Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, R. Rabinovich

WS 2010/11

Mathematical Logic II — Assignment 11

Due: Monday, January 17, 12:00

Exercise 1 4 Points

Let $T \subseteq \mathrm{FO}(\tau)$ be a theory and let $\varphi, \psi \in \mathrm{FO}(\tau)$. Prove that the following statements are equivalent.

(a) There is some Π_1-sentence $\vartheta \in \mathrm{FO}(\tau)$ with $T \models \varphi \rightarrow \vartheta$ and $T \models \vartheta \rightarrow \psi$.

(b) For all models $\mathfrak{A}$ and $\mathfrak{B}$ of T with $\mathfrak{A} \subseteq \mathfrak{B}$, if $\mathfrak{B} \models \varphi$ then $\mathfrak{A} \models \psi$.

Hint: Consider the set $\{\vartheta \in \mathrm{FO}(\tau)_\forall \mid T \models \varphi \rightarrow \vartheta\}$ and use Corollary 2.4 from the Lecture Notes ($\mathfrak{A} \models T_\forall$ if and only if there exists some $\mathfrak{B} \supseteq \mathfrak{A}$ with $\mathfrak{B} \models T$).

Exercise 2 $4 + (1 + 1 + 3) + 6^* + 3^* + 3 + 2$ Points

Let $\mathfrak{A} = (\mathbb{N}, S, 0)$ and let $\mathfrak{B} = (\mathbb{Q}, <)$.

(a) Describe all principal complete 1-types of $\mathfrak{A}$ (see Assingment 10 for a difinition of a principal type).

(b) Consider 1-types of $\mathfrak{B}$.

(i) Give a 1-type that is realised in $\mathbb{Q}$.

(ii) Give a 1-type that is realised in $\mathbb{R}$, but not in $\mathbb{Q}$,

(iii) Give *three* 1-types that are not realised in $\mathbb{R}$.

(c*) Classify all complete 1-types of $\mathfrak{B}$.
Hint: You may find useful that $\mathfrak{B}$ permits quantifier elimination.

(d*) Let p be a complete 1-type of $\mathfrak{B}$ over some finite set $C \subseteq \mathbb{Q}$. Prove that p is a pricipal type.
Hint: Solve (c) first.

(e) Classify all complete 1-types over the empty set of structures (X, f) where $f : X \rightarrow X$ is a bijection. Which of them are pricipal types?

(f) Classify all types over the empty set of $(\mathbb{Z}, S)$ where $S(z) = z + 1$. Which of them are complete? Which are principal?

Exercise 3 $2 + 3$ Points

Are the following strucutres ω-saturated?

(a) $(\mathbb{Q}, <)$,

(b) $(\mathbb{N} \times \mathbb{N}, \sim)$ where $(i, j) \sim (k, l)$ if and only if $i + j = k + l$.

For non-saturated structures give ω-saturated extensions.