Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
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WS 2010/11

Mathematical Logic II — Assignment 12

Due: Monday, January 24, 12:00

Exercise 1 3 Points

Find the least cardinal κ such that the class of all $\{<\}$-structures which are isomorphic to $(\mathbb{Z}, <)$ is axiomatizable in $L_{\kappa\omega}(<)$ or show that there no such cardinal.

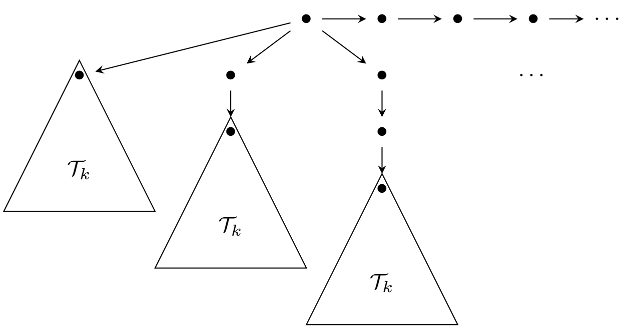
Exercise 2 1 + 4 Points

(a) Prove that $L_{\omega_1\omega}(\tau)$ is uncountable for all signatures τ.

(b) Construct an *uncountable* τ-structure $\mathfrak{B}$ for a suitable *countable* signature τ such that for all *countable* structures $\mathfrak{A}$ holds that $\mathfrak{A} \not\equiv_{L_{\omega_1\omega}(\tau)} \mathfrak{B}$, i.e. $\mathfrak{A}$ and $\mathfrak{B}$ satisfy different sets of $L_{\omega_1\omega}(\tau)$-sentences.

Exercise 3 4 Points

For $k = 1, 2, \ldots$, we define the directed rooted tree $\mathcal{T}_k$ inductively. $\mathcal{T}_1$ consists of disjoint finite paths of lengths $1, 2, 3, \ldots$ that start in the root. For $k > 1$, the tree $\mathcal{T}_{k+1}$ is constructed from $\mathcal{T}_1$ by substituting each leaf of the tree with $\mathcal{T}_k$. Finally, $\mathcal{T}'_k$ is constructed from $\mathcal{T}_k$ by adding an infinite path that starts from the root. (See the picture.) Compute the least ordinal α such that $I_\alpha(\mathcal{T}_k, \mathcal{T}'_k) = \emptyset$ or prove that no such ordinal exists.



Exercise 4 4 Points

Let τ be a finite relational signature and let $\mathfrak{A}$ and $\mathfrak{B}$ be τ-structures with $\mathfrak{A} \cong_\infty \mathfrak{B}$. Let $<\in \tau$ be a binary relation symbol such that $<^{\mathfrak{A}}$ is a well-order. Assume that the universes of $\mathfrak{A}$ and $\mathfrak{B}$ are sets. Prove that $\mathfrak{A} \cong \mathfrak{B}$.

Exercise 5 $5^* + 5^*$ Points

For two linear orders $(A, <)$ and $(B, <)$, let $(A, <) \cdot (B, <) := (A \times B, <)$ where $(a, b) < (a', b')$ if and only if $b < b'$, or $b = b'$ and $a < a'$. (Intuitively, $(A, <) \cdot (B, <)$ consists of $|B|$ many copies of A that are written linearly next to each other.) For $0 < n < \omega$, let $(A, <)^n$ be defined by $(A, <)^1 := (A, <)$ and $(A, <)^n := (A, <)^n \cdot (A, <)$.

(a) Compute the least ordinal α_n such that I has a winning strategy in $G_{\alpha_n}((\mathbb{Z}, <)^n, (\mathbb{Z}, <)^{n+1})$ and describe this strategy.

(b) Compute the least ordinal α_n such that I has a winning strategy in $G_{\alpha_n}((\omega, <)^n, (\omega, <)^{n+1})$ and describe this strategy.