Seminar Logic, Complexity, Games: Algorithmic Meta-Theorems and Parameterized Complexity

Hierarchies in Parameterized Complexity Theory

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1 Introduction

This seminar is based on [1]. We will create a new characterization of the W-hierarchy and the A-hierarchy based on weighted satisfiability problems of certain fragments of propositional logic. Using this characterization, we can then prove further relations between the two hierarchies and even introduce new complexity classes that are interesting on their own.

2 The W-Hierarchy and Propositional Logic

We will start with a reminder on some important definitions.

Definition 1 *For $t \geq 0$ and $d \geq 1$ we will inductively define the classes $\Gamma_{t,d}$ and $\Delta_{t,d}$ by*

$$\begin{aligned}\Gamma_{0,d} &:= \{\lambda_1 \wedge \ldots \wedge \lambda_c \mid c \in [1,d], \lambda_i \textit{ literal for all } i \in [1,c]\}\\ \Delta_{0,d} &:= \{\lambda_1 \vee \ldots \vee \lambda_c \mid c \in [1,d], \lambda_i \textit{ literal for all } i \in [1,c]\}\\ \Gamma_{t+1,d} &:= \{\bigwedge_{i\in[1,n]} \delta_i \mid n \in \mathbb{N}, \delta_i \in \Delta_{t,d} \textit{ for all } i \in [1,n]\}\\ \Delta_{t+1,d} &:= \{\bigvee_{i\in[1,n]} \delta_i \mid n \in \mathbb{N}, \delta_i \in \Gamma_{t,d} \textit{ for all } i \in [1,n]\}\end{aligned}$$

A formula φ is called positive, if it does not contain any negation symbols and is called negative, if it is in negation normal form and there is a negation symbol in front of every variable. For a class A of propositional formulas A^+ denotes the class of all positive formulas in A and A^- denotes the class of all negative formulas in A. We will denote the class of all propositional formulas by PROP.

Here, d denotes the size of the underlying conjunction (disjunction respectively) and t denotes the number of alternations between conjunctions and disjunctions. Note that we do not restrict the size of the conjunction (disjunction) on any level greater than zero.

Example 2:

- $\Gamma_{2,1}$ ($\Delta_{2,1}$) is the class of all formulas in CNF (DNF)
- $\Gamma_{1,d}$ ($\Delta_{1,d}$) is the class of all formulas in d-CNF (d-DNF)
- $\varphi_1 := \Big((X \wedge Y) \vee (X \wedge Z)\Big) \wedge \Big((A \wedge B) \vee (A \wedge C)\Big) \in \Gamma^+_{2,2}$
- $\varphi_2 := \Big((\neg X \vee \neg Y) \wedge (\neg X \vee \neg Z)\Big) \vee \Big((\neg A \vee \neg B) \wedge (\neg A \vee \neg C)\Big) \in \Delta^-_{2,2}$

The classes are defined purely by syntactical properties. In the example above, we have $\neg\varphi_2 \notin \Gamma^+_{2,2}$ since it contains negation symbols but it is equivalent to a formula in $\Gamma^+_{2,2}$ as we have $\neg\varphi_2 \equiv \varphi_1$.

In this first section, we will show an important characterization of the classes $\mathsf{W}[t]$ using only propositional logic. Remember that the class $\mathsf{W}[t]$ is defined over the weighted Fagin-defineability of first order formulas. We will show that this class can be expressed using a parameterized weighted satisfiability problem of certain fragments of PROP. Let Θ be a fragment of PROP. The parameterized weighted satisfiability problem of Θ is the following decision problem:

p-WSAT(Θ):
Input: $\alpha \in \Theta, k \in \mathbb{N}$
Parameter: k
Question: Decide whether α is k-satisfiable

I.e., we are looking for an assignment of a formula $\alpha \in \Theta$ such that exactly k variables are set to true and the rest is set to false. The main theorem of this section reads as follows.

Theorem 3 *For every $t > 1$, the following problems are $\mathsf{W}[t]$-complete.*

1. p-WSAT($\Gamma^+_{t,1}$) *if t is even and* p-WSAT($\Gamma^-_{t,1}$) *if t is odd.*
2. p-WSAT($\Delta_{t+1,d}$) *for every $d \geq 1$.*

Therefore, we have the following characterization of $\mathsf{W}[t]$ *for all* $t \geq 1$*:*

$$\mathsf{W}[t] = [\{\mathsf{p\text{-}WSAT}(\Gamma_{t,d} \cup \Delta_{t,d}) \mid d \geq 1\}]^{\mathsf{fpt}}$$

Note how the parameter t is used here. On one side, it is used in the definition of the class $\mathsf{W}[t]$ for the amount of quantifier alternations in first order formulas. On the other side, it is used for the amount of boolean connectives alternations in propositional formulas. The proof for the simple case of $t = 1$ can be found in [1], here we will prove the more interesting part of $t \geq 2$. This will be done using the following series of lemmas. Most of them will differentiate between the parity of t. For simplicity, we will always only prove one of the statements. The other part is completely analogous if not stated otherwise.

Lemma 4 *Let* $t \geq 1$*. For every* Π_t*-formula* $\varphi(X)$ *there is a* $d \geq 1$ *such that*

$$\mathsf{p\text{-}WD}_\varphi \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma_{t,d})$$

Proof. We will extend the proof for the simple case of $t = 1$ from [1] in the obvious way. There, we had a Π_1-formula and showed that $\mathsf{p\text{-}WD}_\varphi \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(d\text{-CNF}) = \mathsf{p\text{-}WSAT}(\Gamma_{1,d})$ holds. Now we want to increase this statement to arbitrary $t \geq 1$. Let $t \geq 1$ be an even number. Since t is even, we will assume that the quantifier free part of φ is in DNF, so that φ has the form:

$$\varphi(X) = \forall \overline{y}_1 \exists \overline{y}_2 \dots \forall \overline{y}_{t-1} \exists \overline{y}_t \bigvee_{i \in I} \bigwedge_{j \in J} \lambda_{i,j}$$

with literals $\lambda_{i,j}$. We set $d = \max\{2, |J|\}$ and create a reduction from $\mathsf{p\text{-}WD}_\varphi$ to $\mathsf{p\text{-}WSAT}(\Gamma_{t,d})$. For every τ-structure $\mathfrak{A}$ we will introduce a $\Gamma_{t,d}$-formula α such that for all $k \in \mathbb{N}$ we have:

$$(\mathfrak{A}, k) \in \mathsf{p\text{-}WD}_\varphi \iff \alpha \text{ is } k\text{-satisfiable}$$

We will use propositional variables $Y_{\overline{a}}$ to describe whether $\overline{a}$ is in the relation X. Additionally, we have to map the preceding quantifiers to the corresponding boolean operators. An $\exists$-quantifier maps to a disjunction and an $\forall$-quantifier maps to a conjunction over the whole corresponding universe. The resulting formula $\alpha = \alpha(\mathfrak{A}, \varphi)$ looks as follows:

$$\alpha \coloneqq \bigwedge_{\overline{a}_1 \in A^{|\overline{y}_1|}} \bigvee_{\overline{a}_2 \in A^{|\overline{y}_2|}} \cdots \bigvee_{\substack{\overline{a}_t \in A^{|\overline{y}_t|} \\ i \in I}} \gamma_{i,\overline{a}_1,\dots,\overline{a}_t}$$

Here, $\gamma_{i,\overline{a}_1,\dots,\overline{a}_t}$ is a conjunction that describes the evaluation of $\mathfrak{A}$ on the literals, while replacing the X-literals with the corresponding new boolean variables $Y_{\overline{a}}$. To be precise, $\gamma_{i,\overline{a}_1,\dots,\overline{a}_t}$ is the conjunction obtained from $\bigwedge_{j \in J} \lambda_{i,j}$ as follows:

- Replace Literals $(\neg)X y_{\ell_1} \dots y_{\ell_s}$ by $(\neg)Y_{a_{\ell_1} \dots a_{\ell_s}}$
- If $\lambda_{i,j}$ does not contain the relation variable X, then we can evaluate whether we have $\mathfrak{A} \models \lambda_{i,j}(\overline{a}_1, \dots, \overline{a}_t)$ and thus ignore $\lambda_{i,j}$ if $\mathfrak{A} \models \lambda_{i,j}(\overline{a}_1, \dots, \overline{a}_t)$, and omit $\gamma_{i,\overline{a}_1,\dots,\overline{a}_t}$ completely if $\mathfrak{A} \not\models \lambda_{i,j}(\overline{a}_1, \dots, \overline{a}_t)$.

Using this reduction we will result with a formula α with $|\alpha| \in \mathcal{O}(|A|^k \cdot |\varphi|)$. Now, for arbitrary $S \subset A^k$ one can easily verify that we have

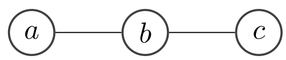
$$\mathfrak{A} \models \varphi(S) \iff \{Y_{\overline{a}} \mid \overline{a} \in S\} \text{ satisfies } \alpha$$

and this directly implies our desired equivalence. In order to ensure that all of the variables really occur in α' we append a tautology that contains all variables:

$$\alpha' := \alpha \wedge \bigwedge_{\overline{a} \in A^k} (Y_{\overline{a}} \vee \neg Y_{\overline{a}}) \qquad \square$$

Let us look at an example for this reduction.

Example 5:
Consider the Π_2-formula $\mathsf{ds}(X) := \forall x \exists y\, (Xy \wedge (Eyx \vee x = y))$ that describes whether X is a dominating set. This formula is equivalent to $\forall x \exists y\, ((Xy \wedge Eyx) \vee (Xy \wedge x = y))$, where the quantifier free part is in DNF. Additionally, we consider the following graph $\mathcal{G}$:



Here, we have to use the propositional variables Y_a, Y_b and Y_c, since our relation X has arity 1. The variable Y_a is true if and only if $a \in X$. Our final formula has the form $\bigwedge_{v \in \{a,b,c\}} \bigvee_{u \in \{a,b,c\}} \psi$ because we translated the $\forall$-quantifier to a conjunction over all elements and the $\exists$-quantifier to a disjunction over all elements of the corresponding universe, which is $\{a, b, c\}$ in our case. The transformation for the quantifier free part tries to evaluate a literal with a fixed assignment for the variables. For example, the conjunction $Xy \wedge Eyx$ together with the assignment $\mathcal{I}(x) := a$ and $\mathcal{I}(y) := b$, gets transformed to Y_b. Here, Xy can be replaced with Y_b since we have $\mathcal{I}(y) = b$ and Eyx can be removed since the structure $\mathcal{G}$ together with the assignment $\mathcal{I}$ satisfies the literal, namely the node a and the node b are adjacent. Satisfying the literal means that we can replace it with 1 and finally we get $Y_b \wedge 1 \equiv Y_b$. In the end, we would receive $\alpha = (Y_a \vee Y_b) \wedge (Y_a \vee Y_b \vee Y_c) \wedge (Y_b \vee Y_c)$ as our final formula.

There exists a straightforward proof to show that $\mathsf{p\text{-}WSAT}(\Gamma_{t,d}) \in \mathsf{W}[t]$, but we opt for another way, that reveals more information about the hierarchy in general. Therefore, we will first prove the following lemma about propositional normalization and then show that $\mathsf{p\text{-}WSAT}(\Delta_{t+1,d})$ is contained in $\mathsf{W}[t]$. The lemma about Propositional Normalization shows that the reduction of parameter d, namely the size of our underlying conjunction/disjunction, to 1 for the weighted satisfiability problem is fixed-parameter tractable. Additionally, we can also reduce to only positive/negative formulas based on the parity of t.

Lemma 6 (Propositional Normalization) *Let $d \geq 1$.*

1. *If $t > 1$ is an even number, then* $\mathsf{p\text{-}WSAT}(\Delta_{t+1,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Delta^{+}_{t+1,1})$ *and* $\mathsf{p\text{-}WSAT}(\Gamma_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma^{+}_{t,1})$
2. *If $t > 1$ is an odd number, then* $\mathsf{p\text{-}WSAT}(\Delta_{t+1,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Delta^{-}_{t+1,1})$ *and* $\mathsf{p\text{-}WSAT}(\Gamma_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma^{-}_{t,1})$

Proof. We will show that for all even $t > 1$ we have $\mathsf{p\text{-}WSAT}(\Gamma_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma^{+}_{t,1})$. In order to prove this, it is enough to prove that for all $d \geq 1$ we have $\mathsf{p\text{-}WSAT}(\Gamma_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma^{+}_{t,d})$ and $\mathsf{p\text{-}WSAT}(\Gamma^{+}_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma_{t,1})$ then we can build the following chain of fpt-reductions:

$$\mathsf{p\text{-}WSAT}(\Gamma_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma^{+}_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma_{t,1}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma^{+}_{t,1})$$

We will start by proving that $\mathsf{p\text{-}WSAT}(\Gamma_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma^+_{t,d})$ holds. Let (α, k) be an instance of $\mathsf{p\text{-}WSAT}(\Gamma_{t,d})$ and let $X_1, \ldots, X_n$ be the variables of α. We will fix some ordering on the variables and introduce variables $X_{i,j}$ and $Y_{i,j,j'}$ with the intended meaning:

- $X_{i,j}$ is true $:\Leftrightarrow$ the i-th variable set to true is X_j
- $Y_{i,j,j'}$ is true $:\Leftrightarrow$ the i-th variable set to true is X_j and the $(i+1)$-th is $X_{j'}$.

The newly created variables $X_{i,j}$ precisely describe which of the variables in our inital formula are set to true and the newly created variables $Y_{i,j,j'}$ describe the distance between those that are set to true. To express a negative literal $\neg X$ positively, we can now say that X is strictly between two successive variables that are set to true (or strictly before the first or after the last variable set to true).

To ensure that our variables have the intended meaning, we use the following construction. Let $\mathcal{Z}_1, \ldots, \mathcal{Z}_m$ be disjoint nonempty sets of propositional variables. We define a formula $\beta_{\overline{\mathcal{Z}}}$ that is satisfied by an assignment of weight m if and only if exactly one variable of each set $\mathcal{Z}_i$ is set to true.

$$\beta_{\overline{\mathcal{Z}}} \coloneqq \bigwedge_{i\in[1,m]} \bigvee_{Z\in\mathcal{Z}_i} Z$$

One can see that $\beta_{\overline{\mathcal{Z}}} \in \Gamma^+_{2,1}$. We will group the new variables into the nonempty sets $\mathcal{X}_i \coloneqq \{X_{i,j} \mid j \in [1,n]\}$ for $i \in [1,k]$ and $\mathcal{Y}_i \coloneqq \{Y_{i,j,j'} \mid 1 \leq j < j' \leq n\}$ for $i \in [1, k-1]$ and introduce formulas $\beta_1, \ldots, \beta_{k-1}$ such that any assignment of weight $(2k-1)$ satisfying $\beta = \beta_{\overline{\mathcal{X}},\overline{\mathcal{Y}}} \wedge \beta_1 \wedge \ldots \wedge \beta_{k-1}$ and setting $X_{1,\ell_1}, \ldots, X_{k,\ell_k}$ to true must set $Y_{i,\ell_1,\ell_2}, \ldots, Y_{k-1,\ell_{k-1},\ell_k}$ to true. This would mean that our variables have the intended meaning. For this we set

$$\beta_i \coloneqq \bigwedge_{j\in[1,n]} \left(\bigvee_{\substack{1\leq j_1<j_2\leq n \\ j_1\neq j}} (X_{i,j} \vee Y_{i,j_1,j_2}) \wedge \bigvee_{\substack{1\leq j_1<j_2\leq n \\ j_2\neq j}} (X_{i+1,j} \vee Y_{i,j_1,j_2}) \right)$$

Intuitively speaking, this formula states that either $X_{i,j}$ is true or some Y_{i,j_1,j_2} with $j \neq j_1$ and it states that either $X_{i+1,j}$ is true or some Y_{i+1,j_1,j_2} with $j \neq j_2$.

Let us now prove that this construction is indeed correct. Consider a satisfying assignment of β of weight $(2k-1)$. Since it satisfies $\beta_{\overline{\mathcal{X}},\overline{\mathcal{Y}}}$ it sets exactly one variable in each $\mathcal{X}_i$ and one variable in each $\mathcal{Y}_i$ to true. Let X_{i,ℓ_i} be the unique variable in each $\mathcal{X}_i$ set to true. Now, fix i and let $Y_{i,\ell,m}$ be the unique variable of $\mathcal{Y}_i$ set to true. If $\ell \neq \ell_i$, then in β_i the first conjunct would not be satisfied. Similarly, if $m \neq \ell_{i+1}$ then the second conjunct would not be satisfied. Note that all of the β_i are equivalent to a $\Gamma^+_{2,1}$ formula and thus β is as well, as a conjunction of several $\Gamma_{t,d}$ formulas is still in $\Gamma_{t,d}$.

A formula that expresses that X_j is either smaller (with respect to the ordering of the variables of α by their indices) than the first variable set to true or between two successive variables set to true or after the last variable set to true, is the following formula $\gamma_j \in \Delta^+_{1,1}$

$$\gamma_j \coloneqq \bigvee_{\substack{j'\in[1,n] \\ j<j'}} X_{1,j'} \vee \left(\bigvee_{i\in[1,k-1]} \bigvee_{\substack{j',j''\in[1,n] \\ j'<j<j''}} Y_{i,j',j''} \right) \vee \bigvee_{\substack{j'\in[1,n] \\ j'<j}} X_{k,j'}$$

Now let α' be the formula obtained from $\alpha \in \Gamma_{t,d}$ by replacing all negative literals $\neg X_j$ with γ_j and all positive literals X_j by $\bigvee_{i\in[1,k]} X_{i,j}$. These are all $\Delta^+_{1,1}$ formulas, so we can apply the distributive law to get a resulting formula in $\Gamma^+_{t,d}$. Thus we have $\alpha' \wedge \beta \in \Gamma^+_{t,d}$ because $\alpha' \in \Gamma^+_{t,d}$ and $\beta \in \Gamma^+_{2,1} \subseteq \Gamma^+_{t,d}$. By our previous analysis, we now have

$$\alpha \text{ is } k\text{-satisfiable} \iff (\alpha' \wedge \beta) \text{ is } (2k-1)\text{-satisfiable}$$

Let us come to the second part of this proof, namely $\mathsf{p\text{-}WSAT}(\Gamma^+_{t,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}WSAT}(\Gamma_{t,1})$. Fix $d \geq 1$. Since t is even, we use conjunctions consisting of d variables in the underlying formulas. The idea of this reduction consists of replacing these conjunctions by single variables for the corresponding set of occurring variables. Let (α, k) be an instance of $\mathsf{p\text{-}WSAT}(\Gamma^+_{t,d})$ and let $\mathcal{X}$ be the set of variables in α. For every nonempty subset $\mathcal{Y} \subseteq \mathcal{X}$ of cardinality at most d we introduce a new variable $S_\mathcal{Y}$ (The number of different variables is bounded by $|\mathcal{X}|^d \leq |\alpha|^d$). The formula

$$\beta_{\text{set}} \coloneqq \bigwedge_{\substack{\emptyset \neq \mathcal{Y} \subset \mathcal{X} \\ |\mathcal{Y}| \leq d}} \left(S_\mathcal{Y} \leftrightarrow \bigwedge_{X \in \mathcal{Y}} X \right) \equiv \bigwedge_{\substack{\emptyset \neq \mathcal{Y} \subset \mathcal{X} \\ |\mathcal{Y}| \leq d}} \left(\left(S_\mathcal{Y} \vee \bigvee_{X \in \mathcal{Y}} \neg X \right) \wedge \left(\bigwedge_{X \in \mathcal{Y}} (\neg S_\mathcal{Y} \vee X) \right) \right)$$

sets the values of the variables correctly and one can see that it is equivalent to a $\Gamma_{2,1}$-formula. Let α_0 be the formula obtained from α by replacing every underlying conjunction $(X_1 \wedge \ldots \wedge X_r)$ by $S_{\{X_1,\ldots,X_r\}}$. Clearly, $\alpha' \coloneqq (\alpha_0 \wedge \beta_{\text{set}})$ is equivalent to a $\Gamma_{t,1}$-formula. Furthermore, let m be the number of nonempty subsets of cardinality $\leq d$ of a set of k elements, that is $m \coloneqq \sum_{i=1}^{d} \binom{k}{i}$. We obtain the desired reduction by showing that

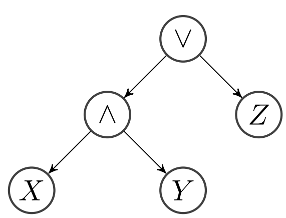
$$\alpha \text{ is } k\text{-satisfiable} \iff \alpha' \text{ is } (k+m)\text{-satisfiable}$$

Assume that we have an assignment of weight k satisfying α that sets $X_1, \ldots, X_k$ to true. Its extension that sets exactly the variables $S_\mathcal{Y}$ of α', where $\mathcal{Y}$ is a nonempty subset of $\{X_1, \ldots, X_k\}$ to true is a weight $k+m$ assignment satisfying α'. Conversely, the formula β_{set} enforces that an assignment of weight $k+m$ satisfying α' must set exactly k variables in $\mathcal{X}$ to true, hence its restriction to the variables in $\mathcal{X}$ is a weight k assignment satisfying α. ☐

Now, we have proven that all of these classes are $\mathsf{W}[t]$-hard under fpt-reductions. Let us now prove that they are contained in $\mathsf{W}[t]$ as well.

Lemma 7 *For every $t > 1$, we have $\mathsf{p\text{-}WSAT}(\Delta_{t+1,d}) \in \mathsf{W}[t]$.*

Proof. By the propositional normalization lemma it suffices to show that $\mathsf{p\text{-}WSAT}(\Delta^+_{t+1,1}) \in \mathsf{W}[t]$ for t even and $\mathsf{p\text{-}WSAT}(\Delta^-_{t+1,1}) \in \mathsf{W}[t]$ for t odd. We will once again only consider the case for t even. In order to show, that a problem is contained in $\mathsf{W}[t]$, we have to give a fpt-reduction to the weighted Fagin-defineability problem of a Σ_{t+1}-sentence. Let $\alpha \in \Delta^+_{t+1,1}$. The parse tree of a formula α is defined as the derivation tree of the syntax of the formula. For example, consider the formula $\varphi \coloneqq (X \wedge Y) \vee Z$. Then we have the following parse tree:



In the parse tree of α, all variables have distance $t+1$ from the root. Consider the directed graph $\mathcal{G} = (G, E^{\mathcal{G}})$ obtained from the parse tree by identifying all leaves corresponding to the same variable in one node. We direct the edges top-down as you can see in the example parse tree above. Let **ROOT** and **LITERAL** be unary relation symbols. Let $\mathbf{ROOT}^{\mathcal{G}}$ just contain the root, and let $\mathbf{LITERAL}^{\mathcal{G}}$ be the set of nodes corresponding to the literals. We introduce a Σ_{t+1}-sentence $\varphi(X)$ with a set variable X such that for any $k \geq 1$ we have

$$\alpha \text{ is } k\text{-satisfiable } \iff (\mathcal{G}, \mathbf{ROOT}^{\mathcal{G}}, \mathbf{LITERAL}^{\mathcal{G}}) \models \varphi(S) \text{ for some } S \subseteq G \text{ with } |S| = k$$

In particular, we even have for all $k \geq 1$ and any variables $Y_1, \ldots Y_k$ of α:

$$\{Y_1, \ldots Y_k\} \text{ satisfies } \alpha \iff (\mathcal{G}, \mathbf{ROOT}^{\mathcal{G}}, \mathbf{LITERAL}^{\mathcal{G}}) \models \varphi(\{Y_1, \ldots Y_k\})$$

Our desired formula $\varphi(X)$ simply mimics the recursive definition of the satisfaction relation of $\Delta^+_{t+1,1}$-formulas. Conjunctions will be mapped to $\exists$-quantifiers and disjunctions to $\forall$-quantifiers. I.e., we can use $\varphi(X)$ as a Σ_{t+1}-sentence equivalent to

$$\forall z (Xz \to \mathbf{LITERAL} z) \wedge$$

$$\exists y_0 (\mathbf{ROOT} y_0 \wedge \exists y_1 (E y_0 y_1 \wedge \forall y_2 (E y_1 y_2 \to \ldots \to \exists y_{t+1} (E y_t y_{t+1} \wedge X y_{t+1}) \ldots)) \qquad \square$$

Now, we can finally prove our main theorem.

Proof of Theorem 3. We have $\Gamma^+_{t,1} \subseteq \Delta_{t+1,d}$ and $\Gamma^-_{t,1} \subseteq \Delta_{t+1,d}$. Therefore, all of the problems mentioned in the theorem are contained in $\mathsf{W}[t]$ by Lemma 7. If t is even, then every parameterized problem $\mathsf{p\text{-}WD}_\varphi$ with a Π_t-formula φ is reducible to $\mathsf{p\text{-}WSAT}(\Gamma^+_{t,1})$ by Lemma 4 and the propositional normalization lemma 6. This shows that $\mathsf{p\text{-}WSAT}(\Gamma^+_{t,1})$ and thus also $\mathsf{p\text{-}WSAT}(\Delta_{t+1,d})$ is $\mathsf{W}[t]$-hard. $\square$

We can also use a slightly modified version of the proof of lemma 7 in order to prove another important relation between the A- and W-hierarchy. Remember that the A-hierarchy is defined using the parameterized model-checking problem for first order formulas. This relation is expressed by the following lemma.

Lemma 8 *For every $t > 1$ and $d \geq 1$, we have* $\mathsf{p\text{-}WSAT}(\Delta_{t+1,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}MC}(\Sigma_t)$.

Proof. Again, we only have to show that $\mathsf{p\text{-}WSAT}(\Delta^+_{t+1,1}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}MC}(\Sigma_t)$ by propositional normalization. We will use the notation from the previous proof and construct for a given $\alpha \in \Delta^+_{t+1,1}$ the structure $(G, E^{\mathcal{G}}, \mathbf{ROOT}^{\mathcal{G}}, \mathbf{LITERAL}^{\mathcal{G}})$ and the formula $\varphi(X)$ that mimics the recursive definition of the satisfaction relation of $\Delta^+_{t+1,1}$-formulas. Now we can replace the set variable X by individual variables $x_1, \ldots, x_k$ and call this new formula ψ. It has the form:

$$\psi(x_1, \ldots, x_k) := \bigwedge_{i \in [1,k]} \mathbf{LITERAL} x_i \wedge$$

$$\exists y_0 (\mathbf{ROOT} y_0 \wedge \exists y_1 (E y_0 y_1 \wedge \forall y_2 (E y_1 y_2 \to \ldots \to \forall y_t (E y_{t-1} y_t \to \bigvee_{i \in [1,k]} E y_t x_i) \ldots))$$

Then we have

$$\alpha \text{ is } k\text{-satisfiable } \Leftrightarrow (\mathcal{G}, \mathbf{ROOT}^{\mathcal{G}}, \mathbf{LITERAL}^{\mathcal{G}}) \models \psi(x_1, \ldots, x_k) \text{ for some distinct } x_1, \ldots, x_k \in G$$

We can ψ easily extend so that we get a formula ψ' of the desired form that ensures that there are k distinct variables satisfying ψ. Then we finally have

$$\alpha \text{ is } k\text{-satisfiable } \Leftrightarrow (\mathcal{G}, \mathbf{ROOT}^{\mathcal{G}}, \mathbf{LITERAL}^{\mathcal{G}}) \models \psi' \qquad \square$$

This reduction directly shows that the class $\mathsf{W}[t]$ is contained in $\mathsf{A}[t]$.

Corollary 9 *For every $t \geq 1$, we have $\mathsf{W}[t] \subseteq \mathsf{A}[t]$.*

3 The A-Hierarchy and Propositional Logic

We also want to create a classification for the A-hierarchy in terms of a weighted satisfiability problem of certain fragments of PROP. When translating the model-checking problem for Σ_ℓ into a weighted satisfiability problem for a class of propositional formulas we have to somehow express the unbounded blocks of quantifiers $\exists x_1 \ldots \exists x_k$. We will represent such a block using an assignment of weight k. So that the quantifier alternations can be represented by assignments and not by the formula itself. This will be represented by an alternating satisfiability problem for classes of propositional logic. Again, let Θ be a fragment of PROP. The parameterized alternating weighted satisfiability problem of Θ is the following decision problem:

$\mathsf{p\text{-}AWSAT}_\ell(\Theta)$:

Input:	$\alpha \in \Theta$, a partition of the propositional variables of α into sets $\mathcal{X}_1, \ldots, \mathcal{X}_\ell$ and $k_1, \ldots, k_\ell \in \mathbb{N}$
Parameter:	$k = k_1 + \ldots + k_\ell$
Question:	Decide whether there is a subset S_1 of $\mathcal{X}_1$ with $\lvert S_1\rvert = k_1$ such that for every subset S_2 of $\mathcal{X}_2$ with $\lvert S_2\rvert = k_2$ there exists … such that the truth value assignment $S_1 \cup \ldots \cup S_\ell$ satisfies α

Obviously, we have $\mathsf{p\text{-}AWSAT}_1(\Theta) = \mathsf{p\text{-}WSAT}(\Theta)$. The main result for the A-hierarchy goes as follows.

Theorem 10 *For every $\ell \geq 1$, we have the following characterization of $\mathsf{A}[\ell]$:*

$$\mathsf{A}[\ell] = [\{\mathsf{p\text{-}AWSAT}_\ell(\Gamma_{1,d} \cup \Delta_{1,d}) \mid d \geq 1\}]^{\mathsf{fpt}}$$

This theorem is an immediate consequence of the following fpt reductions.

- $\mathsf{p\text{-}AWSAT}_\ell(\Gamma_{1,d} \cup \Delta_{1,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}MC}(\Sigma_\ell) \leq^{\mathsf{fpt}} \mathsf{p\text{-}AWSAT}_\ell(\Gamma^-_{1,2})$ if $\ell \geq 1$ is odd.
- $\mathsf{p\text{-}AWSAT}_\ell(\Gamma_{1,d} \cup \Delta_{1,d}) \leq^{\mathsf{fpt}} \mathsf{p\text{-}MC}(\Sigma_\ell) \leq^{\mathsf{fpt}} \mathsf{p\text{-}AWSAT}_\ell(\Delta^+_{1,2})$ if $\ell \geq 1$ is even.

A precise proof of these reductions can be found in [1]. Here, we will only sketch the proof idea.

Proof Sketch. We will first show that the alternating weighted satisfiability problem $\mathsf{p\text{-}AWSAT}_\ell(\Gamma_{1,d} \cup \Delta_{1,d})$ is contained in $\mathsf{A}[\ell]$. For this let $(\alpha, \mathcal{X}_1, \ldots, \mathcal{X}_\ell, k_1, \ldots, k_\ell)$ be an instance of $\mathsf{p\text{-}AWSAT}_\ell(\Gamma_{1,d} \cup \Delta_{1,d})$. Let $k := \sum_{i=1}^{\ell} k_i$ and let $X_1, \ldots, X_m$ be the variables of α.

For our model-checking problem we create the structure $\mathfrak{A}$ with universe $[1, m]$ so that there is a bijection between the propositional variables and the elements in our universe. We expand the structure by unary relations $P_i := \{j \mid X_j \in \mathcal{X}_i\}$ that describe what variables are in which input sets. Using this structure, we can convert the alternating weighted satisfiability problem into a first order formula in a straightforward fashion. The quantifiers of the assignments transfer to the quantifier of the first order formula. Thus we can simply use a formula that looks like the following:

$$\exists x_1 \dots \exists x_{k_1} \Big(\bigwedge_{i \in [1,k_1]} P_1 x_i \wedge \bigwedge_{1 \le i < j \le k_1} x_i \neq x_j \wedge$$
$$\forall x_{k_1+1} \dots \forall x_{k_1+k_2} \Big(\big(\bigwedge_{i \in [k_1+1,k_1+k_2]} P_2 x_i \wedge \bigwedge_{k_1+1 \le i < j \le k_1+k_2} x_i \neq x_j \big) \rightarrow \dots \Big(\psi(x_1, \dots, x_k) \Big) \dots \Big) \Big)$$

The construction of ψ for the quantifier free part is omitted here and can be found in the book. Now we have

$$(\alpha, \mathcal{X}_1, \dots, \mathcal{X}_\ell, k_1, \dots, k_\ell) \in \mathsf{p\text{-}AWSAT}_\ell(\Gamma_{1,d} \cup \Delta_{1,d}) \iff \mathfrak{A} \models \varphi$$

For the second part of this proof, we consider the $\mathsf{A}[\ell]$-hardness of the alternating weighted satisfiability problem, namely $\mathsf{p\text{-}MC}(\Sigma_\ell) \le^{\mathsf{fpt}} \mathsf{p\text{-}AWSAT}_\ell(\Gamma^-_{1,2})$. Given a structure $\mathfrak{A}$ with universe A and a Σ_ℓ-sentence φ we create propositional variables $X_{i,a}$ with the intended meaning that the i-th variable of φ gets the value $a \in A$. We create the sets $\mathcal{X}_1, \dots, \mathcal{X}_\ell$ such that they describe the quantifiers of the formula and thus we can use the alternation of the sets in the alternating weighted satisfiability problem in order to represent the alternation of quantifiers in the first order formula. First, we want to create a formula α such that

$$\mathfrak{A} \models \varphi \iff (\alpha, \mathcal{X}_1, \dots, \mathcal{X}_\ell, k_1, \dots, k_\ell) \in \mathsf{p\text{-}AWSAT}_\ell(\mathsf{PROP})$$

In order to create an $\alpha \in \mathsf{PROP}$ with this property, we need to create another formula that states that in each set $\mathcal{X}_i$ we have for all $j \in [1, k_i]$ exactly one $a \in A$ such that $X_{j,a}$ is set to true. Additionally, we have to transform the quantifier free part such that the new formula is satisfied if and only if the corresponding assignment to the variables satisfies it the quantifier free part. Both of this can be fairly easy done. The tricky part is to get α from PROP to a formula in $\Gamma^-_{1,2}$. We will omit this part here and only refer to the book [1] for further details. □

4 Further Complexity Classes

In the end, we want to fill out a little more space in our parameterized complexity landscape and add some more classes to the hierarchies and in between them.

4.1 W[SAT] and A[SAT]

We have seen that the W-hierarchy can be represented by a weighted satisfiability problem of certain fragments of the class PROP and that the A-hierarchy can be represented by an alternating weighted satisfiability problem of certain fragments of the class PROP. What happens, if we do not only consider certain fragments of PROP but the whole class itself? In other words, how can we place the complexity of p-WSAT(PROP) and p-AWSAT(PROP), where we have no restrictions on the parameters t and ℓ inside of our hierarchy? Here, dropping the ℓ parameter in p-AWSAT(PROP) means that we allow arbitrary quantifier alternations for the assignment.

Definition 11 ($\mathsf{W[SAT]}$) $\mathsf{W[SAT]}$ *is the class of all parameterized problems fpt-reducible to* $\mathsf{p\text{-}WSAT(PROP)}$*, namely*

$$\mathsf{W[SAT]} := [\mathsf{p\text{-}WSAT(PROP)}]^{\mathsf{fpt}}$$

Definition 12 ($\mathsf{A[SAT]}$) $\mathsf{A[SAT]}$ *is the class of all parameterized problems fpt-reducible to* $\mathsf{p\text{-}AWSAT(PROP)}$*, namely*

$$\mathsf{A[SAT]} := [\mathsf{p\text{-}AWSAT(PROP)}]^{\mathsf{fpt}}$$

Clearly we have

$$\mathsf{FPT} \subseteq \mathsf{W}[1] \subseteq \mathsf{W}[2] \subseteq \ldots \subseteq \mathsf{W[SAT]} \subseteq \mathsf{W[P]}$$

and there is no reason to believe that any of these relations are not strict. If we assume that the W-hierarchy is strict, then we also have $\cup_{t\geq 1}\mathsf{W}[t] \subset \mathsf{W[SAT]}$. The converse is also true, i.e., if we assume that there is a $t \geq 1$ such that $\mathsf{W}[t] = \mathsf{W[SAT]}$, then the W-Hierarchy collapses to its t-th level.

4.2 The A-Matrix

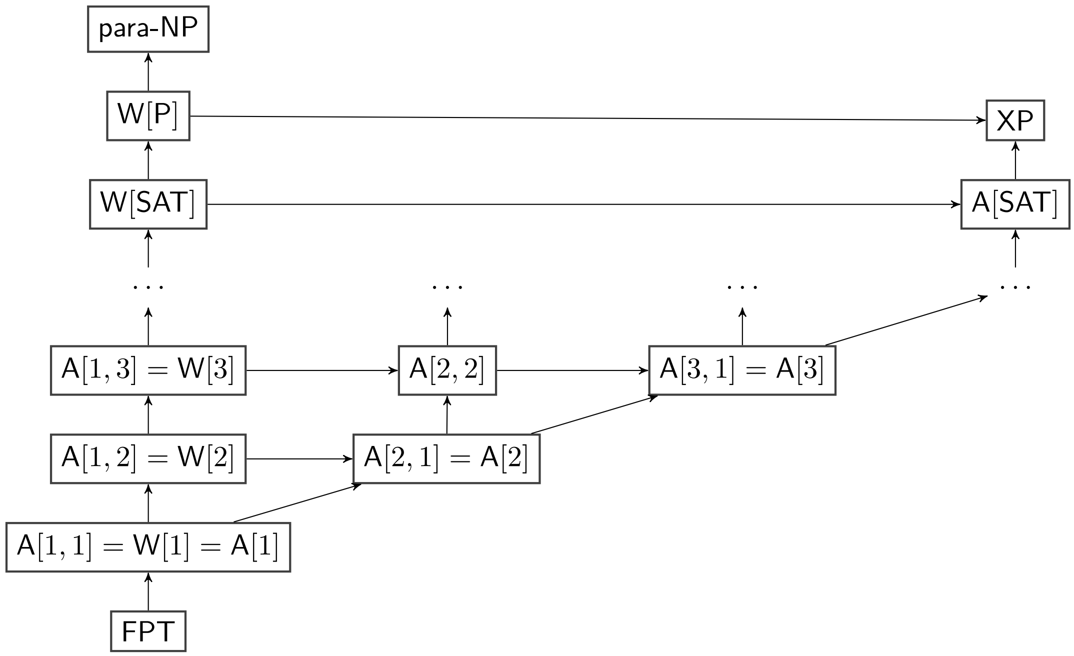
We can also fill the gap between both hierarchies. In the case of the W-hierarchy we use the alternation of propositional connectives in order to increase the computational power and in the case of the A-hierarchy we use the alternation of assignment quantifiers. This means that we can create a two dimensional matrix of complexity classes rather than just two hierarchies on their own.

Definition 13 (A-Matrix) *Let* $\ell, t \geq 1$*. Then we define*

$$\mathsf{A}[\ell, t] := [\{\mathsf{p\text{-}AWSAT}_\ell(\Gamma_{t,d} \cup \Delta_{t,d}) \mid d \geq 1\}]^{\mathsf{fpt}}$$

Again, t denotes the number of alternations of propositional connectives and ℓ denotes the number of alternations of quantifiers in our alternating satisfiability problem. We already know that $\mathsf{W}[t] = \mathsf{A}[1, t]$ for all $t \geq 1$ and that $\mathsf{A}[\ell] = \mathsf{A}[\ell, 1]$ for all $\ell \geq 1$. Using similar methods as we used above, one can prove that for every $\ell \geq 1$ and $t \geq 2$, we have $\mathsf{A}[\ell, t] \subseteq \mathsf{A}[\ell + 1, t - 1]$. This means that quantifier alternation seems to be at least as strong as the alternation of connectives.

Finally, we will give a visual overview over the known containment relations in both of our hierarchies and in the A-matrix. Note that there are further complexity classes inside of this hierarchy that were not mentioned in this seminar but nevertheless are very interesting on their own.



References

[1] Jörg Flum and Martin Grohe. *Parameterized Complexity Theory*. Texts in Theoretical Computer Science. An EATCS Series. Springer, 2006.