Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
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Mathematical Logic II — Assignment 10

Due: Monday, January 10, 12:00

Exercise 1 3 + 5 Points

Let $\mathfrak{A} \subseteq \mathfrak{B}$ be two τ-structures for a signature τ. Prove the following statements.

(a) If for all finite sets $C \subseteq A$ and all $b \in B$ there exists an automorphism f on $\mathfrak{B}$ such that $f(c) = c$ for all $c \in C$ and $f(b) \in A$ then $\mathfrak{A} \preceq \mathfrak{B}$.

(b) The converse does not hold.

Exercise 2 4 Points

Let $T \subseteq \mathrm{FO}(\tau)$ be a theory and let $\varphi, \psi \in \mathrm{FO}(\tau)$ be sentences. Prove that the following statements are equivalent.

(a) There is a Π_1-sentence $\vartheta \in \mathrm{FO}(\tau)$ such that $T \models \varphi \rightarrow \vartheta$ and $T \models \vartheta \rightarrow \psi$.

(b) For all models $\mathfrak{A}, \mathfrak{B}$ of T with $\mathfrak{A} \subseteq \mathfrak{B}$, $\mathfrak{B} \models \varphi$ implies $\mathfrak{A} \models \psi$.

Hint: Consider the set $\{\vartheta \in \mathrm{FO}(\tau) \mid \vartheta \text{ is a } \Pi_1\text{-sentence with } T \models \varphi \rightarrow \vartheta\}$.

Exercise 3 5 Points

Prove that a theory $T \subseteq \mathrm{FO}(\tau)$ is model complete if and only if for each formula $\varphi(\overline{x}) \in \mathrm{FO}(\tau)$ there is some Σ_1-formula $\psi(\overline{x}) \in \mathrm{FO}(\tau)$ with $T \models \forall\overline{x}(\varphi(\overline{x}) \leftrightarrow \psi(\overline{x}))$.

Hint: Consider the set $\Phi := \{\psi(\overline{x}) \in \mathrm{FO}(\tau) \mid \ \psi \in \Sigma_1 \text{ and } T \models \forall\overline{x}(\psi(\overline{x}) \rightarrow \varphi(\overline{x}))\}$.

Exercise 4 3 + 4 + 4 Points

Let $\mathfrak{A}$ be a τ-structure and let $B \subseteq A$. An n-type p of $\mathfrak{A}$ over B is a *principal type* if there exists a formula $\varphi(\bar{x}) \in p$ such that $\mathfrak{A}_B \models \forall\bar{x}(\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$ for all $\psi(\bar{x}) \in p$.

(a) Let p be a complete type of $\mathfrak{A}$ over B that is realised by a tuple $\bar{b} \subseteq B$. Prove that p is a principal type.

(b) Show that all principal types of $\mathfrak{A}$ over B are realised in $\mathfrak{A}$.

(c) Let $\mathfrak{A}$ and $\mathfrak{B}$ be two τ-structures with $\mathfrak{A} \subseteq \mathfrak{B}$. Prove that $\mathfrak{A} \preccurlyeq \mathfrak{B}$ holds if and only if all principal types of $\mathfrak{B}$ over A are realised in $\mathfrak{A}$.