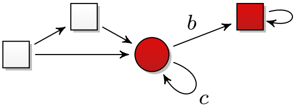
*persistent = positional within each play

Relation to other Strategies



*persistent = positional within each play

Counterexample:



b	2
c	$-$

b	$-$
c	∞

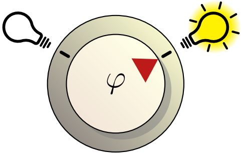
b	1
c	∞

persistent, but not abs.-dom.

Chapter II

Semiring Provenance

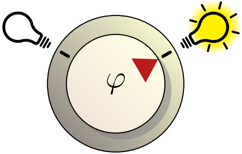
Semiring Semantics with Lightbulbs



Boolean semantics

Semiring semantics

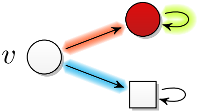
Semiring Semantics with Lightbulbs



Boolean semantics

Semiring semantics

Semiring provenance



$\pi[\![\varphi_{\mathsf{win}}(v)]\!] =$ 

Semirings

Commutative Semiring

$(K, +, \cdot, 0, 1)$ such that $(K, +, 0)$ and $(K, \cdot, 1)$ are commutative monoids, $\cdot$ distributes over $+$, $0 \neq 1$ and $0 \cdot a = 0$.

- $\mathbb{B} = (\{0,1\}, \vee, \wedge, 0, 1)$ — $\widehat{=}$ Boolean semantics
- $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ — count proofs, bag semantics
- $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ — cost computation
- $\mathbb{A} = (\{\mathsf{P} < \mathsf{C} < \mathsf{S} < \mathsf{T} < 0\}, \min, \max, 0, \mathsf{P})$ — access levels

Semirings

Commutative Semiring

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- $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ count proofs, bag semantics
- $\mathbb{T} = (\mathbb{R}_+^\infty, \min, +, \infty, 0)$ cost computation
- $\mathbb{A} = (\{\mathsf{P} < \mathsf{C} < \mathsf{S} < \mathsf{T} < 0\}, \min, \max, 0, \mathsf{P})$ access levels
- Polynomials $(\mathbb{N}[\mathbf{X}], +, \cdot, 0, 1)$ most general semiring

use indeterminates to track facts:

 $= X$,  $= Y$.

instantiate polynomial to obtain value in any other semiring

Semiring Semantics of FO

- Assumptions: finite universe A, relational signature τ.
- $\text{Lit}_A(\tau) := \text{Atoms}_A(\tau) \cup \text{NegAtoms}_A(\tau) \cup \{a \overset{\neq}{=} b \mid a, b \in A\}$.

K-interpretation

$\pi\colon \text{Lit}_A(\tau) \to K$ mapping (in)equalities to their truth values 0 or 1, such that precisely one of $\pi(R\bar{a})$ and $\pi(\neg R\bar{a})$ is 0.

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FO-semantics

$$\pi[\![\varphi \vee \psi]\!] = \pi[\![\varphi]\!] + \pi[\![\psi]\!] \qquad \pi[\![\varphi \wedge \psi]\!] = \pi[\![\varphi]\!] \cdot \pi[\![\psi]\!]$$

$$\pi[\![\exists x\, \varphi(x)]\!] = \textstyle\sum_{a \in A} \pi[\![\varphi(a)]\!] \qquad \pi[\![\forall x\, \varphi(x)]\!] = \textstyle\prod_{a \in A} \pi[\![\varphi(a)]\!]$$

$$\pi[\![\neg\varphi]\!] = \pi[\![\text{nnf}(\neg\varphi)]\!]$$

Semiring Semantics of LFP

To express winning regions in games, we need fixed-point logic.

Q: Can semiring semantics be generalized from FO to LFP?

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To express winning regions in games, we need fixed-point logic.

Q: Can semiring semantics be generalized from FO to LFP?
A: Yes, with some assumptions (Dannert, Grädel, N., Tannen, CSL'21)

- **natural order:**
  $a \leq b \iff \exists c\,(a + c = b)$
- **full continuity:**
  $\leq$ is complete lattice and $+, \cdot$ preserve $\bigsqcup, \bigsqcap$

} well-defined semantics

- **absorption:**
  $a \cdot b \leq b$

} meaninful semantics

✓ $\mathbb{B}$, $\mathbb{T}$, $\mathbb{A}$

✗ $\mathbb{N}$, $\mathbb{N}[\mathbf{X}]$

Modify $\mathbb{N}[\mathbf{X}]$ by

- dropping coefficients,

$$2X^2Y + XY^2 + 5X^2 + 3Z^{10}$$

Absorptive Polynomials

Modify $\mathbb{N}[\mathbf{X}]$ by

- dropping coefficients,
- absorption among monomials (by comparing exponents),

$$2X^2Y + XY^2 + 5X^2 + 3Z^{10}$$

absorbed

Absorptive Polynomials

Modify $\mathbb{N}[\mathbf{X}]$ by

- dropping coefficients,
- absorption among monomials (by comparing exponents),
- allowing ∞ as exponent.

$$2X^2Y + XY^2 + 5X^2 + 3Z^\infty$$

Absorptive polynomials $\mathbb{S}^\infty[\mathbf{X}]$ are

- always finite (Dickson's lemma),
- the most general absorptive, fully-continuous semiring.

Chapter III

Case Study: Strategy Analysis

Case Study

Objective

Apply semiring semantics of LFP to strategy analysis in infinite games

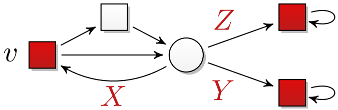
Why Büchi games?

- simplicity, relevance, ...
- non-trivial winning condition for infinite plays

Which formula and semiring?

- winning region formula: $\varphi_{\mathsf{win}}(v) = \mathsf{gfp} \ldots \mathsf{lfp} \ldots$
- absorptive polynomials $\mathbb{S}^\infty[\mathbf{X}]$ to track edges

Case Study – Overview



edge-tracking
$\mathbb{S}^\infty[\mathbf{X}]$-interpretation π

$+$

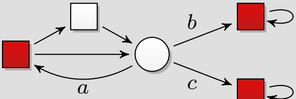
$$\varphi_{\mathsf{win}}(x) := \big[\,\mathsf{gfp}\, Yy.\ [\mathsf{lfp}\, Zz.\ \vartheta(Y,Z,z)](y)\big](x) \qquad \text{where}$$

$$\vartheta(Y,Z,z) := \Big(Fz \ \wedge\ ((V_0 z \wedge \exists u(Ezu \wedge Yu)) \vee (V_1 z \wedge \forall u(Ezu \to Yu)))\Big)$$
$$\vee \Big(\neg Fz \ \wedge\ ((V_0 z \wedge \exists u(Ezu \wedge Zu)) \vee (V_1 z \wedge \forall u(Ezu \to Zu)))\Big)$$

$=$

$$\pi[\![\varphi_{\mathsf{win}}(v)]\!] = X^\infty + Y^2 + Z^2 + YZ$$

Old Memories

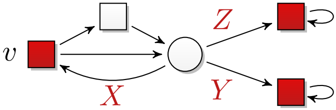


Strategy:	tree ex.	always a	always b	b or c	b or a
Edge profile: a	1	∞	—	—	∞
b	3	—	2	1	∞
c	—	—	—	1	—

absorbed absorbed

The remaining strategies are **absorption-dominant** ("Manhattan").

Sum-of-Strategies Theorem



$\pi[\![\varphi_{\mathsf{win}}(v)]\!] = X^\infty + Y^2 + Z^2 + YZ$

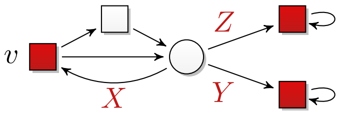
Theorem

For K absorptive and fully continuous, π edge-tracking:

$$\pi[\![\varphi_{\mathsf{win}}(v)]\!] = \sum \left\{ \pi[\![\mathcal{S}]\!] \;\middle|\; \begin{array}{l} \mathcal{S} \text{ is an absorption-dominant} \\ \text{winning strategy from } v \end{array} \right\}$$

product of $\pi(Evw)$ for all edges in the strategy tree
(e.g. minimal access level of $\mathcal{S}$, total cost, . . .)

Results on Strategy Analysis



$$\pi[\![\varphi_{\mathsf{win}}(v)]\!] = X^\infty + Y^2 + Z^2 + YZ$$

Corollary

From the polynomial $\pi[\![\varphi_{\mathsf{win}}(v)]\!]$ in $\mathbb{S}^\infty[\mathbf{X}]$ we can derive:

- whether Player 0 wins
- edge profiles of all absorption-dominant winning strategies
- number and shape of all positional winning strategies
- whether Player 0 can still win if we delete edges
- (how often an edge can occur in a play of a strategy)

Application: Minimal Repairs

Task: Add or delete edges so that Player 0 wins

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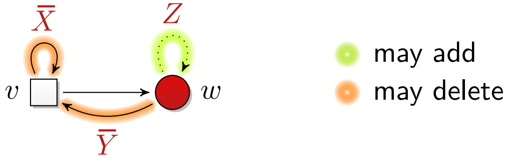


Solution: Use $\mathbb{S}^\infty[\mathbf{X}, \bar{\mathbf{X}}]$, get $\pi[\![\varphi_{\mathsf{win}}(v)]\!] = \bar{X} Z^\infty + \bar{X}^\infty Y^\infty$

delete X, add Z — delete X

Proposition: All minimal repairs occur as monomials
(and all monomials represent repairs)

Further Remarks

Limitation: Cost computation in $\mathbb{T}$

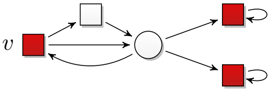
- Reasonable cost measure for a strategy? "Unlocking fees"
- Problem: $\pi[\![\mathcal{S}]\!]$ is product over *all* edges
  (pay multiples times, pay for all plays at once)
- Workaround: Use polynomial (expensive)

Complexity

- $\mathbb{S}^\infty[\mathbf{X}]$: polynomials can (and will) become exponentially large
- Computing $\pi[\![\varphi_{\mathsf{win}}(v)]\!]$: polynomially many semiring operations

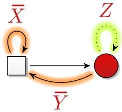
Summary

Case Study:



$+ \quad \varphi_{\mathsf{win}}(v) \quad +$

Results



- $\pi[\![\varphi_{\mathsf{win}}(v)]\!]$ = sum of winning strategies
- Absorptive polynomials: detailed information on absorption-dominant winning strategies
- Extensions: track winning region, parity games, . . .

Absorption-dominant strategies

- minimize multi-set of edges
- positional $\subsetneq$ abs.-dom. $\subsetneq$ persistent