Logic and Games

A Tutorial

Erich Grädel

Outline

Part I: Model Checking Games

- Model checking games for modal logic and first-order logic
- The strategy problem for finite games
- Fragments of first-order logics with efficient model checking
- Fixed point logics: LFP and modal μ-calculus
- Parity games
- Model checking games for fixed point logics

Model checking via games

The model checking problem for a logic L

Given: structure $\mathfrak{A}$

formula $\psi \in L$

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- Falsifier (also called Player 1, or Alter), and
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$\Longrightarrow$ Model checking via construction of winning strategies

ML: propositional modal logic

Syntax: $\psi ::= P_i \mid \neg P_i \mid \psi \wedge \psi \mid \psi \vee \psi \mid \langle a \rangle \psi \mid [a]\psi$

Example: $P_1 \vee \langle a \rangle (P_2 \wedge [b]P_1)$

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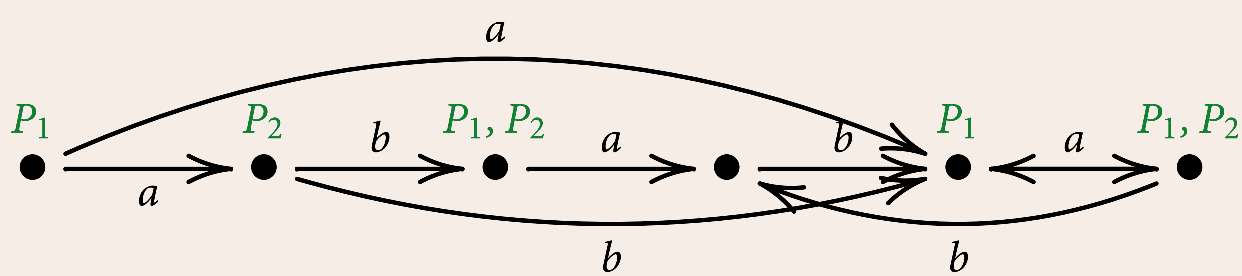
Semantics: transition systems = Kripke structures = labeled graphs

$$\mathcal{K} = (V , (E_a)_{a \in A} , (P_i)_{i \in I})$$

V	$(E_a)_{a \in A}$	$(P_i)_{i \in I}$
states elements	actions binary relations	atomic propositions unary relations



$$[\![\psi]\!]^{\mathcal{K}} = \{v : \mathcal{K}, v \models \psi\} = \{v : \psi \text{ holds at state } v \text{ in } \mathcal{K}\}$$

$$\mathcal{K}, v \models \begin{matrix} \langle a \rangle \psi \\ [a]\psi \end{matrix} \quad :\Longleftrightarrow \quad \mathcal{K}, w \models \psi \text{ for } \begin{matrix} \text{some} \\ \text{all} \end{matrix} \; w \text{ with } (v, w) \in E_a$$

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From position (φ, v), Verifier wants to show that $\mathcal{K}, v \models \varphi$, while Falsifier wants to prove that $\mathcal{K}, v \not\models \varphi$.

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Verifier moves:

$$(\varphi \vee \vartheta, v) \longrightarrow (\varphi, v), \quad (\varphi \vee \vartheta, v) \longrightarrow (\vartheta, v)$$

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If $\mathcal{K}, v \models P_i$ then Verifier has won at (P_i, v), otherwise Falsifier has won.

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Lemma. $\mathcal{K}, v \models \varphi \iff$ Verifier has winning strategy from (φ, v).

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- How complicated are the resulting model checking games?
  - are all plays necessarily finite?
  - if not, what are the winning conditions for infinite plays?
  - structural complexity of the game graphs?
  - do the players always have perfect information?
- How big are the resulting game graphs?

  how does the size of the game depend on different parameters of the input structure and the formula?

Logics and games

First-order logic (FO) or modal logic (ML): Model checking games have

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winning regions computable in linear time wrt. size of game graph

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Fixed-point logics (LFP or L_μ): Model checking games are parity games

- admit infinite plays
- parity winning condition

Open problem: Are winning regions and winning strategies of parity games computable in polynomial time?

Finite games: basic definitions

Two-player games with perfect information and positional winning condition, given by game graph (also called arena)

$$\mathcal{G} = (V, E), \qquad V = V_0 \cup V_1$$

- Player 0 (Ego) moves from positions $v \in V_0$,
  Player 1 (Alter) moves from $v \in V_1$,
- moves are along edges
  a play is a finite or infinite sequence $\pi = v_0 v_1 v_2 \cdots$ with $(v_i, v_{i+1}) \in E$
- winning condition: move or lose!
  Player σ wins at position v if $v \in V_{1-\sigma}$ and $vE = \varnothing$
  Note: this is a purely positional winning condition applying to finite plays only (infinite plays are draws)

Winning strategies and winning regions

Strategy for Player σ: $f : \{v \in V_\sigma : vE \neq \varnothing\} \to V$ with $(v, f(v)) \in E$.

f is **winning from position v** if Player σ wins all plays that start at v and are consistent with f.

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Algorithmic problems: Given a game $\mathcal{G}$

- compute winning regions W_0, W_1
- compute winning strategies

Associated decision problem:

$$\textsc{Game} := \{(\mathcal{G}, v) : \text{Player 0 has winning strategy for } \mathcal{G} \text{ from position } v\}$$

Algorithms for finite games

Theorem

GAME is PTIME-complete and solvable in time $O(|V| + |E|)$.

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- $W_\sigma^{n+1} = \{v \in V_\sigma : vE \cap W_\sigma^n \neq \varnothing\} \cup \{v \in V_{1-\sigma} : vE \subseteq W_\sigma^n\}$
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  (positions with winning strategy in $\leq n + 1$ moves for Player σ)

until $W_\sigma^{n+1} = W_\sigma^n$ (this happens for $n \leq |V|$).

A linear time algorithm for Game

Input: A game $\mathcal{G} = (V, V_0, V_1, E)$

forall $v \in V$ **let** (∗ 1: initialisation ∗)

  $\text{win}[v] := \perp, \quad P[v] := \{u : (u, v) \in E\}, \quad n[v] := |vE|$

forall $\sigma \in \{0, 1\}, v \in V_\sigma$ (∗ 2: calculate win ∗)

  if $n[v] = 0$ **then** **Propagate**$(v, 1 - \sigma)$

return win **end**

procedure **Propagate**(v, σ)

  if $\text{win}[v] \neq \perp$ **then** **return**

  $\text{win}[v] := \sigma$ (∗ 3: mark v as winning for Player σ ∗)

  forall $u \in P[v]$ **do** (∗ 4: propagate change to predecessors ∗)

    $n[u] := n[u] - 1$

    if $u \in V_\sigma$ or $n[u] = 0$ **then** **Propagate**(u, σ)

  enddo

GAME and the satisfiability of propositional Horn formulae

Propositional Horn formulae: conjunctions of clauses of form

$$X \leftarrow X_1 \wedge \cdots \wedge X_n \qquad \text{and} \qquad 0 \leftarrow X_1 \wedge \cdots \wedge X_n$$

Theorem. SAT-HORN is PTIME-complete and solvable in linear time.

(actually, GAME and SAT-HORN are essentially the same problem)

Game and the satisfiability of propositional Horn formulae

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1) Game $\leq_{\text{log-lin}}$ Sat-Horn:

For $\mathcal{G} = (V_0 \cup V_1, E)$ construct Horn formula ψ with clauses

$$u \leftarrow v \qquad \text{for all } u \in V_0 \text{ and } (u, v) \in E$$

$$u \leftarrow v_1 \wedge \cdots \wedge v_m \qquad \text{for all } u \in V_1,\ uE = \{v_1, \ldots, v_m\}$$

The minimal model of ψ is precisely the winning region of Player 0.

$$(\mathcal{G}, v) \in \text{Game} \iff \psi_{\mathcal{G}} \wedge (0 \leftarrow v) \text{ is unsatisfiable}$$

2) Sat-Horn $\leq_{\text{log-lin}}$ Game:

Define game $\mathcal{G}_\psi$ for Horn formula $\psi(X_1, \dots, X_n) = \bigwedge_{i \in I} C_i$

Positions: $\{0\} \cup \{X_1, \dots, X_n\} \cup \{C_i : i \in I\}$

Moves of Player 0: $X \longrightarrow C$ for $X = \text{head}(C)$

Moves of Player 1: $C \longrightarrow X$ for $X \in \text{body}(C)$

Note: Player 0 wins iff play reaches clause C with $\text{body}(C) = \varnothing$

Player 0 has winning strategy from position $X \iff \psi \models X$

Hence,

$$\text{Player 0 wins from position 0} \iff \psi \text{ unsatisfiable.}$$

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nondeterministic algorithms, with states divided into
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Acceptance condition: game with Players $\exists$ and $\forall$, played on computation graph $C(M, x)$ of M on input x

Positions: configurations of M

Moves: $C \longrightarrow C'$ for C' successor configuration of C

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M accepts x $\quad :\Longleftrightarrow \quad$ Player $\exists$ has winning strategy for game on $C(M, x)$

Alternating versus deterministic complexity classes

Alternating time $\equiv$ deterministic space

Alternating space $\equiv$ exponential deterministic time

$$
\begin{array}{ccccccccc}
\textsc{Logspace} & \subseteq & \textsc{Ptime} & \subseteq & \textsc{Pspace} & \subseteq & \textsc{Exptime} & \subseteq & \textsc{Expspace} \\
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Alternating logspace algorithm for GAME: Play the game !

Evaluation game for FO

FO: $\psi ::= R_i\bar{x} \mid \neg R_i\bar{x} \mid x = y \mid x \neq y \mid \psi \wedge \psi \mid \psi \vee \psi \mid \exists x\psi \mid \forall x\psi$

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Winning condition: φ atomic / negated atomic

Verifier / Falsifier wins at $\varphi(\bar{a}) \iff \mathfrak{A} \models / \not\models \varphi(\bar{a})$

Complexity of FO model checking

To decide whether $\mathfrak{A} \models \psi$, construct the game $\mathcal{G}(\mathfrak{A}, \psi)$ and check whether Verifier has winning strategy from initial position ψ.

Efficient implementation: on-the-fly construction of game while solving it

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Size of game graph can be exponential: $|\mathcal{G}(\mathfrak{A}, \psi)| \leq |\psi| \cdot |A|^{\mathrm{width}(\psi)}$

width(ψ): maximal number of free variables in subformulae

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alternating time: $O(|\psi| + \text{qd}(\psi) \log |A|)$ qd(ψ): quantifier-depth of ψ

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deterministic time: $O(|\psi| \cdot |A|^{\mathrm{width}(\psi)})$

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Fragments of FO with model checking complexity $O(|\psi| \cdot \|\mathfrak{A}\|)$:
— **ML** : propositional modal logic
— **FO^2** : formulae of width two
— **GF** : the guarded fragment of first-order logic

The guarded fragment of first-order logic (GF)

Fragment of first-order logic with only guarded quantification

$$\exists \bar{y}(\alpha(\bar{x}, \bar{y}) \land \varphi(\bar{x}, \bar{y})) \qquad \forall \bar{y}(\alpha(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}, \bar{y}))$$

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with guards α : atomic formulae containing all free variables of φ

Generalizes modal quantification: $\mathrm{ML} \subseteq \mathrm{GF} \subseteq \mathrm{FO}$

$$\langle a \rangle \varphi \equiv \exists y(E_a xy \wedge \varphi(y)) \qquad [a]\varphi \equiv \forall y(E_a xy \rightarrow \varphi(y))$$

Guarded logics generalize and, to some extent, explain the good algorithmic and model-theoretic properties of modal logics.

Model-theoretic and algorithmic properties of GF

- Satisfiability for GF is decidable (Andréka, van Benthem, Németi)
- GF has finite model property (Grädel)
- GF has (generalized) tree model property:
  every satisfiable formula has model of small tree width (Grädel)
- Extension by fixed points remains decidable (Grädel, Walukiewicz)
- . . .
- Guarded logics have small model checking games:
  $\|\mathcal{G}(\mathfrak{A}, \psi)\| = O(|\psi| \cdot \|\mathfrak{A}\|)$
  $\Longrightarrow$ efficient game-based model checking algorithms

Advantages of game based approach to model checking

- intuitive top-down definition of semantics
  (very effective for teaching logic)
- versatile and general methodology,
  can be adapted to many logical formalisms
- isolates the real combinatorial difficulties of an evaluation problem,
  abstracts from syntactic details.
- if you understand games, you understand alternating algorithms
- closely related to automata based methods
- algorithms and complexity results for many logic problems follow
  from results on games

Logics and games

First-order logic (FO) or **modal logic (ML):** Model checking games have

- only finite plays
- positional winning condition

Winning regions computable in linear time wrt. size of game graph

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$\Longrightarrow$ we have to consider the theory of infinite games

Parity games

$G = (V, E, \Omega), \qquad V = V_0 \cup V_1, \qquad \Omega : V \longrightarrow \mathbb{N}$

Player 0 moves at positions $v \in V_0$, Player 1 at positions $v \in V_1$

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Winning regions W_0, W_1:

$$W_i = \{v \in V : \text{Player } i \text{ has winning strategy from position } v\}$$

Least fixed point logics

Extend a basic logical formalism by least and greatest fixed points

FO	(first-order logic)	$\longrightarrow$	LFP	(least fixed point logic)
ML	(modal logic)	$\longrightarrow$	L_μ	(modal μ-calculus)
GF	(guarded fragment)	$\longrightarrow$	μGF	(guarded fixed point logic)
conjunctive queries		$\longrightarrow$	Datalog / Stratified Datalog	

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Idea: Capture recursion.

For any definable monotone relational operator

$$F_\varphi : T \mapsto \{\bar{x} : \varphi(T, \bar{x})\}$$

make also the least and the greatest fixed point of F_φ definable:

$$[\mathbf{lfp}\, T\bar{x} \,.\, \varphi(T, \bar{x})](\bar{z}) \qquad [\mathbf{gfp}\, T\bar{x} \,.\, \varphi(T, \bar{x})](\bar{z})$$

$$\mu X \,.\, \varphi \qquad \nu X \,.\, \varphi$$

Greatest fixed points (in LFP)

$[\mathbf{gfp}\, T\overline{x}\,.\,\varphi(T,\overline{x})](\overline{a})$: $\overline{a}$ contained in greatest T with $T = \{\overline{x} : \varphi(T,\overline{x})\}$

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Inductive construction of the greatest fixed point on a structure $\mathfrak{A}$:

$$T^0 := A^k \quad \text{(all tuples of appropriate arity)}$$

$$T^{\alpha+1} := F_\varphi(T^\alpha)$$

$$T^\lambda := \bigcap_{\alpha<\lambda} T^\alpha \quad (\lambda \text{ limit ordinal})$$

$\Longrightarrow$ decreasing sequence of stages $(T^\alpha \supseteq T^{\alpha+1})$,
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Fact: $T^\infty = \mathbf{gfp}(F_\varphi)$ (Knaster, Tarski)

Example: Bisimulation

$\mathcal{K} = (V, E, P_1, \dots, P_m)$ transition system

Bisimilarity on $\mathcal{K}$ is the greatest equivalence relation $Z \subseteq V \times V$ such that:

if $(u, v) \in Z$ then

– u and v have the same atomic properties

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Thus, bisimilarity is the greatest fixed point of the refinement operator

$$Z \quad \mapsto \quad \{(u, v) : \mathcal{K} \models \varphi(Z, u, v)\} \quad \text{where}$$

$$\varphi := \bigwedge_{i \leq m} P_i u \leftrightarrow P_i v \wedge$$
$$\forall x\,(Eux \rightarrow \exists y(Evy \wedge Zxy)) \;\wedge\; \forall y(Evy \rightarrow \exists x(Eux \wedge Zxy))$$

u and v are bisimilar in $\mathcal{K} \quad \Longleftrightarrow \quad \mathcal{K} \models [\mathbf{gfp}\, Zuv \,.\, \varphi](u, v)$

Least fixed point logic LFP

Syntax. LFP extends FO by fixed point rule:

- For every formula $\psi(T, x_1 \dots x_k) \in \text{LFP}[\tau \cup \{T\}]$,
  T k-ary relation variable, occuring only positive in ψ,
  build formulae $[\mathbf{lfp}\ T\overline{x}\,.\,\psi](\overline{x})$ and $[\mathbf{gfp}\ T\overline{x}\,.\,\psi](\overline{x})$

Semantics. On τ-structure $\mathfrak{A}$, $\psi(T, \overline{x})$ defines monotone operator

$$\begin{aligned} \psi^{\mathfrak{A}} : \mathcal{P}(A^k) &\longrightarrow \mathcal{P}(A^k) \\ T &\longmapsto \{\overline{a} : (\mathfrak{A}, T) \models \psi(T, \overline{a})\} \end{aligned}$$

- $\mathfrak{A} \models [\mathbf{lfp}\ T\overline{x}\,.\,\psi(T, \overline{x})](\overline{a}) \quad :\Longleftrightarrow \overline{a} \in \mathbf{lfp}(\psi^{\mathfrak{A}})$
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Modal μ-calculus L_μ

Syntax. L_μ extends ML by fixed point rule:

- With every formula $\psi(X)$, where X occurs only positive in ψ L_μ also contains the formulae $\mu X.\psi$ and $\nu X.\psi$

Semantics. On transition system $\mathcal{K}$, $\psi(X)$ defines operator

$$\psi^{\mathcal{K}} : X \longmapsto [\![\psi]\!]^{(\mathcal{K},X)} = \{v : (\mathcal{K}, X), v \models \psi\}$$

$\psi^{\mathcal{K}}$ is monotone, and therefore has a least and a greatest fixed point

$$\mathbf{lfp}(\psi^{\mathcal{K}}) = \bigcap\{X : \psi^{\mathcal{K}}(X) \subseteq X\}, \qquad \mathbf{gfp}(\psi^{\mathcal{K}}) = \bigcup\{X : X \subseteq \psi^{\mathcal{K}}(X)\}$$

- $[\![\mu X.\psi]\!]^{\mathcal{K}} := \mathbf{lfp}(\psi^{\mathcal{K}}), \qquad [\![\nu X.\psi]\!]^{\mathcal{K}} := \mathbf{gfp}(\psi^{\mathcal{K}})$

Inductive generation of fixed points

$\psi(X)$ defines operator $\psi^{\mathcal{K}} : X \mapsto \{v : (\mathcal{K}, X), v \models \psi\}$

$$X^0 := \varnothing \qquad\qquad Y^0 := V$$

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$$X^\lambda := \bigcup_{\alpha<\lambda} X^\alpha \qquad (\lambda \text{ limit ordinal}) \qquad Y^\lambda := \bigcap_{\alpha<\lambda} Y^\alpha$$

$X^0 \subseteq \cdots \subseteq X^\alpha \subseteq X^{\alpha+1} \subseteq \cdots$ $\qquad$ $Y^0 \supseteq \cdots \supseteq Y^\alpha \supseteq Y^{\alpha+1} \supseteq \cdots$

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These inductive sequences reach fixed points

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for some α, β, with $|\alpha|, |\beta| \leq |V|$

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L_μ: Examples

- $\mathcal{K}, w \models \nu X . \langle a \rangle X \iff$ there is an infinite a-path from w in $\mathcal{K}$

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- Logics of knowledge: multi-modal propositional logics where $[a]\varphi$ stands for "agent a knows φ"

  add common knowledge:

  everybody knows φ, and everybody knows that everybody knows φ, and everybody knows that everybody knows that everybody knows ...

  expressible as a greatest fixed point: $C\varphi \;\equiv\; \nu X \,.\, (\varphi \wedge \bigwedge_a [a]X)$

Finite games and LFP

- GAME is definable in LFP / L_μ

Player 0 has winning strategy for game $\mathcal{G}$ from position v

$$\Longleftrightarrow$$

$$\mathcal{G} = (V, V_0, V_1, E) \models [\mathbf{lfp}\, Wx\,.\,(V_0x \wedge \exists y(Exy \wedge Wy)) \\ \vee (V_1x \wedge \forall y(Exy \rightarrow Wy)](v)$$

$$\Longleftrightarrow$$

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- $\textsc{Game}$ is complete for LFP
(via quantifier-free reductions on finite structures)

Importance of the modal μ-calculus

- encompasses most of the popular logics used in hardware verification: LTL, CTL, CTL*, PDL,. . . , and also many logics from other fields: game logic, description logics, etc.

- reasonably good algorithmic properties:
  - satisfiability problem decidable (Exptime-complete)
  - efficient model checking for practically important fragments of L_μ
  - automata-based algorithms

- nice model-theoretic properties:
  - finite model property
  - tree model property

- L_μ is the bisimulation-invariant fragment of MSO

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Disadvantage: Fixed-point formulae are hard to read

Model checking games for LFP and L_μ

LFP-game: extend FO-game by moves

$$\begin{aligned} [\mathbf{fp}\, T\overline{x}\,.\,\varphi](\overline{a}) &\longrightarrow \varphi(T, \overline{a}) \qquad (\mathbf{fp} \in \{\mathbf{lfp}, \mathbf{gfp}\}) \\ T\overline{b} &\longrightarrow \varphi(T, \overline{b}) \end{aligned}$$

Similarly for L_μ: extend ML-game by moves

$$\begin{aligned} (\lambda X\,.\,\varphi, u) &\longrightarrow (\varphi, u) \qquad (\lambda \in \{\mu, \nu\}) \\ (X, w) &\longrightarrow (\varphi, w) \end{aligned}$$

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Infinite plays possible

need winning condition for infinite plays

Model checking game for L_μ: Example

$$\psi = \mu X.P \lor \Box X \qquad \equiv \qquad [\mathbf{lfp}\, Tx\,.\, Px \lor \forall y(Exy \rightarrow Ty)](x)$$

$$\mathcal{K}: \qquad \begin{tikzcd} \underset{a}{\bullet} \arrow[r] \arrow[rrr, bend left] & \underset{b}{\bullet} \arrow[r] & \underset{c}{\bullet} \arrow[r] & \overset{P}{\underset{d}{\bullet}} \end{tikzcd} \qquad \text{from node } a$$

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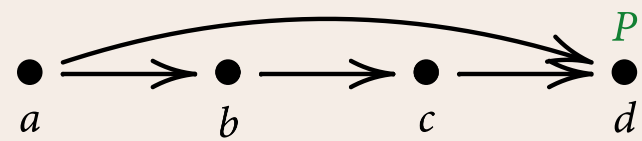

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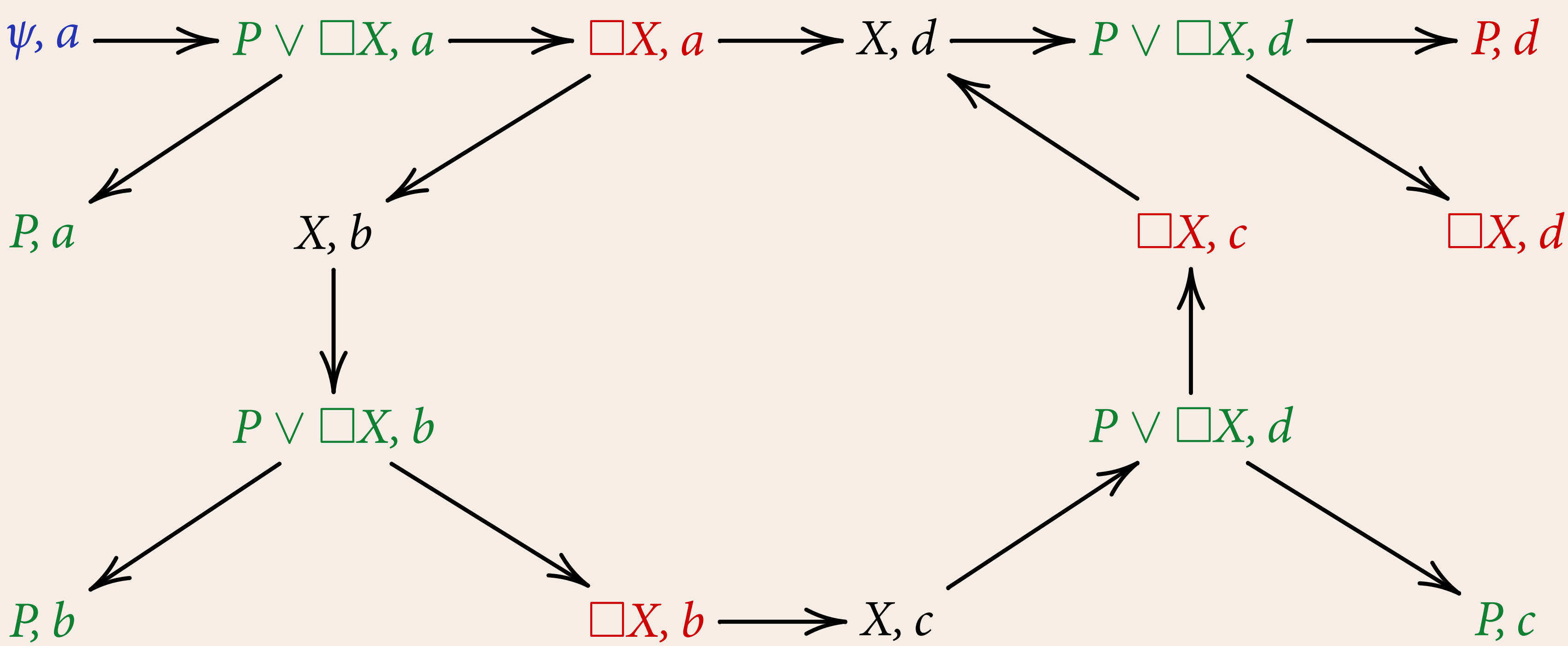

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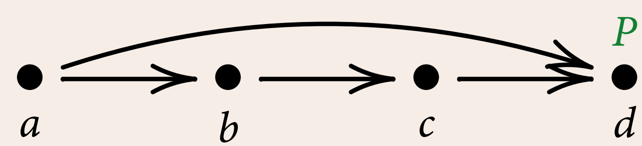

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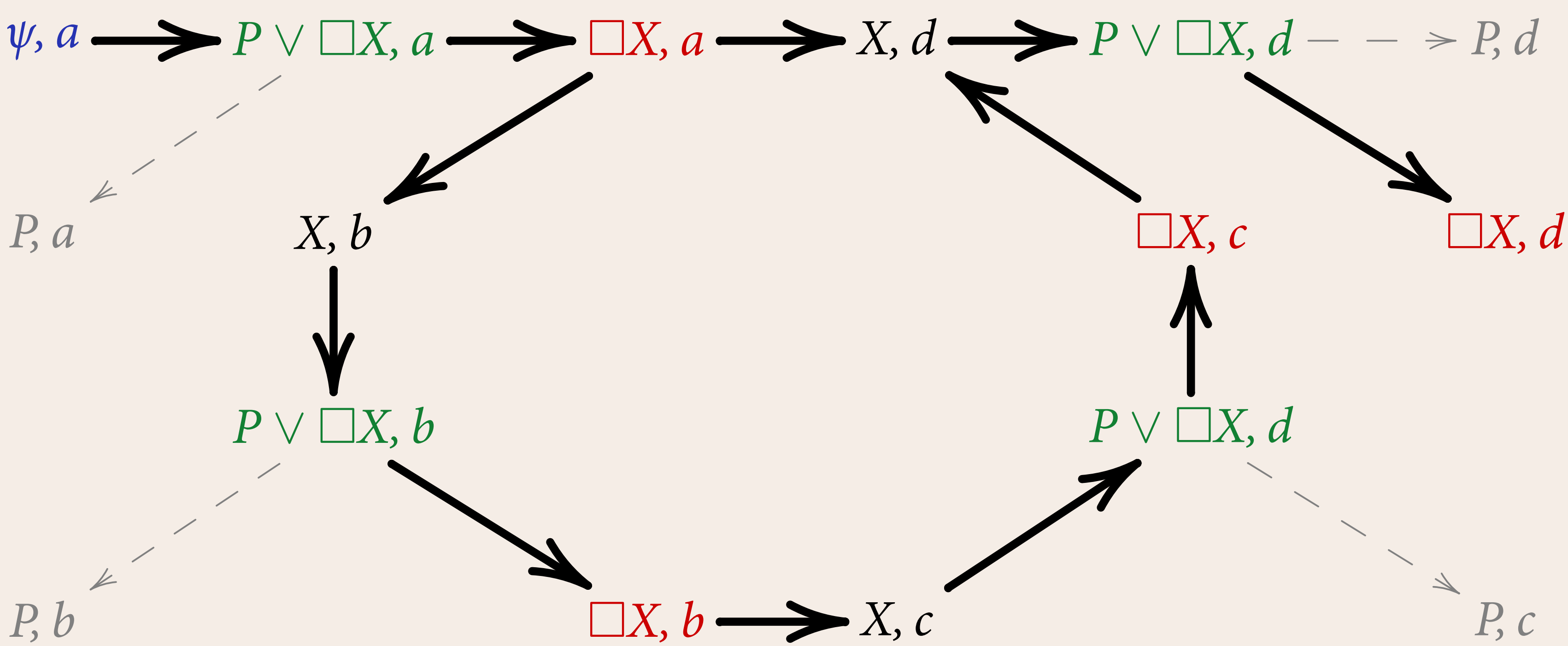

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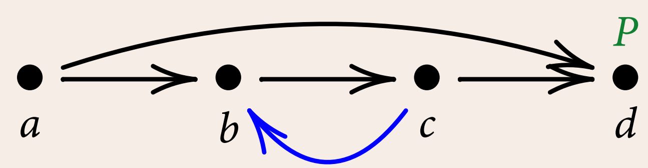

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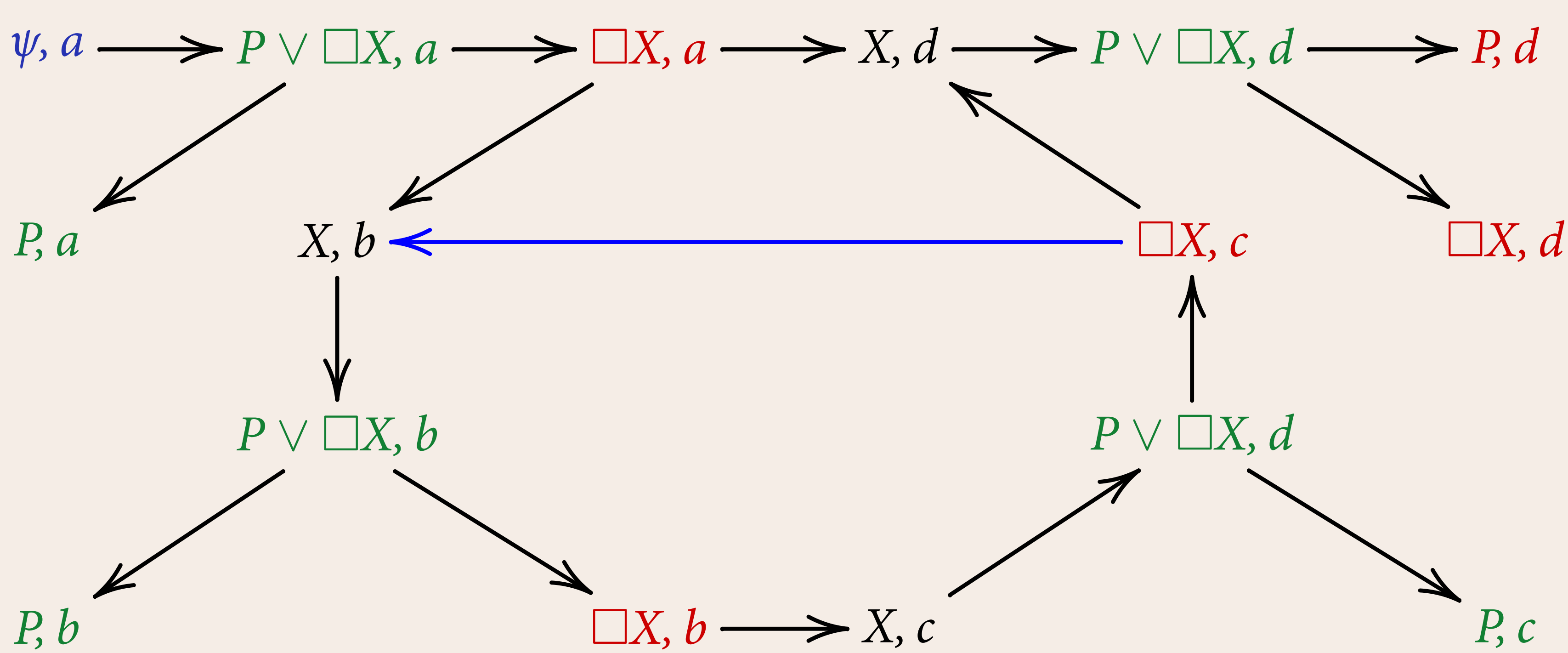

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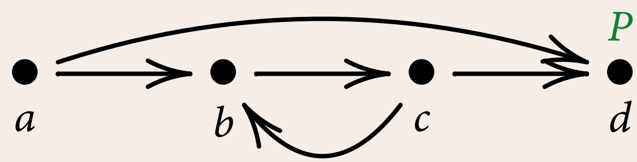

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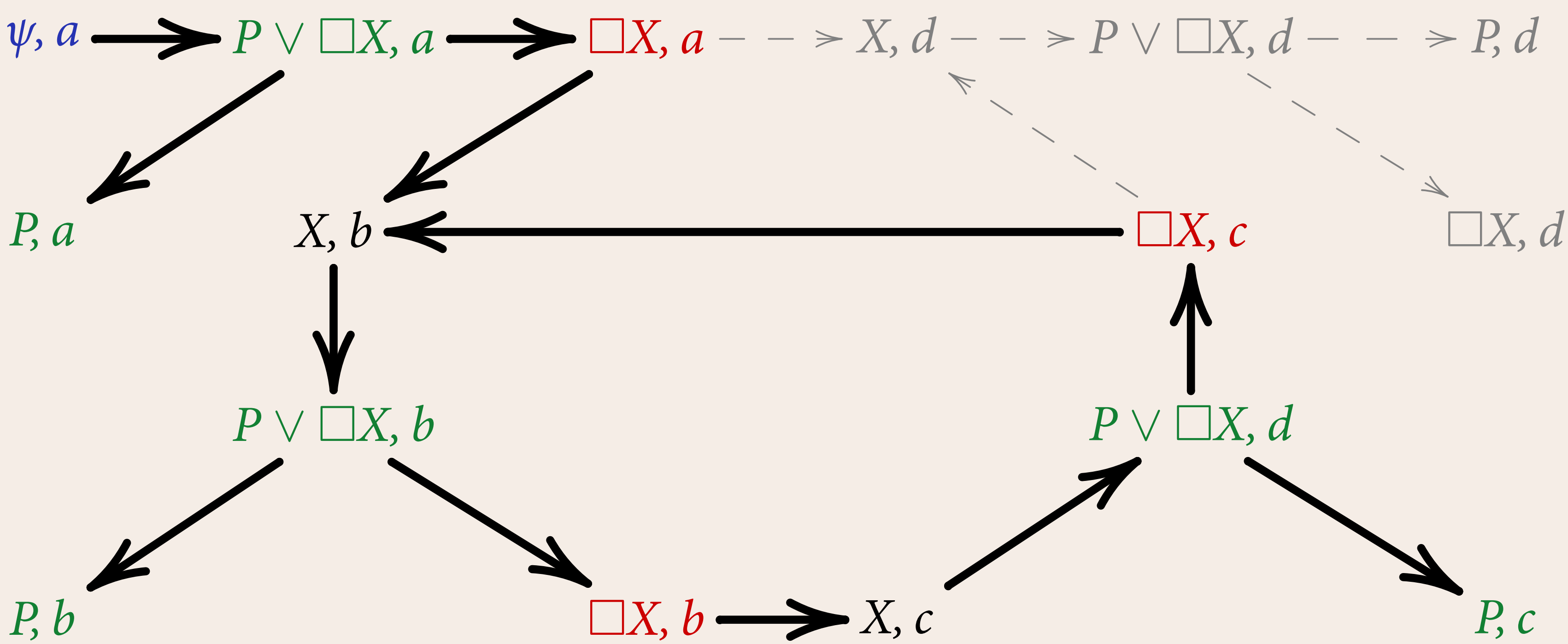

Winning conditions

On formulae $[\mathbf{lfp}\ T\overline{x}\,.\,\psi(T,\overline{x})](\overline{a})$ or $\mu X.\psi$ (where ψ has no fixed points), Verifier must win in a finite number of steps.

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The outermost fixed point on cycle determines the winner

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Parity game, with following priority assignment:

- $\Omega(T\bar{a})$ is $\begin{cases} \text{even} & \text{if } T \text{ } \mathbf{gfp}\text{-variable} \\ \text{odd} & \text{if } T \text{ } \mathbf{lfp}\text{-variable} \end{cases}$
- $\Omega(T\bar{a}) \leq \Omega(T'\bar{b})$ if T' depends on T
  (i.e. if T free in $[\mathbf{fp}\ T'\bar{x}\,.\,\varphi(T', T, \bar{x})](\bar{a})$)
- $\Omega(\varphi)$ maximal, for other formulae φ

Analogous for L_μ

Model checking game with nested cycles: Example

$\psi = \nu X \underbrace{\mu Y \,.\, \Diamond((P \wedge X) \vee Y)}_{\varphi} \equiv$ on some path, P occurs infinitely often

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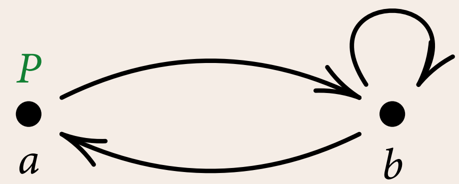

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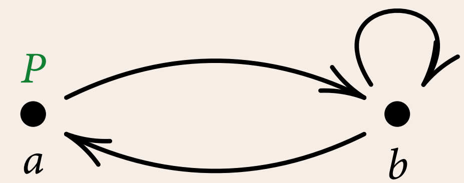
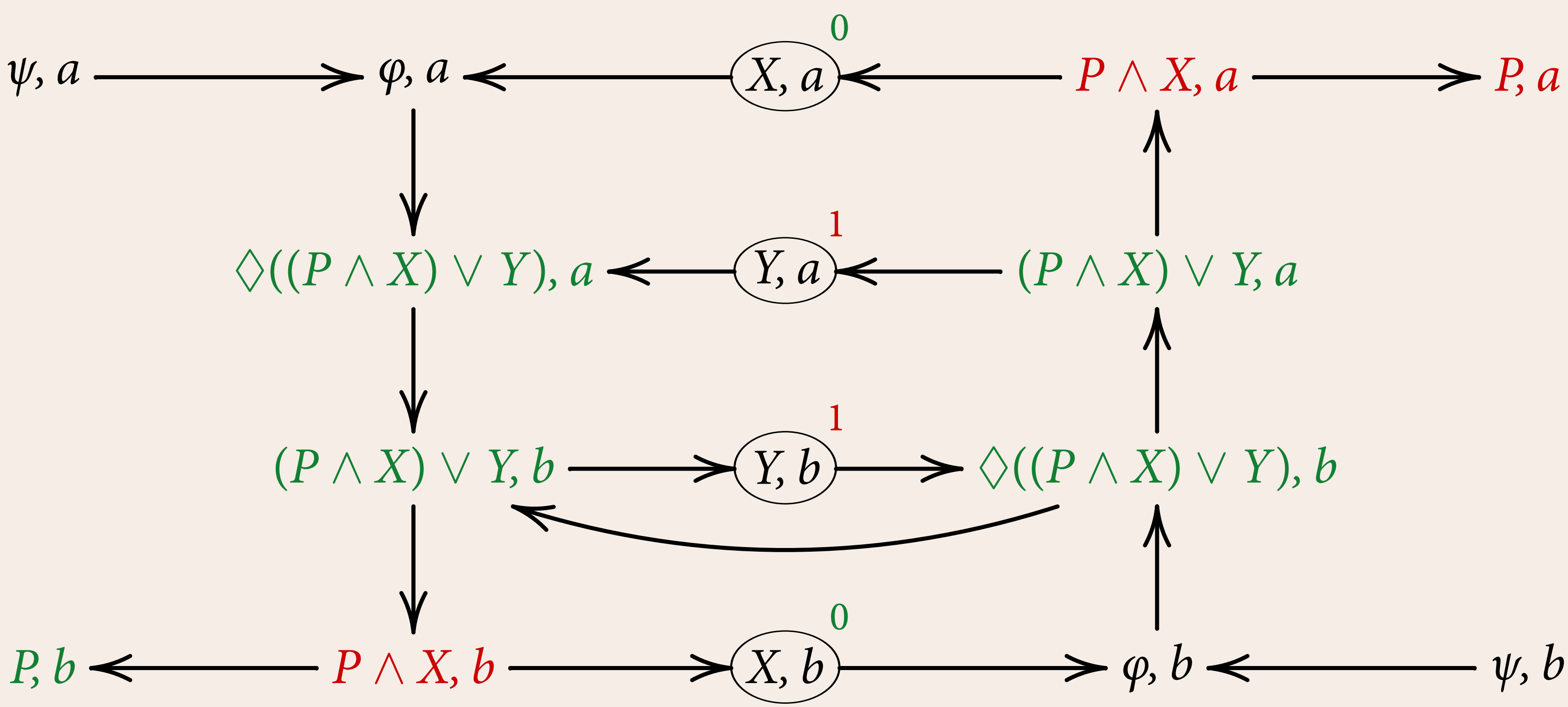
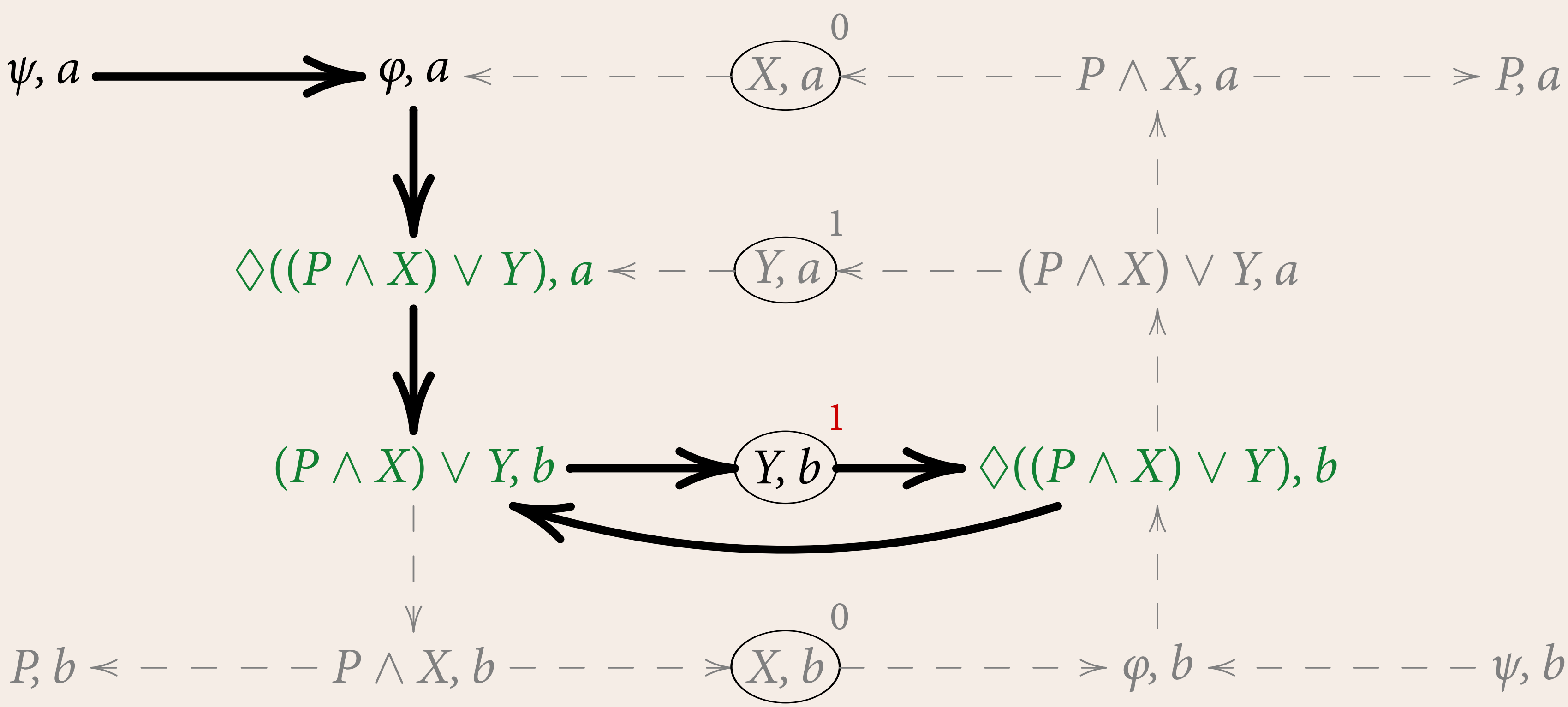




Model checking game with nested cycles: Example

Bad cycles for Verifier: Least priority is odd



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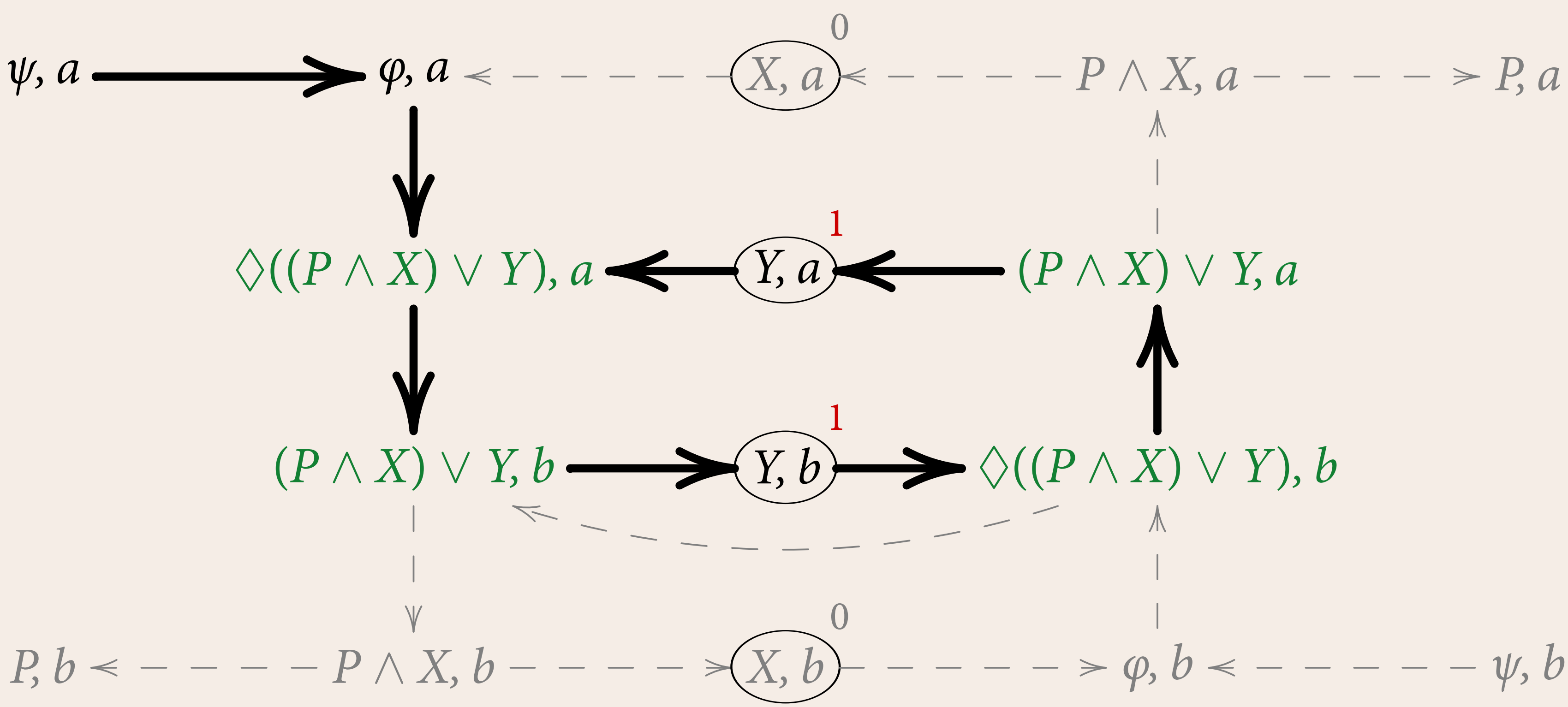
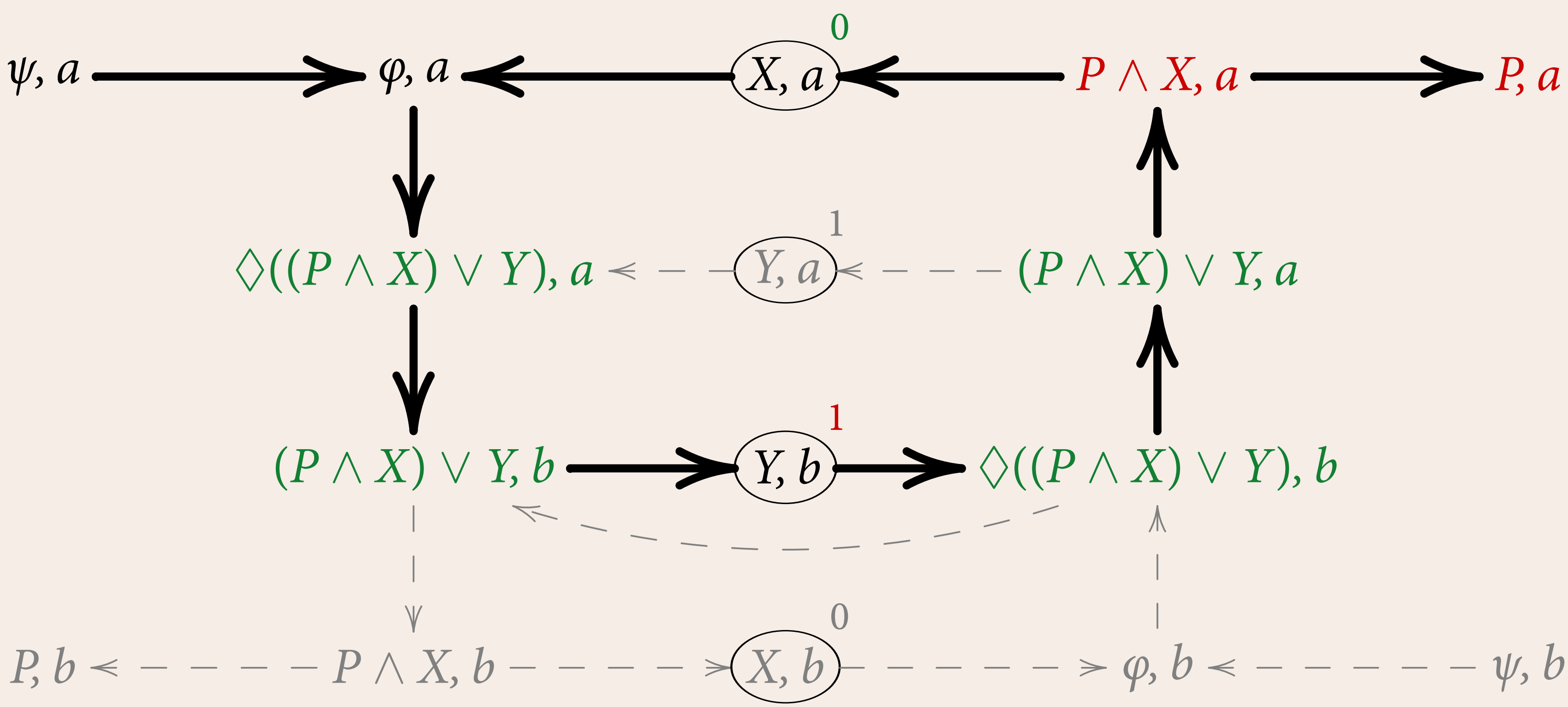


Model checking game with nested cycles: Example

Winning strategy for Verifier



Defining winning regions of parity games in L_μ

Describe parity game with d priorities by transition system

$\mathcal{G} = (V, E, E_0, \dots, E_{d-1}, A_0, \dots, A_{d-1})$ where

$$E_i = \{u : \Omega(u) = i \text{ and Ego (Player 0) moves from } u\}$$

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Define the formula

$$\mathrm{Win}_d := \nu X_0\, \mu X_1\, \nu X_2 \cdots \lambda X_{d-1} \bigvee_i \Big((E_i \wedge \Diamond X_i) \vee (A_i \wedge \Box X_i) \Big)$$

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Proof. The model checking game for Win_d on $\mathcal{G}$ coincides (up to the presence of additional 'stupid' moves) with the game $\mathcal{G}$ itself !