Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, R. Rabinovich

WS 2010/11

Mathematical Logic II — Assignment 6

Due: Monday, November 29, 12:00

Exercise 1 3 Points

Show using the Axiom of Choice that for all sets a and b there is an injective function $f : a \to b$ or $f : b \to a$.

Exercise 2 2 + 2 Points

Compute the cardinality of the following sets:

(a) $\{\alpha \in \text{On} \mid \alpha \text{ is a successor cardinal} < \aleph_1\}$,

(b) $\{\alpha \in \text{On} \mid \alpha \text{ is a limit ordinal} < \aleph_1\}$.

Exercise 3 3 + 3 Points

A set x is Dedekind-finite if no proper subset of x has the same cardinality as x. Prove or disprove:

(a) The set x is Dedekind-finite if and only if it is finite.

(b) The set x is finite if and only if every function $f : x \to x$ that is surjective or injective is already bijective.

Exercise 4* 4* Points

Let x be a set with $|x| \leq \kappa$ for some $\kappa \in \text{Cn}^\infty$ (where Cn^∞ is the class of limit cardinals). Let $|y| \leq \kappa$ for all $y \in x$. Prove that $|\bigcup x| \leq \kappa$.

Exercise 5 5* + 2 + 2 + 2 + 2 + 2 + 2 + 5* Points

Let A be a set and let $\leq$ be a linear order on A. A subset X of A is *cofinal in A* if for every $a \in A$ there is some $x \in X$ such that $a \leq x$ holds. Let α be an ordinal. The cofinality $\text{cf}(\alpha)$ of α is the least ordinal such that there is a function $f : \text{cf}(\alpha) \to \alpha$ with a non-bounded image in α. (That means, for all $\gamma \in \alpha$ there is some $\delta \in \text{cf}(\alpha)$ such that $f(\delta) \geq \gamma$.) An ordinal α is regular if α is a limit ordinal and $\text{cf}(\alpha) = \alpha$.

(a*) Prove that every linear order $(A, \leq)$ has a cofinal well-founded subset.

(b) Compute $\text{cf}(\alpha)$ for $\alpha = \omega$, $\alpha = \omega \cdot 2$ and for every successor ordinal α.

(c) Prove that $\text{cf}(\alpha)$ is a limit ordinal if α is a limit ordinal.

(d) Prove that for every $\alpha \in \text{On}$ there is a strongly monotone function $f : \text{cf}(\alpha) \to \alpha$ that is unbounded in α.

(e) Prove that $\text{cf}(\text{cf}(\alpha)) = \text{cf}(\alpha)$ holds for all $\alpha \in \text{On}$.

(f) Prove that $\text{cf}(\alpha) \in \text{Cn}$ holds for all $\alpha \in \text{On}$.

(g) Prove that $\text{cf}(\aleph_\omega) = \omega$.

(h*) Prove that all infinite successor cardinals are regular.