Logic and Games
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Contents

Appendix A - Ordinal Numbers

The standard basic notion used in mathematics is the notion of a set, and all mathematical theorems follow from *the axioms of set theory*. The standard set of axioms, which (among others) guarantee the existence of an empty set, an infinite set, and the powerset of any set, and that no set is a member of itself (i.e. $\forall x\ \neg x \in x$) is called the *Zermelo-Fränkel Set Theory ZF*. Furthermore, it is consequence of ZF that every set a contains an $\in$-minimal element $b \in a$, i.e. $b \cap a = \emptyset$. This implies that there are no infinite $\ni$-sequences $x_1 \ni x_2 \ni x_3 \ni \dots$, because otherwise the set $\{x_1, x_2, x_3, \dots\}$ would not contain an $\in$-minimal element. It is standard in mathematics to use ZF extended by *the axiom of choice AC*, which together are called ZFC.

Since everything is a set in mathematics, there is a need to represent numbers as sets. The standard way to do this is to start with the empty set, let $0 = \emptyset$, and proceed by induction, defining $n + 1 = n \cup \{n\}$. Here are the first few numbers in this coding:

- $0 = \emptyset$,
- $1 = \{\emptyset\}$,
- $2 = \{\emptyset, \{\emptyset\}\}$,
- $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = 2 \cup \{2\}$,
- $4 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\} = 3 \cup \{3\}$.

Observe that for each number n (as a set) it holds that

$$m \in n \implies m \subseteq n \text{ for every set } m. \tag{4.1}$$

Sets satisfying property (4.1) are called *transitive* sets, because (4.1) is equivalent to

$$x \in y \in n \implies x \in n \text{ for every set } x, y.$$

For example, the set $a := \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\} \neq 3$ is a transitive set, but a does not occur on our list of natural numbers. Intuitively, the problem is that $\{\{\emptyset\}\} \notin \emptyset$ and $\emptyset \notin \{\{\emptyset\}\}$, so $\in$ is not trichotomous on a. This is why, $\in$ does not constitute a linear order on a. Now, we define a more general class of numbers, the so-called *von Neumann ordinal* numbers.

Definition 4.45. A set α is an *ordinal* if

(1) α is transitive, i.e. $x \in y \in \alpha \implies x \in \alpha$ for every x, y, and

(2) $\in$ is trichotomous on α, i.e. for every $x, y \in \alpha$ either $x = y$ or $x \in y$ or $y \in x$.

$\mathrm{On} := \{\alpha : \alpha \text{ is an ordinal}\}$ is the class of all ordinals.

We are going to prove in Theorem 4.47 that for all ordinal α, β it holds that either

$$a = b \text{ or } a \in b \text{ or } a \ni b.$$

It is even the cases, that the class of ordinal numbers forms a *well-founded* order (w.r.t. $\in$). This means, that $\in$ is a linear order on the class of ordinals and that every non-empty class X of ordinal number contains an $\in$-minimal ordinal $\alpha \in X$, i.e. $\alpha \in \beta$ for every $\beta \in X \setminus \{\alpha\}$. Note, that this also implies that the class On is a proper class, which means that On is not a set itself (otherwise On would satisfy Definition 4.45 and, hence, $\mathrm{On} \in \mathrm{On}$ in contradiction to the ZFC axioms).

It is easy to check that the natural numbers we defined above are ordinal numbers: Indeed, if n is a natural number, then we have that $n = \{0, \dots, n-1\}$ and, consequently, for every $i \in n$ follows that $i = \{0, \dots, i-1\} \subseteq \{0, \dots, i-1, i, \dots n-1\} = n$. Similarly, it is easy to see that for every $m, k \in n$ that either $m = k$ or $m \in k$ or $k \in m$ holds. It is worth mentioning that the relation $\in$ coincides with the usual order $<$ on natural numbers.

Except for natural numbers, are there any other ordinal numbers? In fact, we shall see that there are infinite many ordinals which are

infinitely large. For example, consider ω which is defined by

$$\omega = \bigcup_n n = \bigcup_n \{0, \dots, n-1\} = \{0, 1, 2, 3, \dots\}.$$

ω is the set of all natural numbers, but it is easy to verify that it satisfies Definition 4.45 and, hence, ω is also an ordinal number. But it does not stop here! It is always possible to apply the $+1$ operation, which is defined as

$$\alpha + 1 := \alpha \cup \{\alpha\}.$$

Lemma 4.46. Let α be an ordinal and $\beta \in \alpha$. Then β and $\alpha + 1$ are ordinals as well.

Proof. First, we prove that β is an ordinal. To do this, we need to prove that β satisfies (1) and (2) of Definition 4.45.

(1) For this, let $d \in c \in \beta$. We need to show that $d \in \beta$. Due to $b \in c \in \beta \in \alpha$ and the transitivity of α (Definition 4.45 (1)), it follows that $b, c \in \alpha$. Thus, $\beta, c, d \in \alpha$. By Definition 4.45 (2), we can conclude that $\beta = d$ or $\beta \in d$ or $d \in \beta$ holds.
$\beta = d$ is impossible, because $\beta = d$ would implies that $d \in c \in \beta = d$ and, thus, $c \ni d \ni c \ni \dots$ but due to the ZFC axioms there are no infinite $\ni$-sequences. Similarly, $\beta \in d$ is also wrong since otherwise $d \in c \in \beta \in d$. Therefore, $d \in \beta$ has to be true.

(2) It remains to show that Definition 4.45 (2) is true for β. But this is trivial because, due to Definition 4.45 (1), it is the case that $\beta \subseteq \alpha$ and condition (2) is assumed to be true for α.

Now we demonstrate that $\alpha + 1$ is an ordinal number.

(1) Transitivity of $\alpha + 1$: Let $c \in b \in \alpha + 1$. Our goal is to prove that $c \in \alpha + 1$. Since $\alpha + 1 = \alpha \cup \{\alpha\}$, we can distinguish the following two cases. If $b = \alpha$, then $c \in b \in \alpha$ and, by using the transitivity of α, we can deduce that

$$c \in \alpha \subseteq \alpha \cup \{\alpha\} = \alpha + 1.$$

Otherwise, $b \neq \alpha$ and then $b \in \alpha$ (because $b \in \alpha + 1$). By transitivity of α, we obtain $c \in \alpha \subseteq \alpha + 1$.

(2) Trichotomy: Let $x, y \in \alpha + 1$. We need to prove that $x = y$ or $x \in y$ or $y \in x$ holds. If both $x, y \in \alpha$, then there is nothing to prove. Hence, $x \notin \alpha$ or $y \notin \alpha$. W.l.o.g. we assume that $x \notin \alpha$. Since $x \in \alpha + 1$, it follows that $x = \alpha$. If $y \in \alpha$, then we are done. If, on the other hand, $y \notin \alpha$, then $x = \alpha = y$. Thus, we obtain $x \in y$ or $x = y$.

Q.E.D.

But does it make sense to say that $\omega + 1$ is the *next* ordinal, is there an order on ordinals?

In fact, the ordinal numbers are linearly ordered by $\in$.

Theorem 4.47. For every ordinal α, β either $\alpha = \beta$ or $\alpha \in \beta$ or $\beta \in \alpha$. Furthermore, $\alpha \subseteq \beta$ holds, if and only if $\alpha \in \beta$ or $\alpha = \beta$.

Before can prove this theorem, we need some lemmas first.

Lemma 4.48. If X is a non-empty class of ordinals, then

$$\bigcap X := \{x \ : \ x \in a \text{ for every } a \in X\}$$

is an ordinal.

Proof. Since X is non-empty, there is an ordinal $\alpha \in X$ and, then, $\bigcap X \subseteq \alpha$. Because α is a set, it is possible to prove (by using the ZFC axioms) that $\bigcap X$ is a set. Now it suffices to prove that $\bigcap X$ satisfies the two conditions from Definition 4.45:

(1) Transitivity: Let $a \in b \in \bigcap X$. Then $a \in b \in \gamma$ for all $\gamma \in X$. Since X is a class of ordinals, it follows that $a \in \gamma$ for all $\gamma \in X$ and, finally, $a \in \bigcap X$.

(2) Trichotomy: Let $a, b \in \bigcap X$. Then $a, b \in \alpha$ and, because α is an ordinal, $a \in b$ or $a = b$ or $b \in a$.

Q.E.D.

The transitivity of ordinals allows us to prove that elements of ordinals are subsets. Of course, the converse is not true in general, because not every subset of an ordinal is an element. However, proper

subsets that are ordinals turn out to be elements. As usual we write $\alpha \subset \beta$ as a shorthand for $\alpha \subseteq \beta$ and $\alpha \neq \beta$.

Lemma 4.49. Let α, β be ordinals and $\alpha \subset \beta$. Then $\alpha \in \beta$.

Proof. Towards a contradiction, we assume there are some ordinals $\alpha \subset \beta$ with $\alpha \notin \beta$.

In order to obtain a contradiction, we prove that there is an infinite $\ni$-sequence $\beta_0 \ni \beta_1 \ni \beta_2 \ldots$ of ordinals starting at β such that $\alpha \subset \beta_i$ but $\alpha \notin \beta_i$ for all $i \in \{0, 1, 2, \ldots\}$.

We start with $\beta_0 := \beta$. Now, consider the set

$$\beta_0 \setminus \alpha := \{y \in \beta_0 \ : \ y \notin \alpha\}.$$

We define $\gamma := \bigcap(\beta_0 \setminus \alpha)$. Due to $\alpha \subset \beta = \beta_0$, there is a $\mu \in \beta_0 \setminus \alpha$. As a result, $\beta_0 \setminus \alpha \neq \emptyset$ and γ is an ordinal (by Lemma 4.48).

Claim 4.50. $\alpha \subseteq \gamma$.

Proof. Let $\delta \in \alpha$. We are going to prove that $\delta \in \gamma$.

Since $\alpha \subset \beta_0$ we have $\delta \in \beta_0$. Let $\mu' \in \beta_0 \setminus \alpha$ be picked arbitrarily. As a result $\mu', \delta \in \beta_0$ and, by Definition 4.45 (2), it follows that

$$\mu' = \delta \text{ or } \mu' \in \delta \text{ or } \delta \in \mu'.$$

We observe that $\mu' \neq \delta$, because $\mu' \notin \alpha$ but $\delta \in \alpha$. Furthermore, $\mu' \notin \delta$, because otherwise $\mu' \in \delta \in \alpha$ and (since α is an ordinal) $\mu' \in \alpha$ but $\mu' \notin \alpha$.

Therefore, it must be the case that $\delta \in \mu'$. $\mu' \in \beta_0 \setminus \alpha$ was chosen arbitrarily, so $\delta \in \bigcap(\beta_0 \setminus \alpha) = \gamma$. Q.E.D.

Now we have

$$\alpha \subseteq \gamma = \bigcap \beta_0 \setminus \alpha.$$

Recall that $\mu \in \beta_0 \setminus \alpha$ and, therefore, $\gamma = \bigcap(\beta_0 \setminus \alpha) \subseteq \mu$. Together with $\alpha \subseteq \gamma$ this leads to $\alpha \subseteq \mu$. Since $\mu \in \beta_0$ and $\alpha \notin \beta_0$, it follows that $\alpha \subset \mu$. Furthermore, $\alpha \notin \mu$, because otherwise $\alpha \in \mu \in \beta_0$ and then $\alpha \in \beta_0$ (because β_0 is an ordinal) in contradiction to $\alpha \notin \beta_0$.

All in all, we managed to prove that $\mu \in \beta_0$ is an ordinal (due to Lemma 4.46) with $\alpha \subset \mu$ but $\alpha \notin \mu$. Hence, we can set $\beta_1 := \mu$.

By repetition, we can construct the desired sequence $\beta_0 \ni \beta_1 \ni \beta_2 \ni \ldots$, but this contradicts the ZFC axioms!

Q.E.D.

Now we have all the tools we need to finally prove Theorem 4.47.

Proof (of Theorem 4.47). First we prove that $\alpha \subseteq \beta \Longleftrightarrow \alpha = \beta \vee \alpha \in \beta$.

The direction "$\Longleftarrow$" follows intermediately from Definition 4.45 (1), while "$\Longrightarrow$" is Lemma 4.49.

Now we demonstrate that $\in$ is a linear order on the class of ordinal numbers. Towards a contradiction, assume that there are ordinals α, β that are incomparable w.r.t. $\in$, i.e., we have

$$\alpha \neq \beta \text{ and } \alpha \notin \beta \text{ and } \beta \notin \alpha. \tag{4.2}$$

Consider $\alpha \cap \beta$. By Lemma 4.48, $\alpha \cap \beta$ is an ordinal. Furthermore, $\alpha \cap \beta \subseteq \alpha$ and $\alpha \cap \beta \subseteq \beta$. If $\alpha = \alpha \cap \beta$, then $\alpha \subseteq \beta$ and by Lemma 4.49 either $\alpha = \beta$ or $\alpha \in \beta$ in contradiction to (4.2). Thus, $\alpha \neq \alpha \cap \beta$ and, similarly, $\beta \neq \alpha \cap \beta$.

But then, $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \subset \beta$, which implies that $\alpha \cap \beta \in \alpha$ and $\alpha \cap \beta \in \beta$, which leads to $\alpha \cap \beta \in \alpha \cap \beta$, but due to the ZFC axioms this is not possible! Contradiction!

So, $\in$ is in fact a linear order on the class of ordinal numbers. Q.E.D.

Recall that On is the class of all ordinals. Theorem 4.47 tells us that $\in$ is a linear order on On. More general, $\in$ is a *well-founded order* on On. An order $(A, <)$ is a *well-founded order*, if

(1) $(A, <)$ is a linear order and
(2) for every non-empty set $X \subseteq A$ there is a $<$-minimal element $x \in X$, i.e., $x < y$ for every $y \in X$.

For example, $(\mathbb{N}, <)$ is a well-founded order but $(\mathbb{Z}, <)$ or $(\mathbb{Q}_{\geq 0}, <)$ are not well-founded orders.

It is not difficult to see that ordinal numbers are well-founded orders (w.r.t. $\in$). Indeed, if $X \subseteq$ On is a non-empty class of ordinals, then $\gamma := \bigcap X$ is an ordinal (by Lemma 4.48) and $\gamma \subseteq x$ for all $x \in X$.

It remains to prove that $\gamma \in X$: Otherwise $\gamma \notin X$ and then $\gamma \subset x$ for all $x \in X$. This leads to $\gamma \in x$ for all $x \in X$ (by Theorem 4.47). Thus, $\{\gamma\} \subseteq x$ and, as a consequence, $\gamma + 1 \subseteq x$ for all $x \in X$. But then $\gamma + 1 \subseteq \bigcap X = \gamma \implies \gamma \in \gamma$ which violates the ZFC axioms! Hence, $\bigcap X = \gamma \in X$.

Now we turn our attention towards the construction of bigger ordinals. For this, we need the following lemma which states that ordinal numbers are closed under unions.

Lemma 4.51. Let x be a set of ordinals, i.e., every $\alpha \in x$ is an ordinal. Then

$$\bigcup x := \{\beta \ : \ \beta \in \alpha \text{ for some } \alpha \in x\}$$

is an ordinal number.

Proof. Using the ZFC axioms, it is possible to prove that $\bigcup x$ is a set. Hence, it remains to show that (1) and (2) of Definition 4.45 are satisfied.

(1) Transitivity of $\bigcup x$: If $a \in b \in \bigcup x$, then there is a $c \in x$ such that $a \in b \in c$ and, by transitivity, $a \in c$ which implies that $a \in \bigcup x$.
(2) Trichotomy: If $a, b \in \bigcup x$. Then there are some $c, d \in x$ such that $a \in c$ and $b \in d$. Applying Lemma 4.46 yields that a, b are ordinals and, by Theorem 4.47, either $a = b$ or $a \in b$ or $b \in a$.

Q.E.D.

$\omega := \bigcup_n n$, the union of all natural numbers, is again an ordinal number. To prove this, we observe that $\omega = \bigcup \omega$ and use Lemma 4.51 (that ω is a set is a consequence of the axiom of infinity).

What is the next ordinal number after ω? We can again apply the $+1$ operation in the same way as for natural numbers, so

$$\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, \{0, 1, 2, \ldots\}\}.$$

Of course it is now possible to construction ordinals like $\omega + 2 := (\omega + 1) + 1, \omega + 3, \ldots$ and then we can build the union

$$\omega + \omega = \bigcup_{i \in \omega} \omega + i = \{0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots\},$$

which is an ordinal because of Lemma 4.51. The fact that $\omega \cup \{\omega + i \ : \ i \in \omega\}$ is a set can be proven by using the axiom of replacement.

To get an intuition on how ordinals look like, consider the following examples of infinite ordinals: $\omega + 1$, $\omega + \omega = 2\omega$, $3\omega, \ldots, \omega \cdot \omega = \omega^2$, $\omega^3, \ldots, \omega^\omega$.

For some ordinals α it is the case that $\alpha = \beta + 1$ for some β. However, it is not possible to find an ordinal γ such that $\gamma + 1 = \omega$ holds (Why?).

Definition 4.52. Let $\alpha \neq 0$ be an ordinal. If $\beta + 1 \in \alpha$ for every $\beta \in \alpha$, then we call α an *limes* ordinal.

It is easy to see that λ is an limes ordinal, if and only if $\lambda \neq 0$ and $\bigcup \lambda = \lambda$.

Ordinals that are not limes ordinals are called successor ordinals because of the following theorem.

Theorem 4.53. Let $\alpha \neq 0$ be an ordinal that is not an limes ordinal. Then there is an ordinal β such that $\beta + 1 = \alpha$.

Proof. By Definition 4.52 there is a $\beta \in \alpha$ such that $\beta + 1 \notin \alpha$. By Theorem 4.47, either $\beta + 1 = \alpha$ or $\beta + 1 \ni \alpha$.

So, we only need to show that $\beta + 1 \not\ni \alpha$ holds. Otherwise $\alpha \in \beta + 1 = \beta \cup \{\beta\}$. Clearly, $\alpha \notin \{\beta\}$ because $\alpha = \beta \in \alpha$ would violate the ZFC axioms. But then $\alpha \in \beta \in \alpha$ which contradicts the ZFC axioms as well. Hence $\beta + 1 \ni \alpha$ is impossible which leads to $\beta + 1 = \alpha$. Q.E.D.

Ordinals are intimately connected to well-orders. In fact any well-ordering $(A, <)$ is isomorphic to some $(\alpha, \in)$ where α is an ordinal. For example, $(\mathbb{N}, <)$ is isomorphic to $(\omega, \in)$ and $\omega + \omega$ represents $(\{0,1\} \times \mathbb{N}, <_{\text{lex}})$ where $<_{\text{lex}}$ is the lexicographical order.

The well-ordering of ordinals allows to define and prove the principle of *transfinite induction*. This principle states that On, the class of *all ordinals*, is generated from $\emptyset$ by taking the successor $(+1)$ and the union on limit steps, as shown on the examples before.

The principle of *transfinite induction* allows us to define sets X_α where α is an ordinal number. Since On is a well-order, we only need to

describe how X_α is constructed under the assumption that X_β is already defined for every $\beta \in \alpha$.

For example, it is possible to define (via transfinite induction) the winning region of player 0 in a reachability game (V, V_0, V_1, E). To do this, we define sets W^0_α for every ordinal number α:

$$\begin{aligned} W^0_0 &:= \emptyset, \\ W^0_{\alpha+1} &:= \left\{ x \in V_0 \ : \ xE \cap W^0_\alpha \neq \emptyset \right\} \cup \left\{ x \in V_1 \ : \ xE \subseteq W^0_\alpha \right\}, \\ W^0_\lambda &:= \bigcup_{\beta \in \lambda} W^0_\beta \text{ for limes ordinals } \lambda. \end{aligned}$$

Now it is easy to verify that $\bigcup_{\alpha \in \text{On}} W^0_\alpha$ is the winning region of Player 0.

4.9 Cardinal Numbers

Besides ordinals, we sometimes need cardinal numbers which are special ordinal number that can be used to measure the size of sets. We say that two sets x, y have the same cardinality, if there is a bijection between x and y.

Definition 4.54. An ordinal κ is a *cardinal number*, if for every $\alpha \in \kappa$ there is no bijection between κ and α. Furthermore, we say that a cardinal number κ is *the* cardinality of a set x, if there is a bijection between x and κ. In this case we let $|x| := \kappa$.

$\text{Cn} := \{\kappa \in \text{On} \ : \ \kappa \text{ is a cardinal number}\}$ is the class of all cardinal numbers.

But is it guaranteed that we really find a cardinal number for every possible set out there? The next theorem answers this question.

Theorem 4.55. For every set x there is a cardinal number $|x|$.

Proof. Consider the class Y of ordinals, which is given by

$$\begin{aligned} Y &:= \{\alpha \in \text{On} \ : \ \text{there is a bijection } f : x \to \alpha\} \\ &= \{\alpha \in \text{On} \ : \ \text{there is a bijection } f : \alpha \to x\}. \end{aligned}$$

If Y is non-empty, then $|x| := \bigcap Y \in Y$ is the desired cardinal number.

Now we prove that $X \neq \emptyset$ is indeed the case. By the axiom of choice, there is a choice function g for x, i.e., for every $y \subseteq x$ with $y \neq \emptyset$ we have $g(y) \in y$.

Using transfinite induction, we define for every ordinal α an object x_α by

$$x_\alpha := \begin{cases} g(y_\alpha) & \text{if } y_\alpha := x \setminus \{x_\beta \ : \ \beta \in \alpha\} \neq \emptyset \\ x & \text{if } y_\alpha = \emptyset \end{cases}$$

It is easy to see that for every $x_\alpha \neq x$ we have that x_α is an element of x but $x_\alpha \neq x_\beta$ for every $\beta \in \alpha$.

If $x_\alpha = x$ holds for some ordinal α, then there is a minimal ordinal $\alpha' \subseteq \alpha$ such that $x_{\alpha'} = x$ and, by definition of $x_{\alpha'}$, this means that $x = \{x_\beta \ : \ \beta \in \alpha'\}$. Furthermore, the function $f : \alpha' \to x$, $\beta \mapsto x_\beta$ is a bijection between x and α'. This implies that $\alpha' \in Y$.

So, it only remains to prove that $x_\alpha = x$ for some ordinal α. Towards a contradiction, we assume that $x_\alpha \neq x$ for every ordinal α. Then every $x_\alpha \in x$ and, therefore, the mapping $f : \mathrm{On} \to x' := \{x_\alpha \ : \ \alpha \in \mathrm{On}\}$, $\alpha \mapsto x_\alpha$ is a bijection between On and x'. Since x is a set, $x' \subseteq x$ is a set as well. Therefore, by the axiom of replacement,

$$f^{-1}[x'] := \left\{f^{-1}(y) \ : \ y \in x'\right\} = \mathrm{On}$$

is a set. As a result, On satisfies Definition 4.45 and, consequently, $\mathrm{On} \in \mathrm{On}$ which violates the ZFC axioms! Contradiction!

Q.E.D.

It is worth mentioning that the enumeration $(x_\alpha)_{\alpha \in |x|}$ induces a well-ordering $<$ on x by

$$x_\alpha < x_\beta \iff \alpha \in \beta.$$

Corollary 4.56 (Well-ordering theorem)**.** Every set x can be well-ordered, i.e., there is a well-order $<$ on x.

Every finite ordinal number is a cardinal number but there are also

infinite cardinal numbers. For example, $\beth_0 := \omega$ is the smallest infinite cardinal number and, by using the power set, we can construct strictly larger cardinal numbers:

$$\beth_{\alpha+1} := 2^{\beth_\alpha} := |\mathcal{P}(\beth_\alpha)|,$$
$$\beth_\lambda := \bigcup_{\beta \in \lambda} \beth_\beta \text{ for limes ordinals } \lambda.$$

Please observe that $\beth_1 = |\mathcal{P}(\omega)| = |\mathbb{R}|$.

Whether there exists cardinal numbers between $\beth_0$ and $\beth_1$ is called the continuum hypothesis (CH) which has turned out to be independent of ZFC, i.e., neither (CH) nor $\neg$(CH) are consequences of ZFC.