We give a simple brute-force solution for Exercise 4 (a).

Exercise 4 0 Points

A set a is *inductive* if $\emptyset \in a$ and for all $x \in a$, $x \cup \{x\} \in a$. Let $\omega = \bigcap\{x \mid x \text{ is inductive}\}$.

(a) Show that ω is a set.

Solution The intersection of every nonempty class is a set. Indeed, let $A \neq \emptyset$ be a class, then $\bigcap A = \{x \mid x \in y \in A \text{ for some set } y\}$. As $A \neq \emptyset$, there is some set $z \in A$. Then $\bigcap A = \{x \in z \mid x \in y \in A \text{ for some set } y\}$. Now it is easy to see that every limit stage is inductive, so the class of inductive sets is not empty.

We show: if s is a limit stage and $x \in s$ then $x \cup \{x\} \in s$. As s is a limit stage, we have $x \in s' \in s$ for some stage s'. Then $\{x\} \subseteq s'$, so $x, \{x\} \in \mathcal{P}(s')$. Now we show that if two sets are in some stage then so is their union. The result follows then.

We show first that if $a \in s$ then $\bigcup a \in s$ for any set a and any stage s. Let $b \in \bigcup a$. Then $b \in c \in a \in s$ for some $c \in a$. By transitivity of s, $b \in s$.

Now, $\{x, \{x\}\} \in \mathcal{P}(\mathcal{P}(s'))$ and $\{x, \{x\}\} \in s$, as s is a limit stage. Then $\bigcup\{x, \{x\}\} \in s$ and $x \cup \{x\} \subseteq \bigcup\{x, \{x\}\} \in s$, so $\{x, \{x\}\} \in s$ because s is hereditary as a stage.