Semiring Provenance for Fixed-Point Logic

Katrin Dannert, Erich Grädel, Matthias Naaf, Val Tannen

January 25, CSL 2021

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Provenance: RWTH Aachen, Germany — University of Pennsylvania, Philadelphia, USA

Provenance Analysis of Logic

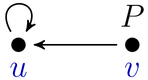
$$\mathfrak{A} \models \varphi$$

Why does the formula hold?

How many proofs are there?

Which literals contribute to its truth?

Provenance Analysis by Semirings Semantics

$\mathfrak{A}:$ 

$$\varphi(x) = (\neg Px \vee Exx) \wedge \exists y\, Py$$

Evaluation: $\varphi(u) = (\neg Pu \vee Euu) \wedge (Pu \vee Pv)$

$\neg Pu \to \top$, $Euu \to \top$, $Pu \to \bot$, $Pv \to \top$

Provenance Analysis by Semirings Semantics

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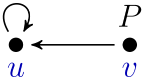

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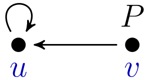

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Evaluation: $\varphi(u) = (\neg Pu \vee Euu) \wedge (Pu \vee Pv)$

- $\mathbb{B}$: $(\top \vee \top) \wedge (\bot \vee \top) = \top$
- $\mathbb{N}[X]$: $(x + y) \cdot (0 + z) = xz + yz$

additional truth values as provenance information

Development

Origin: Database Theory

- [Green, Karvounarakis, Tannen, PODS 2007]:
  Provenance semirings

From Databases to Logic and Games

- [Grädel, Tannen, 2017]:
  Semirings for logics with negation
- [Grädel, Tannen, 2020]:
  Reachability games and logics with least fixed points only

This Talk

- Logic with least and greatest fixed points

Semiring Semantics for FO

Assumptions

- Finite universe A (elements a, $\mathbf{a}$)
- Finite relational signature τ
- Formula φ in negation normal form

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Commutative Semiring
$(K, +, \cdot, 0, 1)$

Semiring Interpretations

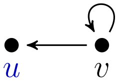
Instantiated literals $\{R\mathbf{a}, \neg R\mathbf{a}, \dots\}$

$$\pi \colon \mathsf{Lit} \to K$$

$$\pi[\![\varphi \wedge \vartheta]\!] = \pi[\![\varphi]\!] \cdot \pi[\![\vartheta]\!] \qquad \pi[\![\exists x\, \varphi(x)]\!] = \textstyle\sum_{a \in A} \pi[\![\varphi(a)]\!]$$

In this talk: $\pi(R\mathbf{a}) > 0$, $\pi(\neg R\mathbf{a}) = 0$ (or vice versa)

Semiring Semantics for LFP

$\mathfrak{A}:$ 

$$\varphi(x) = \big[\mathsf{lfp}\, R\, x.\; x = u \lor \exists y(Exy \land Ry)\big](x)$$

$\mathfrak{A}$

$$\emptyset$$

$$\downarrow$$

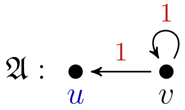
$$\{u\}$$

$$\downarrow$$

$$\{u, v\}$$

monotone, complete lattice

Semiring Semantics for LFP



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$\mathfrak{A}$
$\emptyset$
$\downarrow$
$\{u\}$
$\downarrow$
$\{u, v\}$

monotone, complete lattice

$\mathbb{N}$
$(0, 0)$
$(1, 0)$
$(1, 1)$
$(1, 2)$
$(1, 3)$
$(1, 4)$
$\vdots$
$?$

chain

Semiring Semantics for LFP

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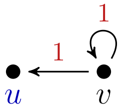

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$\mathfrak{A}$

$$\emptyset \;\downarrow\; \{u\} \;\downarrow\; \{u, v\}$$

monotone, complete lattice

$\mathbb{N}^\infty$

$$(0,0)$$
$$(1,0)$$
$$(1,1)$$
$$(1,2)$$
$$(1,3)$$
$$(1,4)$$
$$\vdots$$
$$(1,\infty)$$

chain

Semirings

Commutative Semiring
$(K, +, \cdot, 0, 1)$

associative, commutative
$a \cdot 0 = 0$
$a \cdot (b + c) = ab + ac$

Properties

- idempotent: $a + a = a$
- absorption: $a + ab = a$
- naturally ordered: $a \leq a + b$ is a partial order
- continuous: $\bigsqcup C$, $\bigsqcap C$ exist for chains $C \subseteq K$, compatible with $+, \cdot$

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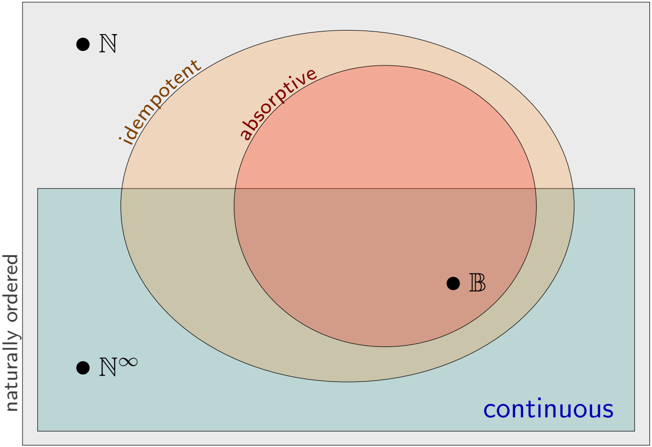
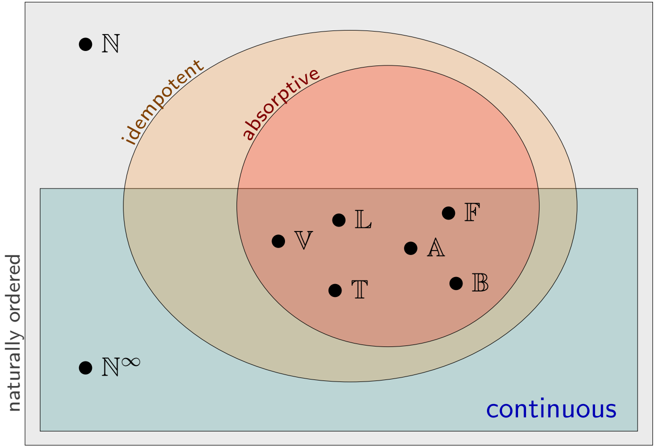
monotone ✓

complete ✓

Application Semirings



Application Semirings



Viterbi (confidence)
$\mathbb{V} = ([0,1], \max, \cdot, 0, 1)$

Access control
$\mathbb{A} = (\{\mathsf{P} > \mathsf{C} > \mathsf{S} > \mathsf{T} > 0\}, \max, \min, 0, \mathsf{P})$

Overview

1. Semiring Semantics for LFP
2. Obstacle: Greatest Fixed Points

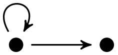
   Which semirings give reasonable information?

3. Main Results

   Most general semiring

   Explain semantics by model-checking games

A Simple$^?$ Example

$\mathfrak{A}:$ 

$$\varphi(x) = \underbrace{\big[\mathsf{gfp}\, R\, x.\ \exists y (Exy \land Ry)\big](x)}_{\text{exists infinite path from } x}$$

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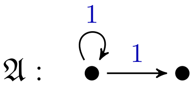

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$\mathbb{N}^\infty$ (counting): ∞ ✗

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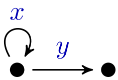
$\mathfrak{A}$: 

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$\mathbb{N}^\infty$ (counting):	∞	✗
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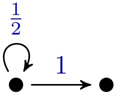

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$\mathbb{V}$ (confidence):	$1 \mapsto \frac{1}{2} \mapsto \frac{1}{4} \mapsto \frac{1}{8} \quad \cdots \quad 0$	✓

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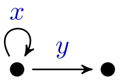

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$\mathbb{S}^\infty[X]$ (tracking): x^∞ ✓

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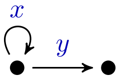
$\mathfrak{A}:$ 

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$\mathbb{N}^\infty$ (counting):	∞	✗
$\mathbb{N}^\infty[\![X]\!]$ (tracking):	0	✗

not abs.

$\mathbb{V}$ (confidence):	$1 \mapsto \frac{1}{2} \mapsto \frac{1}{4} \mapsto \frac{1}{8} \cdots 0$	✓
$\mathbb{S}^\infty[X]$ (tracking):	x^∞	✓

absorptive

Why Absorption?

- Gives meaningful results in (these) examples
- Application semirings are absorptive
- **Symmetry!**

The following are equivalent:

- absorption: $a + ab = a$
- greatest element $\top = 1$
- decreasing mult.: $ab \leq a$

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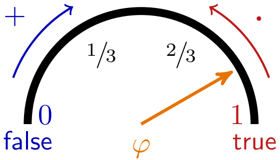

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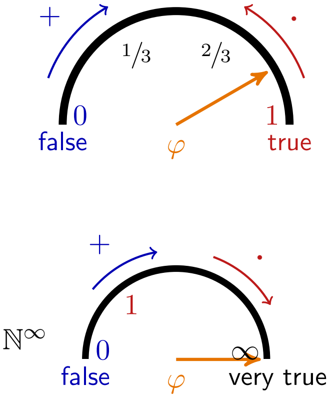

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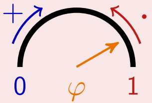

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Absorptive Polynomials $\mathbb{S}^\infty[X]$

- Introduced by [Deutsch, Milo, Roy, Tannen, ICDT 2014]
- Exponents in $\mathbb{N}^\infty$, no coefficients
- Absorption order on monomials: shorter $\geq$ longer

$$1 \geq x, \qquad xy \geq xy^2, \qquad x^5 \geq x^\infty$$

- Absorptive polynomials are antichains (always finite!)

$$x^\infty + y, \qquad x^2y + xy^2 + x^4$$

Homomorphisms

Theorem

Let $h\colon K \to K'$ a semiring homomorphism, π a K-interpretation.
This diagram commutes ...

$$\begin{array}{ccc} \pi & \stackrel{h}{\longmapsto} & h \circ \pi \\ \downarrow \varphi & & \downarrow \varphi \\ \pi[\![\varphi]\!] & \stackrel{h}{\longmapsto} & h \circ \pi[\![\varphi]\!] \end{array}$$

... if h is continuous.

Universal Property of $\mathbb{S}^\infty[X]$

Main Result I

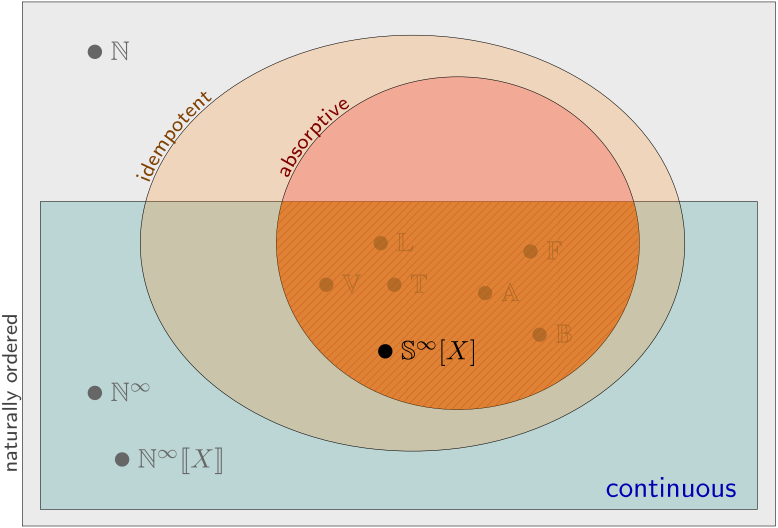
Let K' be an absorptive, continuous semiring.

Any assignment $h\colon X \to K'$ uniquely extends to a continuous homomorphism $h\colon \mathbb{S}^\infty[X] \to K'$.

Difficult part: $h\big(\prod C\big) \stackrel{!}{=} \prod h(C)$

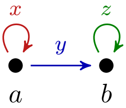
- Use Kőnig's lemma (polynomials are finite)

Situation



$\mathbb{N}$
idempotent
absorptive
$\mathbb{L}$
$\mathbb{F}$
$\mathbb{V}$
$\mathbb{T}$
$\mathbb{A}$
$\mathbb{B}$
$\mathbb{S}^\infty[X]$
naturally ordered
$\mathbb{N}^\infty$
$\mathbb{N}^\infty[\![X]\!]$
continuous

Model-Checking Games

$\mathfrak{A}:$ 

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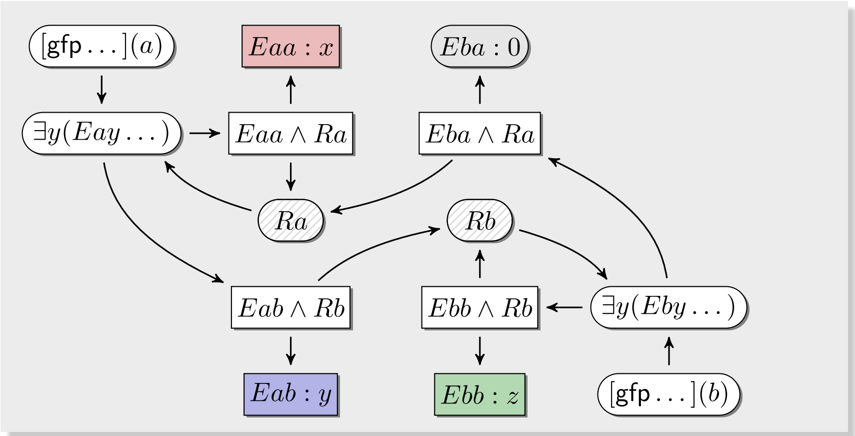

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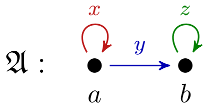

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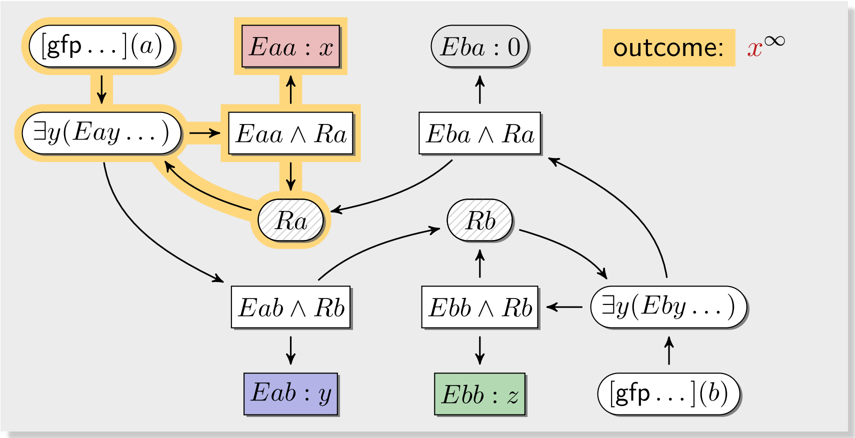

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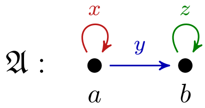

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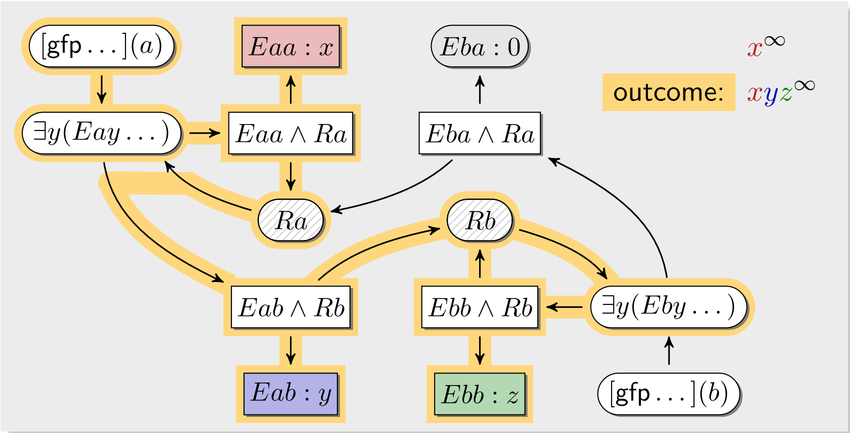

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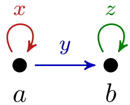

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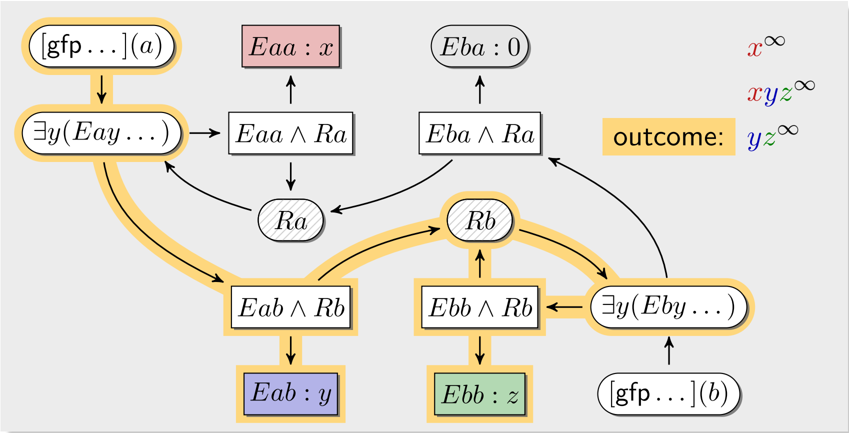

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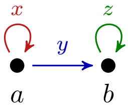

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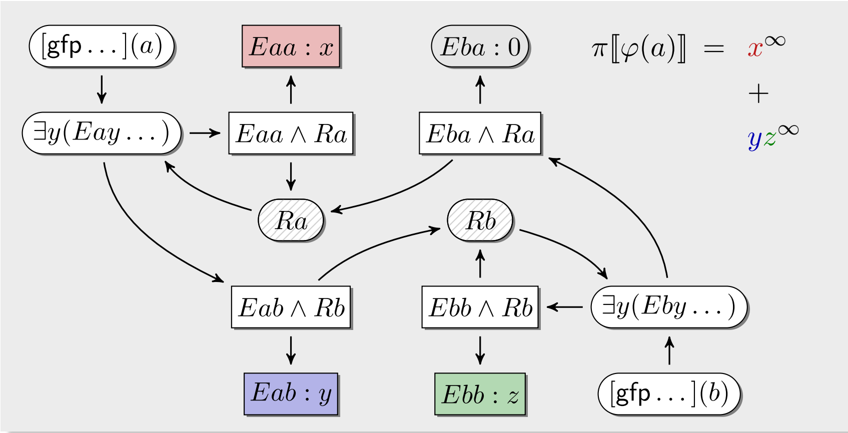

Semiring Semantics via Strategies

Main Result II

For any K-interpretation π with K absorptive and continuous,

$$\pi[\![\varphi]\!] = \sum \{\pi[\![\mathcal{S}]\!] \mid \mathcal{S} \text{ winning strategy in } \mathcal{G}(\pi, \varphi)\}$$

outcome

Intuition: Provenance value = sum of witnesses

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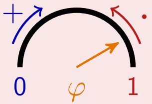
Absorption for reasonable information.



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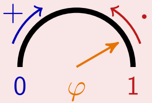

3. Main Results

   $\mathbb{S}^\infty[X]$: most-general absorptive, continuous semiring.

   $\pi[\![\varphi]\!] = \sum\{\pi[\![\mathcal{S}]\!] \mid \mathcal{S}$ winning strategy in $\mathcal{G}(\pi, \varphi)\}$.

Future work: Infinite games, Algorithms

Thanks for
your attention