Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
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WS 2018/19

3. Übung Logic and Games

Due: Tuesday the 30th October at 12:00 in the lecture or at our chair.

Aufgabe 1 7 Punkte

For each $k \in \mathbb{N}$ the logic FO^k is defined as

$$\mathrm{FO}^k := \{\varphi \in \mathrm{FO} \ : \ \mathrm{width}(\varphi) \leq k\}\,.$$

Prove that the model-checking problem for FO^2 is PTIME-hard by reducing GAME in LOGSPACE to the model-checking problem for FO^2. For this purpose construct for each reachability game

$$\mathcal{G} := (V, V_0, V_1, E)$$

a formula $\varphi_{\mathcal{G}}(x) \in \mathrm{FO}^2(\{V_0, V_1, E\})$ such that

$$\mathcal{G} \models \varphi_{\mathcal{G}}(v) \iff v \in W_0$$

holds for every $v \in V$ and explain why this formula can be computed in $\mathcal{O}(\log(|V|))$.
Hint: You may assume that the players in $\mathcal{G}$ move alternately, meaning that for every edge $(v, w) \in E$ we have $v \in V_0 \iff w \in V_1$.

Aufgabe 2 7 Punkte

We say that in a reachability game $\mathcal{G} := (V, V_0, V_1, E)$ a node v has *finite degree* if vE is a finite set. Consider again the inductive definition for computing the winning regions:

$$W_\sigma^0 := \{v \in V_{1-\sigma} \ : \ vE = \varnothing\}$$

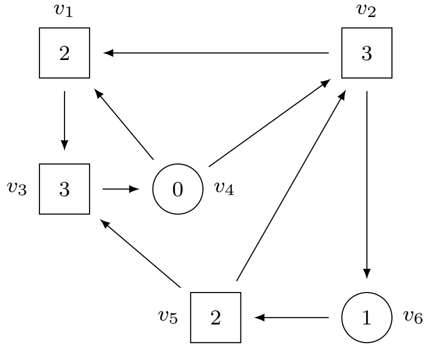
$$W_\sigma^{n+1} := \{v \in V_\sigma \ : \ vE \cap W_\sigma^n \neq \varnothing\} \cup \{v \in V_{1-\sigma} \ : \ vE \subseteq W_\sigma^n\}\,.$$

Prove or disprove that $W_\sigma = \bigcup_{n \in \mathbb{N}} W_\sigma^n$ holds in (possibly) infinitely large reachability games where each node has finite degree.

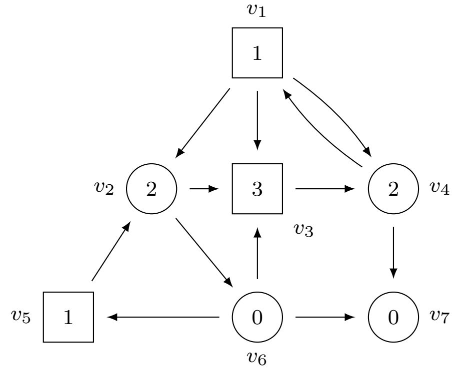
Aufgabe 3 8 Punkte

Compute the winning regions and corresponding winning strategies in the following parity games. Circular nodes belong to player 0 while rectangular nodes belong to player 1. The priorities of the nodes are given by the numbers in the nodes.

a)



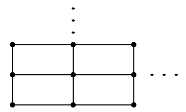
b)



Aufgabe 4 8 Punkte

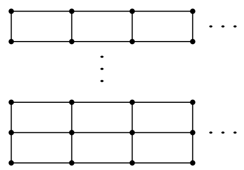
We consider the following variation on the chocolate game (see Assignment 2, Exercise 3) where we change the size of the chocolate bar but not the rules of the game, meaning that again the players alternate, picking a piece of chocolate and removing all pieces to its top right. The player who picks the last piece loses. Show which player wins the following variants of the chocolate game:

(a) For the $(\omega \times \omega)$-chocolate game we consider a 'square-shaped' chocolate bar which is infinite both to the top and to the right. Formally the possible moves can be represented by the grid $\{0, 1, 2, \dots\} \times \{0, 1, 2, \dots\}$.



Hint: Give an explicite winning strategy for one of the players.

(b) For $n \geq 1$ we consider the $(n \times \omega)$-chocolate game where the players alternate in choosing from a 'rectangular' chocolate bar consisting of n rows and infinitely many columns. We can formally view this as a grid $\{0, 1, 2, \dots, n\} \times \{0, 1, 2, \dots\}$.



Hint: The case $n = 2$ is central to the solution of the problem.