Semiring Provenance in the Infinite

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Semiring provenance

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Propagate these annotations through a query, keeping track of whether pieces of information are used jointly or alternatively.

- $+$ interprets alternative use of information ($\vee$, $\exists$, unions)
- $\cdot$ interprets joint use of information ($\wedge$, $\forall$, joins)
- $0 \in S$ interprets false assertions and elements $s \neq 0$ provide annotations for true assertions.
- untracked information is interpreted by $1 \in S$.

This can give detailed insights about which combinations of facts are responsible for the truth of a statement and further information about confidence scores, cost analysis, number of evaluation strategies, access levels, $\ldots$.

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Infinity: Is semiring semantics confined to finite domains, or can it be extended to infinite ones? The obvious problem is the treatment of quantifiers

$$\pi[\![\exists x \varphi(x,\bar{b})]\!] := \sum_{a \in A} \pi[\![\varphi(a,\bar{b})]\!] \qquad \text{and} \qquad \pi[\![\forall x \varphi(x,\bar{b})]\!] := \prod_{a \in A} \pi[\![\varphi(a,\bar{b})]\!]$$

Semirings with Infinitary Operations?

In collaboration with Val, and with my students Katrin Dannert and especially Matthias Naaf, we have proposed new approaches for dealing with negation and fixed points, based on negation normal forms, semirings of polynomials with dual indeterminates, fully continuous and absorptive semirings, and generalised absorptive polynomials.

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Is there a reasonable algebraic notion of such infinitary semirings ?

Basic Requirements for an Infinitary Operation

Expand a commutative monoid $S = (S, +, 0)$ by operation $\sum_{i \in I} s_i$, for arbitrary index sets I.

Partition invariance (infinite associativity): For each partition $(I_j)_{j \in J}$ of I

$$\sum_{i \in I} s_i = \sum_{j \in J} \sum_{i \in I_j} s_i.$$

Bijection invariance (infinite commutativity): For every bijection $\sigma : J \to I$

$$\sum_{i \in I} s_i = \sum_{j \in J} s_{\sigma(j)}.$$

Compatibility with the finite: For each finite index set $I = \{i_0, \ldots, i_n\}$

$$\sum_{i \in I} s_i = s_{i_0} + \cdots + s_{i_n}.$$

The need for further requirements

The three basic conditions do not exclude pathological infinitary operations.

Example: $(\mathbb{N} \cup \{\infty\}, +, 0)$ with $\sum_{i \in I} s_i = \infty$ for all infinite I (and compatible with + for finite I).

This violates, for instance, the following two natural properties.

Neutrality: $\sum$ respects the neutral element if $\sum_{i \in I} s_i = \sum_{i \in I, s_i \neq 0} s_i$.

Idempotence: $\sum$ respects idempotent elements if $s + s = s$ implies that $\sum_{i \in I} s = s$ for all $I \neq \emptyset$.

We can guarantee these, and other relevant conditions by imposing

Compactness properties: Informally, compactness means that the value of an infinitary sum only depends on the set of values of its finite subsums.

Compactness: pros and cons

Compactness has a number of important consequences:

- compact operators respect the neutral element and idempotent elements
- compactness implies that the monoid is aperiodic and naturally ordered

Problem. Compactness is sometimes difficult to verify, and some relevant infinitary semirings violate compactness. An important example is the universal infinitary semiring $\mathbb{N}^\infty[\![X^\infty]\!]$

We therefore prefer to use a weaker condition. Notice that bijection invariance implies that $\sum_{i\in I} s = \sum_{j\in J} s$ whenever $|I| = |J|$. Compactness implies a stronger property.

Unique infinite powers: For every element $s \in S$ there exists a unique element $\infty \cdot s$ with

$$\sum_{i\in I} s = \infty \cdot s \quad \text{for all infinite I.}$$

Distributive laws

The two algebraic operations in a semiring are related by the distributive law $s(r+t) = sr + st$.
The generalisation to infinitary operations comes in two variants:

Weak distributivity: $s \cdot \sum_{i \in I} s_i = \sum_{i \in I} (s \cdot s_i)$.

Strong distributivity: For every index set I and every collection $(J_i)_{i \in I}$ of index sets

$$\prod_{i \in I} \sum_{j \in J_i} s_j = \sum_{f \in F} \prod_{i \in I} s_{f(i)},$$

where F is the set of all choice functions $f : I \to \bigcup_{i \in I} J_i$ such that $f(i) \in J_i$ for all $i \in I$.

Weak distributivity is straightforward, but there are issues with strong distributivity:
- It is used to prove that products are monotone wrt. to the natural order.
- It is needed for certain results, such as the Sum-of-Proof-Trees Theorem.
- In the tropical semiring, strong distributivity only holds for countable products.

Monotonicity

It is an important property of naturally ordered semirings that addition and multiplication are monotone. Since this is needed in many results, we also want to have it for infinitary operations.

Monotonicity: For all families $(s_i)_{i\in I}$ and $(t_i)_{i\in I}$ such that $s_i \leq t_i$, also

$$\sum_{i\in I} s_i \leq \sum_{i\in I} t_i \qquad \text{and} \qquad \prod_{i\in I} s_i \leq \prod_{i\in I} t_i.$$

Monotonicity for the infinitary sum is implied by partition invariance, but monotonicity of infinitary products does not seem to follow from weaker properties than strong distributivity.

Indeed there exist "pathological" semirings that satisfy all properties that we discussed, including compactness, except monotonicity and strong distributivity.

We can live without strong distributivity but not without monotonicity.

Infinitary Semirings

Definition. An infinitary semiring is a commutative, naturally ordered semiring $S = (S, +, \cdot, 0, 1)$, together with infinitary operations $\sum$ and $\prod$ that satisfy:

- partition invariance and hence also bijection invariance,
- compatibility with finite addition and multiplication,
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- there are unique infinite powers,
- weak distributivity, and
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- weak distributivity, and
- monotonicity.

The notion of homomorphisms must be extended to κ-homomorphisms which also preserve infinitary sums and products of sequences of length $< \kappa$.

Examples of infinitary semirings

Finite semirings in which both operations are aperiodic expand to infinitary semirings.

Infinite lattice semirings expand to infinitary semirings if the underlying order is a complete lattice in which finite infima distribute over arbitrary suprema.

$\mathbb{N}^\infty$, with the natural definitions of $\sum$ and $\prod$ is a strongly distributive infinitary semiring.

Infinitary absorptive semirings: Take an absorptive semiring S whose natural order $(\mathscr{S}, \leq)$ is a complete lattice, and which is (fully) continuous: suprema $\bigsqcup C$ and infima $\bigsqcap C$ of chains are compatible with addition and multiplication.

Define infinitary operations by taking suprema of finite subsums and infima of finite subproducts:

$$\sum_{i \in I} s_i := \bigsqcup_{\substack{I_0 \subseteq I \\ I_0 \text{ finite}}} \Big(\sum_{i \in I_0} s_i \Big) \quad \text{and} \quad \prod_{i \in I} s_i := \bigsqcap_{\substack{I_0 \subseteq I \\ I_0 \text{ finite}}} \Big(\prod_{i \in I_0} s_i \Big).$$

Universal semirings

Fundamental question: Which combinations of atomic facts are responsible for the truth of a statement, and how often is a fact used in the evaluation ?

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Let X be a set of indeterminates, which are used to label the facts that we want to track: $\alpha \mapsto X_\alpha$ (untracked facts are mapped to 0 or 1).

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Universality: Any function $f : X \to S$ into an arbitrary semiring S extends uniquely to a semiring homomorphism $h : \mathbb{N}[X] \to S$.

Question: Can we generalise $\mathbb{N}[X]$ to an infinitary semiring with an analogous universality property?

The semiring $\mathbb{N}^\infty[\![X^\infty]\!]$ of generalised power series

How to expand $\mathbb{N}[X]$? We must be able to

- add the same monomial infinitely often: $x + x + x + \dots$
  allow coefficients in $\mathbb{N}^\infty$
- add infinitely many different monomials: $x + x^2 + x^3 + \dots$
  use formal power series instead of polynomials
- multiply the same variable infinitely often: $x \cdot x \cdot x \cdot \dots$
  allow exponents in $\mathbb{N}^\infty$.

For finite sets X of indeterminates, and an appropriate definition for infinitary products, this indeed makes $\mathbb{N}^\infty[\![X^\infty]\!]$ the free strongly distributive infinitary semiring.

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For infinite sets X, this does not work: $\mathbb{N}^\infty[\![X^\infty]\!]$ is not an infinitary semiring (it does not have unique infinite powers).

Semiring provenance for first-order logic

Infinitary semirings admit to extend most of the results on semiring provenance for FO to infinite domains, sometimes with minor modifications.
An interesting example is the analysis of proof trees for model checking problems $\mathfrak{A} \models \psi$.

Sum-of-Proof-Trees Theorem. Let A be domain of cardinality $< \kappa$, and let S be a κ-distributive infinitary semiring. For every interpretation $\pi : \mathrm{Lit}_A(\tau) \to S$, and every $\psi \in \mathrm{FO}$

$$\pi[\![\psi]\!] = \sum \Big\{ \pi[\![T]\!] \ : \ T \text{ is a proof tree for } \psi \text{ and } \pi \Big\}$$

Here, the valuation of a proof tree T is $\pi[\![T]\!] := \prod_{\alpha \in \mathrm{Lit}_A(\tau)} \pi(\alpha)^{\#_\alpha(T)}$ where $\#_\alpha(T)$ denotes the number of leaves of T labelled with the literal α.

Semiring provenance for first-order logic

This nicely combines with universal semirings, dual indeterminates and model-compatible interpretations.

Corollary. Let $\pi : \mathrm{Lit}_A(\tau) \to \mathbb{N}^\infty[\![X^\infty, \overline{X}^\infty]\!]$ be model-compatible and let $\psi \in \mathrm{FO}(\tau)$. Then the power series $\pi[\![\psi]\!]$ describes all proof trees that verify ψ using premises from the literals that π maps to indeterminates or to 1.

Specifically, each monomial $c\,x_1^{e_1} \cdots x_k^{e_k}$ in $\pi[\![\psi]\!]$ stands for c distinct proof trees that use e_1 times the literal annotated by x_1, …, and e_k times the literal annotated by x_k, where $x_1, \dots, x_k \in X \cup \overline{X}$. In particular, when $\pi[\![\psi]\!] = 0$ no proof tree exists, and hence there is no model of ψ that is compatible with π.