Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
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Mathematical Logic II — Assignment 4

Due: Monday, November 15, 12:00

Exercise 1 1 + 2 + 2 + 2 Points

One can define the pair (x, y) of the sets x and y as $\{\{x\}, \{x, y\}\}$. A formalisation of triples (a, b, c) as sets x_{abc} is *adequate* if $(a, b, c) = (a', b', c') \Leftrightarrow x_{abc} = x_{a'b'c'}$. Are the following formalisations of triples adequate:

(a) $(x, y, z) = ((x, y), z)$,

(b) $(x, y, z) = \{\{x, [0]\}, \{y, [1]\}, \{z, [2]\}\}$,

(c) $(x, y, z) = \{a, \{b\}, \{\{c\}\}\}$,

(d) $(x, y, z) = \{\{x\}, \{x, y\}, \{x, y, z\}\}$?

Exercise 2 2 Points

For classes A, B and C, let $R \subseteq A \times B$ and $S \subseteq B \times C$ be binary relations. The *composition* $S \circ R \subseteq A \times C$ of R and S is defined by

$$S \circ R = \{\langle a, c\rangle \mid \text{ there is some } b \in B \text{ with } \langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in S\}.$$

We define the relation id_A by $\{\langle a, a\rangle \mid a \in A\}$. Let $R^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in R\}$. Prove or disprove that $R^{-1} \circ R = \mathrm{id}_A$ holds for all relations $R \subseteq A \times B$.

Exercise 3 3 Points

Let $(A, \leq)$ be an ordering and $X \subseteq A$. An element $a \in A$ is a *lower bound* of X if $a \leq x$ for all $x \in X$. If a is a lower bound of X and $a \geq b$ for all lower bounds b of X then a is an *infimum* of X. An element $a \in A$ is *minimal* if there is no element $c \in A$ with $c \leq a$ and $c \neq a$.

We consider $(B, \subseteq)$ with $B = \{x \subseteq \omega \mid x \text{ is finite or } \omega \backslash x \text{ is finite}\}$. (Formally, a set x is finite if there is a bijection $f : x \to n$ from this set in a natural number $n \in \omega$.)

Is there a subset of B without a minimal element? Construct a subset of B that has a lower bound, but no infimum.

Exercise 4 3 + 3 Points

Let A be a class. A *closure operator* on A is a function $c : \mathcal{P}(A) \to \mathcal{P}(A)$, such that for all $x, y \in \mathcal{P}(A)$ holds:

- $x \subseteq c(x)$,
- $c(c(x)) = c(x)$ und

- $x \subseteq y$ implies $c(x) \subseteq c(y)$.

Let $(A, \leq)$ be a partial ordering. An *upper bound* is defined analogously to the lower bound. We define for sets $X \subseteq A$:

- $U(X) = \{a \in A \,|\, a$ is an upper bound for $x\}$ and
- $L(X) = \{a \in A \,|\, a$ is a lower bound for $x\}$.

Prove or disprove:

(a) $c : X \mapsto L(U(X))$ is a closure operator on A.

(b) Building transitive closure $\mathrm{TC} : X \mapsto \mathrm{TC}(X)$ is a closure operator on A.