The Model Theory of Semiring Semantics

Erich Grädel

(joint work with Val Tannen and with many students in my research group:
Clotilde Bizière, Sophie Brinke, Hayan Helal, Lovro Mrkonjić, Matthias Naaf, Richard Wilke)

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- $+$ interprets alternative use of information ($\vee$, $\exists$, unions)
- $\cdot$ interprets joint use of information ($\wedge$, $\forall$, joins)
- $0 \in S$ interprets false assertions and elements $s \neq 0$ provide annotations for true assertions.
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In this way, we compute for a query ψ a valuation $\pi[\![\psi]\!] \in S$.

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Cost: How to minimize the cost for obtaining the output based on prizes for input items?

From provenance to semiring semantics

Provenance analysis aims to explain how a particular result depends on the specific input items. The explanations provided by semiring provenance and the applications to cost calculations, confidence scores, clearance levels, repairs, etc. are interesting not only for databases.

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This also provides a general method for the strategy analysis in finite and infinite games.

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Infinity: Is semiring semantics confined to finite domains, or can it be extended to infinite ones? The obvious problem is the treatment of quantifiers

$$\pi[\![\exists x\varphi(x,\overline{b})]\!] := \sum_{a\in A} \pi[\![\varphi(a,\overline{b})]\!] \qquad \text{and} \qquad \pi[\![\forall x\varphi(x,\overline{b})]\!] := \prod_{a\in A} \pi[\![\varphi(a,\overline{b})]\!]$$

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Model theory: To what extend do standard logical results and model-theoretic methods survive in semiring semantics, and how does this depend on algebraic properties of the underlying semiring.

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I have used these here for baking onion tarts.

Red Semirings

White Semirings

Semirings: from simpler to more complicated ones

We are only interested in commutative semirings that are naturally ordered by addition: $a \leq b \;:\Longleftrightarrow\; \exists c(a+c=b)$ is antisymmetric, and therefore a partial order.

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Fully idempotent semirings:

- Min-max semirings $(S,\max,\min,0,1)$, induced by a total order $(S,<)$. Relevant examples are $\mathbb{F}=([0,1],\max,\min,0,1)$ and the security semiring induced by $\mathbb{A}=\{0<\mathsf{T}<\mathsf{S}<\mathsf{C}<\mathsf{P}=1\}$ where P is "public", C is "confidential", S is "secret", T is "top secret".

- Lattice semirings $(S,\sqcup,\sqcap,0,1)$, induced by a bounded distributive lattice $(S,\leq)$.

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- The tropical semiring $\mathbb{T}=([0,\infty],\min,+,\infty,0)$ for cost interpretations.

- The Viterbi semiring $\mathbb{V}=([0,1],\max,\cdot,0,1)$ for confidence scores.

- The Łukasiewicz semiring $\mathbb{L}=([0,1],\max,\otimes,0,1)$ with $a\otimes b:=\max(a+b-1,0)$ is popular in the study of many-valued logics, and gives a different notion of confidence or degrees of truth.

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Semirings that are neither absorptive nor idempotent:

- The natural semiring $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$ for counting proofs and strategies.

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This justifies a general strategy for computing semiring valuations:

- Compute valuations in the universal semiring $\mathbb{N}[X]$
- Specialize via homomorphisms to other semirings

Other provenance semirings

Simpler and “less informative” semirings with specific algebraic properties:

$$
\begin{tikzcd}[column sep=small]
 & & \begin{array}{c}\mathbb{N}[X] \\ 2x^2y+xy+5y^2+xz\end{array} \arrow[dl, "\text{drop coeff.}"'] \arrow[dr, "\text{drop exponents}"] & & \\
 & \begin{array}{c}\mathbb{B}[X] \\ x^2y+xy+y^2+xz\end{array} \arrow[dl, "\text{absorb}"'] \arrow[dr, "\text{drop exp.}"] & & \begin{array}{c}\mathrm{Trio}[X] \\ 3xy+5y+xz\end{array} \arrow[dl, "\text{drop coeff.}"] & \\
\begin{array}{c}\mathbb{S}[X] \\ xy+y^2+xz\end{array} \arrow[dr, "\text{drop exp.}"] & & \begin{array}{c}\mathbb{W}[X] \\ xy+y+xz\end{array} \arrow[dl, "\text{absorb}"] \arrow[dr, "+\,=\,\cdot"] & & \\
 & \begin{array}{c}\mathrm{PosBool}[X] \\ y+xz\end{array} & & \begin{array}{c}\mathrm{Which}[X] \\ xyz\end{array} &
\end{tikzcd}
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- Provenance for logic is intimately connected to provenance analysis for games.
- New kinds of applications: Missing answers, repairs, etc.

Semiring interpretations

Fix a commutative semiring S.

Let A be a finite universe and $\tau = \{R_1, \dots, R_m\}$ be a finite relational vocabulary.

$\mathrm{Lit}_A(\tau)$: all fully instantiated literals $R\overline{a}$ and $\neg R\overline{a}$ with $R \in \tau$ and $\overline{a} \in A^k$.

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Further, let π map equalities $a = b$ and $a \neq b$ to their truth values 0 or 1.

We call $\pi : \mathrm{Lit}_A(\tau) \to S$ model-defining if, for all atoms $R\overline{a}$, precisely one of the values $\pi(R\overline{a})$ and $\pi(\neg R\overline{a})$ is zero. Then π specifies a unique structure $\mathfrak{A}_\pi$.

Semiring semantics for first-order logic

We can extend any S-interpretation $\pi : \mathrm{Lit}_A(\tau) \to S$ to a S-valuation $\pi : \mathrm{FO}(\tau) \to S$ giving values $\pi[\![\varphi]\!] \in S$ to all $\varphi \in \mathrm{FO}(\tau)$.

$$\pi[\![\varphi \vee \psi]\!] := \pi[\![\varphi]\!] + \pi[\![\psi]\!] \qquad \pi[\![\varphi \wedge \psi]\!] := \pi[\![\varphi]\!] \cdot \pi[\![\psi]\!]$$

$$\pi[\![\exists x \varphi(x)]\!] := \textstyle\sum_{a \in A} \pi[\![\varphi(a)]\!] \qquad \pi[\![\forall x \varphi(x)]\!] := \textstyle\prod_{a \in A} \pi[\![\varphi(a)]\!]$$

$$\pi[\![\neg\varphi]\!] := \pi[\![\mathrm{nnf}(\neg\varphi)]\!].$$

Semirings of dual-indeterminate polynomials

Annotate atoms by indeterminates in X, and negated atoms by indeteminates in $\overline{X}$, with a bijection $X \leftrightarrow \overline{X}$ mapping $x \in X$ to its complementary token $\overline{x} \in \overline{X}$.

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$\mathbb{N}[X, \overline{X}] := \mathbb{N}[X \cup \overline{X}]/(X\overline{X})$ is the quotient semiring of $\mathbb{N}[X \cup \overline{X}]$ by the congruence generated by the equations $x \cdot \overline{x} = 0$. Corresponds to polynomials in $\mathbb{N}[X \cup \overline{X}]$ such that no monomial contains complementary tokens.

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Universality. Any map $h : (X \cup \overline{X}) \to S$ into a semiring S, with $h(x) \cdot h(\overline{x}) = 0$ for $x \in X$, extends uniquely to a semiring homomorphism $h : \mathbb{N}[X,\overline{X}] \to S$.

Also the other provenance semirings can be extended by dual indeterminates to get semirings like $\mathbb{B}[X,\overline{X}], \mathbb{W}[X,\overline{X}], \mathbb{S}[X,\overline{X}]$ etc.

Proof trees and evaluation strategies

An evaluation tree for a sentence $\psi \in \mathrm{FO}$ and a semiring interpretation $\pi : \mathrm{Lit}_A(\tau) \to S$ is the same thing as a strategy in the associated evaluation game.

Let $\#_\alpha(T)$ denote the number of leaves of the tree T labelled by the literal α.

Valuation of T:

$$\pi[\![T]\!] := \prod_{\alpha \in \mathrm{Lit}_A(\tau)} \pi(\alpha)^{\#_\alpha(T)}.$$

A proof tree for $\psi \in \mathrm{FO}$ and $\pi : \mathrm{Lit}_A(\tau) \to S$ is an evaluation tree with $\pi(T) \neq 0$

Theorem. For every semiring interpretation $\pi : \mathrm{Lit}_A(\tau) \to S$ and every $\psi \in \mathrm{FO}$

$$\pi[\![\psi]\!] = \sum \left\{ \pi[\![T]\!] \;:\; T \text{ is a proof tree for } \psi \text{ and } \pi \right\}$$

Proof trees and dual-indeterminate polynomials

Consider a model-defining semiring interpretation $\pi : \mathrm{Lit}_A(\tau) \to \mathbb{N}[X, \overline{X}]$ that maps each literal to either an indeterminate in $X \cup \overline{X}$ or to a truth value 0 or 1 ,

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is a sum of monomials $m x_1^{e_1} \cdots x_k^{e_k}$. Each such monomial tells us that there are precisely m proof trees establishing that $\mathfrak{A}_\pi \models \psi$ which

- use among the tracked literals only those labelled by $x_1, \dots, x_k$,
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In particular $\mathfrak{A}_\pi \models \psi$ if, and only if, $\pi[\![\psi]\!] \neq 0$.

Provenance information for classes of structures

Model-compatible interpretations $\pi : \mathrm{Lit}_A(\tau) \to \mathbb{N}[X,\overline{X}]$. For every atom $R\overline{a}$, either

(1) $\pi(R\overline{a}) = x$ and $\pi(\neg R\overline{a}) = \overline{x}$, for some $x \in X$, or

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A model-compatible interpretation is consistent with at least one τ-structure on A, but in general with a larger set of such structures.

$\mathrm{Must}_\pi := \{\varphi \in \mathrm{Lit}_A(\varphi) : \pi(\varphi) = 1\}$ (true in all models of π)

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Conclusion. For the resulting valuation $\pi : \mathrm{FO}(\tau) \to \mathbb{N}[X, \overline{X}]$, the provenance polynomial $\pi[\![\psi]\!]$ describes all proof trees for ψ whose leaves are in $\mathrm{Must}_\pi \cup \mathrm{May}_\pi$. Every monomial corresponds to one proof tree, and gives precise information about the literals on which the proof tree depends, giving a complete description of all models of ψ that are compatible with π.

Fixed-point logics

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Semiring provenance for Datalog had already been done in papers by Green, Karvounarakis, Tannen 2007 and Deutch, Milo, Roy, Tannen 2014, based on ω-continuous semirings. The universal one are semirings $\mathbb{N}^\infty[\![X]\!]$ of formal power series.

With dual indeterminates, this leads to semirings $\mathbb{N}^\infty[\![X,\overline{X}]\!]$ which provide semiring semantics for semipositive Datalog and the positive fragment posLFP of fixed-point logic.

However the general fixed-point logics LFP and L_μ may have arbitrary interleavings of least and greatest fixed points, and ω-continuous semirings are not adequate for these.

Semiring semantics for fixed-point logic

What are the algebraic conditions required for semirings for fixed-point logics?

Full continuity: each chain $C \subseteq S$ has a supremum $\bigsqcup C$ and an infimum $\bigsqcap C$ in S, with $a + \bigsqcup C = \bigsqcup(a + C)$, $a \cdot \bigsqcup C = \bigsqcup(a \cdot C)$ and analogously for $\bigsqcap C$.

Fully continuous semirings suffice to get a well-defined semantics for LFP, but for a meaningful semantics that provides insights why a formula holds, an additional condition is necessary.

Absorption: $a + ab = a$ for all $a, b \in S$. This makes multiplication decreasing: $a \cdot b \leq a$ and $a \leq 1$.

Theorem. (Dannert-G.-Naaf-Tannen 2021)
In absorptive, fully chain-continuous semirings S, each monotone function $f : S \to S$ has a least fixed point $\mathbf{lfp}(f)$ and a greatest fixed point $\mathbf{gfp}(f)$. Together with the semiring semantics for FO, this provides meaningful semiring semantics for LFP.

Semirings for LFP

Many common application semirings are fully continuous and absorptive such as the tropical semiring, min-max semirings, the Lukasiewicz semiring. However, the general provenance semirings $\mathbb{N}[X]$ and $\mathbb{N}^\infty[\![X]\!]$ are neither fully continuous nor absorptive.

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Instead, the general semirings for LFP are the semirings $\mathbb{S}^\infty[X]$ of generalized absorptive polynomials

$$f = x^2y^3z + x^\infty y + z^\infty$$

- no coefficients
- exponents in $\mathbb{N} \cup \{\infty\}$.
- absorption among monomials (those with larger exponents are absorbed).

Semirings $\mathbb{S}^\infty[X]$ and $\mathbb{S}^\infty[X,\overline{X}]$ have universality properties that make them the “right” general semirings for fixed-point logics. (Dannert, G., Naaf, Tannen, CSL 21)

Semirings with Infinitary Operations

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Answers will be given in the talk by Lovro Mrkonjić on Wednesday.

The model theory of semiring semantics

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- Ehrenfeucht-Fraïssé games: a sound and complete method for logical equivalences.
- Locality. By Theorems of Hanf and Gaifman, first-order formulae can only express local properties. In fact, every first-order formula is equivalent to one in Gaifman normal form.

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- Locality. By Theorems of Hanf and Gaifman, first-order formulae can only express local properties. In fact, every first-order formula is equivalent to one in Gaifman normal form.
- Compactness: $\Phi \models \psi$ if, and only if, $\Phi_0 \models \psi$ for some finite $\Phi_0 \subseteq \Phi$.

The definitions involved in these results generalise to semiring semantics. But what about the results themselves, and the associated methods?

Compactness

One of the most important results on first-order logic:

Compactness Theorem: For every class of sentences $\Phi \subseteq \mathrm{FO}$ and every sentence $\psi \in \mathrm{FO}$
- Φ is satisfiable if, and only if, every finite subset $\Phi_0 \subseteq \Phi$ is satisfiable
- $\Phi \models \psi$ if, and only if, there exists a finite subset $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \models \psi$.

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The appropriate setting is provided by absorptive semirings S that admit infinitary operations.

Satisfiability: Φ is S-satisfiable if there is a model-defining S-interpretation π with $\pi[\![\Phi]\!] \neq 0$.
Entailment: $\Phi \models_S \psi$ means that $\pi[\![\Phi]\!] \leq \pi[\![\psi]\!]$ for every model-defining S-interpretation π.

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Sophie Brinke will tell you about semirings for which compactness holds, or fails, in terms of satisfiablity and in terms of entailment.

Locality

Hanf's Theorem: A locality criterion for m-equivalence of two structures based on the number of local substructures of any given isomorphism type.

Gaifman normal form: Every $\psi \in \mathrm{FO}$ is equivalent to a Boolean combination of local formulae and sentences "there exist m disjoint neighbourhoods of radius r satisfying a local property $\varphi^{(r)}$".

In semiring semantics, we have the following results (Bizière, G, Naaf 2023):

- Hanf's Theorem generalises to all fully idempotent semirings, but fails for others.
- $\exists y(Uy \wedge y \neq x)$ does not have a Gaifman normal form over any $S \neq \mathbb{B}$. Over some semirings, Gaifman's Theorem also fails for sentences: $\exists z \forall x \exists y(Uy \vee x = z)$ in the tropical semiring.
- Positive result: Gaifman normal forms for sentences exist over min-max semirings, and even lattice semirings.

Elementary equivalence versus isomorphism

Both notions naturally generalize to semiring interpretations $\pi : \mathrm{Lit}_A(\tau) \to S$

$\pi_A \equiv \pi_B$ if $\pi_A[\![\varphi]\!] = \pi_B[\![\varphi]\!]$ for all $\varphi \in \mathrm{FO}$

$\pi_A \cong \pi_B$ if $\dots$.

In Boolean semantics, for finite structures, we have that $\mathfrak{A} \equiv \mathfrak{B} \Longleftrightarrow \mathfrak{A} \cong \mathfrak{B}$.

This fails in semiring semantics, for some semrings.

Theorem (G., Mrkonjic, 2021) There exist finite S-interpretations $\pi_A \ncong \pi_B$ (for instance in min-max semirings with ≥ 3 elements) such that $\pi_A \equiv \pi_B$.

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This fails in semiring semantics, for some semrings.

Theorem (G., Mrkonjic, 2021) There exist finite S-interpretations $\pi_A \not\cong \pi_B$ (for instance in min-max semirings with ≥ 3 elements) such that $\pi_A \equiv \pi_B$.

Indeed, finite semiring interpretations are not always first-order definable up to isomorphism.
And even if they are, they may need an infinite axiom system.
And even if, as in the tropical semiring, a finite axiom system suffices, a single axiom might not.

How to prove elementary equivalence

Let π_A, π_B be two S-interpretations. We want to prove that $\pi_A \equiv \pi_B$ although π_A and π_B are quite different.

Find a separating set of homomorphisms $h : S \to \mathbb{B}$ such that for all $s,t \in S$ we have that $h(s) \neq h(t)$ for some $h \in H$. Prove that $h \circ \pi_A \equiv h \circ \pi_B$ for all $h \in H$. Since these are $\mathbb{B}$-interpretations we can do this by standard methods.

Claim. This implies $\pi_A \equiv \pi_B$

Otherwise there exists ψ such that $\pi_A[\![\psi]\!] = s \neq t = \pi_B[\![\psi]\!]$. But then

$$(h \circ \pi_A)[\![\psi]\!] = h(\pi_A[\![\psi]\!]) = h(s) \neq h(t) = h(\pi_B[\![\psi]\!]) = (h \circ \pi_B)[\![\psi]\!]$$

which is impossible since $h \circ \pi_A \equiv h \circ \pi_B$.

Example

Let $S = \mathrm{PosBool}[X]$. Every $Y \subseteq X$ induces a unique homomorphism $h_Y : \mathrm{PosBool}[X] \to \mathbb{B}$ with $h_Y(x) = \top$ for $x \in Y$ and $h_Y(x) = \bot$ for $x \in X \setminus Y$. For $p \in \mathrm{PosBool}[X]$, we have that $h_Y(p) = \top$ if, and only if, p contains a monomial with only variables from Y.

$\{h_Y : Y \subseteq X\}$ is a separating set of homorphisms.

Claim. The following two $\mathrm{PosBool}[x,y]$-interpretations π_{xy}, π_{yx} are elementarily equivalent.

π_{xy} :

A	P	Q	$\neg P$	$\neg Q$
a	0	y	x	0
b	x	0	0	y
c	y	x	0	0
d	0	0	y	x

π_{yx} :

A	P	Q	$\neg P$	$\neg Q$
a	y	0	0	x
b	0	x	y	0
c	x	y	0	0
d	0	0	x	y

Proof

The separating set of homomorphisms $h : \mathrm{PosBool}[x,y] \to \mathbb{B}$ consists of $h_\varnothing, h_{\{x\}}, h_{\{y\}}$ and $h_{\{x,y\}}$.

For each of these, we have to show that $h \circ \pi_{xy} \equiv h \circ \pi_{yx}$

For $h_\varnothing$ this is trivial.

$h_\varnothing \circ \pi_{xy}$:

A	P	Q	$\neg P$	$\neg Q$
a	$\perp$	$\perp$	$\perp$	$\perp$
b	$\perp$	$\perp$	$\perp$	$\perp$
c	$\perp$	$\perp$	$\perp$	$\perp$
d	$\perp$	$\perp$	$\perp$	$\perp$

$h_\varnothing \circ \pi_{yx}$:

A	P	Q	$\neg P$	$\neg Q$
a	$\perp$	$\perp$	$\perp$	$\perp$
b	$\perp$	$\perp$	$\perp$	$\perp$
c	$\perp$	$\perp$	$\perp$	$\perp$
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Proof: $h = h_{\{x\}}$

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$h_{\{x\}} \circ \pi_{xy}$:

A	P	Q	$\neg P$	$\neg Q$
a	$\bot$	$\bot$	$\top$	$\bot$
b	$\top$	$\bot$	$\bot$	$\bot$
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d	$\bot$	$\bot$	$\bot$	$\top$

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$h_X \circ \pi_{xy}$:

A	P	Q	$\neg P$	$\neg Q$
a	$\bot$	$\top$	$\top$	$\bot$
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0-1 laws and almost sure valuations in semiring semantics

(Joint work with Hayyan Helal, Matthias Naaf, and Richard Wilke, LICS 2022)

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Reminder: the classical 0-1 law for first-order logic

Fix a constant $0 < p < 1$.

$G_{n,p}$: random graphs with universe $[n] = \{0, \dots, n-1\}$ where, independently for each pair $i < j$, we decide randomly whether the edge $\{i, j\}$ exists (with probability p) or not (with probability $1 - p$)

$\mu_{n,p}(\psi) := \Pr[G \models \psi]$ for random graphs $G \in G_{n,p}$.

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Theorem (0-1 law for FO): (Glebskii et al. 1969, Fagin 1976)
For every $\psi \in \mathrm{FO}$, the sequence $(\mu_{n,p}(\psi))_{n\in\omega}$ converges exponentially fast to either 0 or 1.

Informally: Every $\psi \in \mathrm{FO}$ is either almost surely false or almost surely true.

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Informally: Every $\psi \in \mathrm{FO}$ is either almost surely false or almost surely true.

This holds not just for graphs, but generally for relational structures.

Proof by extension axioms

Extension axioms: Every configuration of k elements can be extended in every consistent way to $k+1$ elements.

For graphs: For all $i \leq k$ and every collection of nodes $v_1, \dots, v_k$ there is a node w with edges to $v_1, \dots, v_i$ but not to $v_{i+1} \dots v_k$.

- Every extension axiom is almost surely true (with exponential convergence).
- The theory T of all extension axioms is ω-categorical: it has a unique countable model.
- Hence T is complete and, by compactness, for every $\psi \in \mathrm{FO}$, either $T_0 \models \psi$ or $T_0 \models \neg\psi$ for some finite collection $T_0 \subseteq T$. In the first case ψ is almost surely true, in the second case almost surely false.

Random semiring interpretations

Random structures naturally generalise to random S-interpretations.

Fix a probability distribution p on $S \setminus \{0\}$.

Independently, for each atom $R\overline{a} \in \mathrm{Lit}_{[n]}(\tau)$:
- decide by coin flip whether $R\overline{a}$ or $\neg R\overline{a}$ is true
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From the classical 0-1 law, we conclude (for semirings without divisors of 0) that for every $\psi \in \mathrm{FO}$, $\pi[\![\psi]\!] = 0$ almost surely or $\pi[\![\psi]\!] \neq 0$ almost surely.

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But we aim at more informative results.

Questions

The classical 0-1 law partitions FO into almost surely false and almost surely true sentences.

- Do we get partitions $(\Phi_j)_{j\in S}$ of FO so that sentences in Φ_j evaluate to j almost surely?
- If yes, are all classes Φ_j non-empty, or do the almost sure valuations concentrate on just a few values.
- For which semirings does this work? How does the partition into the classes Φ_j depend on the underlying semiring?
- What is the complexity of computing the almost sure valuation of a given $\psi \in$ FO?

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For simplicity, we first consider finite min-max semirings $(S, \max, \min, 0, 1)$, induced by a finite total order $(S, <)$.

Extension properties

The *k*-extension property: If a configuration of k points is realised then all consistent extensions to $k+1$ points are also realised.

- $\mathrm{Lit}_k(\tau)$: τ-literals $R\overline{x}$ in the variables $x_1, \dots, x_k$,
- Configurations: consistent assignments $\rho : \mathrm{Lit}_k(\tau) \to S$.

Lemma. For finite semirings S, random S-interpretations almost surely have the k-extension property (for every fixed k).

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Lemma. For finite semirings S, random S-interpretations almost surely have the k-extension property (for every fixed k).

The proof is simple, and uses the same arguments as in the Boolean case.

Bollobas: “The first-order 0-1 law looks sophisticated but follows from shallow computations”

Algebraic descriptions

We know: For $\pi : \mathrm{Lit}_A(\tau) \to S$, the valuations $\pi[\![\psi]\!]$ can be described by polynomials with indeterminates from $\mathrm{Lit}_A(\tau)$.

$$\psi := \forall x\Big(\neg Exx \vee (Exx \wedge \exists^{\neq} yExy)\Big)$$

$$f_{\psi,A} = \prod_{a\in A}\Big(\neg Eaa + \Big(Eaa \cdot \sum_{b\in A\setminus\{a\}} Eab\Big)\Big)$$

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If $\psi \in \mathrm{FO}^k$ and π has the k-extension property, we can do better:

We find a polynomial f_ψ with indeterminates in $\mathrm{Lit}_k(\tau)$, which only depends on k, but not on n.

Polynomials for almost sure valuations

Idea: Use polynomials with indeterminates in $\mathrm{Lit}_k(\tau)$ rather than $\mathrm{Lit}_A(\tau)$.

- Indeterminates: $\mathbf{X}^{(k)} = \{X_\alpha, X_{\neg\alpha} : \alpha \in \mathrm{Lit}_k(\tau)\}$
- Coefficients from the three-element semiring $E = \{0, e, 1\}$ with $e + e = e \cdot e = e$ and $e + 1 = 1$.

ψ	f_ψ
$x_i = x_j$	1 or 0 (depending on whether $i = j$)
$\beta, \neg\beta$	$X_\beta, X_{\neg\beta}$
$\varphi \vee \vartheta$	$f_\varphi + f_\vartheta$
$\varphi \wedge \vartheta$	$f_\varphi \cdot f_\vartheta$
$\exists^{\neq} y\, \varphi(\bar{x}, y)$	$\sum_{t \in T} f_\varphi(\mathbf{X}, t(\mathbf{Y}))$, for consistent $t : \mathbf{Y} \to \{0,1\}$
$\forall^{\neq} y\, \varphi(\bar{x}, y)$	$\prod_{t \in T} f_\varphi(\mathbf{X}, s(\mathbf{Y}))$, for consistent $t : \mathbf{Y} \to \{0,e\}$

Polynomials for quantified formulae

Question: how to construct polynomials for $\exists^{\neq} y\ \varphi(\bar{x}, y)$ and $\forall^{\neq} y\ \varphi(\bar{x}, y)$?

For $\varphi(\bar{x}, y)$ we have a polynomial $f_\varphi(\mathbf{X}, \mathbf{Y})$ where $\mathbf{Y}$ contains the indeterminates for the literals involving y.

If $\varphi(\bar{a}, y)$ is satisfiable then (by the extension property) we find a b realising a maximal extension: true literals involving b get the maximal value 1 of the semiring.

$$\psi(\bar{x}) = \exists^{\neq} y\ \varphi(\bar{x}, y) \quad \longmapsto \quad f_\psi(\mathbf{X}) = \sum_{t \in T} f_\varphi(\mathbf{X}, t(\mathbf{Y}))$$

where T is the set of consistent assignments $t : \mathbf{Y} \to \{0, 1\}$ (which, out of any pair $(\beta, \neg\beta)$ of complementary literals, map one to 0, the other to 1).

Example

$$\psi := \exists^{\neq} x \exists^{\neq} y \big((\neg Exx \wedge Exy) \vee (Exx \wedge \neg Exy) \big)$$

Use variables $X := X_{Exx}$, $Y := X_{Exy}$ and $\overline{X}, \overline{Y}$ for the negations.

formula	polynomial
$\varphi := (\neg Exx \wedge Exy) \vee (Exx \wedge \neg Exy)$	$\overline{X} \cdot Y + X \cdot \overline{Y}$
$\exists^{\neq} y \varphi$	$(\overline{X} \cdot 0 + X \cdot 1) + (\overline{X} \cdot 1 + X \cdot 0) = X + \overline{X}$
$\psi := \exists^{\neq} x \exists^{\neq} y \varphi$	$f_\psi := (0+1) + (1+0) = 1$

Example

$$\psi := \exists^{\neq} x \exists^{\neq} y \big((\neg Exx \land Exy) \lor (Exx \land \neg Exy) \big)$$

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$\varphi := (\neg Exx \land Exy) \lor (Exx \land \neg Exy)$	$\overline{X} \cdot Y + X \cdot \overline{Y}$
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The sentence ψ evaluates almost surely to the maximal truth value 1.

Universal quantifiers and small positive values

Existential quantifiers: use sums and consistent assignments to $\{0,1\}$

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Recall that 1 is the maximal value in the semiring.

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Recall that 1 is the maximal value in the semiring.

Universal quantifiers: use products and consistent assignments to $\{0,e\}$ where e stands for the smallest positive value $\min(S\setminus\{0\})$.

$$\vartheta(\bar{x}) = \forall^{\neq} y\, \varphi(\bar{x},y) \quad \longmapsto \quad f_\vartheta(\mathbf{X}) = \prod_{t\in T_e} f_\varphi(\mathbf{X}, t(\mathbf{Y}))$$

where T_e is the set of consistent assignments $t:\mathbf{Y}\to\{0,e\}$.

Example

$$\psi := \forall^{\neq} x\big(\neg Exx \vee (Exx \wedge \exists^{\neq} yExy)\big)$$

Use variables $X := X_{Exx}$, $Y := X_{Exy}$ and $\overline{X}, \overline{Y}$ for the negations.

formula	polynomial
Exy	Y
$\exists^{\neq} yExy$	$0+1=1$
$Exx \wedge \exists^{\neq} yExy$	$X \cdot 1 = X$
$\neg Exx \vee (Exx \wedge \exists^{\neq} yExy)$	$\overline{X} + X$
$\psi := \forall^{\neq} x \cdots$	$f_\psi := (e+0)\cdot(0+e) = e$

The sentence ψ evaluates almost surely to the minimal positive truth value in the given semiring.

The 0-1 law and the almost sure valuations

Let $(S, \max, \min, 0, 1)$ be a finite min-max semiring.
Let $\psi(\bar{x}) \in \mathrm{FO}^k(\tau)$, with associated polynomial $f_\psi(\mathbf{X}) \in E[\mathbf{X}]$.

Theorem.
If π is a S-interpretation with the k-extension property, and ρ is the atomic type of the tuple $\bar{a}$, then $\pi[\![\psi(\bar{a})]\!] = f_\psi[\rho]$.

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S-interpretations almost surely have the k-extension property. Thus, every $\psi \in \mathrm{FO}$ has a unique almost sure valuation, with only three possible values:

- if $f_\psi = 0$ then $\pi[\![\psi]\!] = 0$ almost surely,
- if $f_\psi = 1$ then $\pi[\![\psi]\!] = 1$ almost surely,
- if $f_\psi = e$ then $\pi[\![\psi]\!] = \min(S \setminus \{0\}$ almost surely.

Example: Secret facts

An interpretation π into the security semiring $\mathbb{A} = (\{0 < \mathsf{T} < \mathsf{S} < \mathsf{C} < \mathsf{P} = 1\}$ labels facts as "public" (P), "confidential" (C), "secret" (S), or "top secret" (T).

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Assume that access restrictions are assigned randomly. The 0-1 law says that, if ψ is almost surely true, then either

- $f_\psi = 1$ and ψ can almost surely be verified with public information. This is typically the case for existential statements $\exists x \varphi(x)$. Or
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Thus, clearance for just confidential or secret information is completely useless! :-)

Complexity

The split of $\mathrm{FO}(\tau)$ into $f_\psi = 0, 1, e$ is independent of the semiring.

$f_\psi \in \{1, e\}$ if, and only if, ψ is almost surely true (in the Boolean setting).

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The problem of computing the almost sure valuation of first-order sentences is

- PSPACE-hard (even just deciding whether $f_\psi = 1$ or $f_\psi = e$)
- PSPACE-complete: evaluate f_ψ in alternating polynomial time.

Beyond finite min-max semirings

The 0-1 law extends (with certain variations) to other classes of semirings:

Finite and infinite lattice semirings $(S, \sqcup, \sqcap, \bot, \top)$.

For infinite semirings, different kinds of extension properties must be considered, realising maximal extensions and "small" extensions of types.

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The natural semiring $(\mathbb{N}, +, \cdot, 0, 1)$, but the proof and the almost sure valuations are different:

Other extensions properties are needed, saying that there are many witnesses realising types with sufficiently large values.

Instead of polynomials, we have to use ∞-expressions using ∞ as coefficient and exponent.

Outlook

Our work is just a first step in the study of random semiring interpretations.

We have assumed a fixed probability distribution on the semiring. This corresponds to the $G_{n,p}$-model of random graphs for constant p. The study of logic on random structures has considered many different scenarios:

- probability distributions that depend on the size of the universe
- probability distributions on special classes of structures
- different logics

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0-1 laws and convergence laws imply non-definability results: properties with a different probabilistic behaviour than the one of formulae are inexpressible. Can we use our results to prove non-definability results for numerical parameters in semiring semantics?

Ehrenfeucht-Fraïssé Games

(Joint work with Sophie Brinke and Lovro Mrkonjić, CSL 2024)

$G_m(\mathfrak{A},\mathfrak{B})$: m-move EF-game on τ-structures $\mathfrak{A}$ and $\mathfrak{B}$

i-th move: Spoiler (I) selects $a_i \in \mathfrak{A}$ or $b_i \in \mathfrak{B}$, Duplicator (II) answers with $b_i \in \mathfrak{B}$ or $a_i \in \mathfrak{A}$.

after m moves, II has won if $\{(a_1,b_1),\dots,(a_m,b_m)\}$ is a local isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$.

Theorem. For any two structures $\mathfrak{A}$ and $\mathfrak{B}$, the following are equivalent

(1) $\mathfrak{A} \equiv_m \mathfrak{B}$

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The game $G(\mathfrak{A},\mathfrak{B})$: I selects $m \in \mathbb{N}$. Then $G_m(\mathfrak{A},\mathfrak{B})$ is played.

$$\text{II wins } G(\mathfrak{A},\mathfrak{B}) \iff \text{II wins } G_m(\mathfrak{A},\mathfrak{B}) \text{ for all } m \iff \mathfrak{A} \equiv_m \mathfrak{B} \text{ for all } m \iff \mathfrak{A} \equiv \mathfrak{B}.$$

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Question: What about $G_m(\pi_A,\pi_B)$ versus $\pi_A \equiv_m \pi_B$ for semiring interpretations π_A and π_B?

Soundness and Completeness

The game G_m is sound for $\equiv_m$ on a semiring S if for all S-interpretations π_A and π_A:

$$\text{II wins } G_m(\pi_A, \pi_B) \quad \Longrightarrow \quad \pi_A \equiv_m \pi_B$$

G_m is complete for $\equiv_m$ on a semiring S if for all S-interpretations π_A and π_A:

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By the Ehrenfeucht-Fraïssé-Theorem G_m is sound and complete for $\equiv_m$ on the Boolean semiring $\mathbb{B}$.

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However, for other semirings the games need be neither sound nor complete.

To what extent do the games work for semirings?

Question: For which semirings are the EF-games G_m and G sound, for which are they complete?

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There are also other variants of model comparison games. for which we pose the same question;

The m-move bijection game $BG_m(\pi_A, \pi_B)$: (Hella, for logics with counting quantifiers)

i-th move: Duplicator selects a bijection $h : A \to B$ with $h(a_j) = b_j$ for $j < i$

Spoiler selects a new pair (a_i, b_i) where $b_i = h(a_i)$.

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The parametrised m-counting game $CG^n_m \pi_A, \pi_B)$:

i-th move: Spoiler selects a set $X \subseteq A$ or $X \subseteq B$ with $|X| \leq n$.

Duplicator answers with $Y \subseteq B$ or $Y \subseteq A$ such that $|Y| = |X|$.

Spoiler selects an element of Y, Duplicator answers with an element of X.

This gives the new pair (a_i, b_i).

Note that $CG^1_m = G_m$

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- On $\mathbb{N}$ and $\mathbb{N}[X]$, all these games are complete.
- All these games are incomplete on $\mathbb{V}, \mathbb{T}, \mathbb{L}, \mathbb{D}, \mathbb{N}^\infty, \mathbb{W}[X], \mathbb{S}[X], \mathbb{B}[X]$, and $\mathbb{S}^\infty[X]$.

Ehrenfeucht-Fraïssé Games for Application Semirings

		$S \ncong \mathbb{B}$ fully idempotent	$\mathbb{V} \cong \mathbb{T}$	$\mathbb{L} \cong \mathbb{D}$	$\mathbb{N}$	$\mathbb{N}^\infty$
Soundness of	G_m for $\equiv_m$	✓	✗	✗	✗	✗
	CG^n_m for $\equiv_m$	✓	✗	✗	✗	✗
	BG_m for $\equiv_m$	✓	✓	✓	✓	✓
	G for $\equiv$	✓	✗	✗	✓	✓
Completeness of	G_m for $\equiv_m$	✗	✗	✗	✓	✗
	CG^n_m for $\equiv_m$	✗	✗	✗	✓	✗
	BG_m for $\equiv_m$	✗	✗	✗	✓	✗
	G for $\equiv$	✗	✗	✗	✓	✗

Ehrenfeucht-Fraïssé Games for Provenance Semirings

		PosBool$[X]$	$\mathbb{W}[X]$	$\mathbb{S}[X]$, $\mathbb{B}[X]$	$\mathbb{N}[X]$	$\mathbb{S}^\infty[X]$
Soundness of	G_m for $\equiv_m$	✓	✗	✗	✗	✗
	CG^n_m for $\equiv_m$	✓	✓	✗	✗	✗
	BG_m for $\equiv_m$	✓	✓	✓	✓	✓
	G for $\equiv$	✓	✓	✓	✓	✓
Completeness of	G_m for $\equiv_m$	✗	✗	✗	✓	✗
	CG^n_m for $\equiv_m$	✗	✗	✗	✓	✗
	BG_m for $\equiv_m$	✗	✗	✗	✓	✗
	G for $\equiv$	✗	✗	✗	✓	✗