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3 Quantum Algorithms

3.1 The Deutsch-Jozsa algorithm

Suppose that your task is to decide whether a function $f : \{0,1\}^n \to \{0,1\}$ is either constantly equal to 0 or it is *balanced*, i.e. $f(x) = 1$ for precisely half of all inputs $x \in \{0,1\}^n$ (either one of these two cases is guaranteed to hold). If you decide correctly, you are awarded 1 000 €. On the other hand, a false answer is fatal. To help you find the right answer, you can repeatedly ask for the value of f for a given input x. Each such query will set you back 2 €.

Classically, there is a good chance to find the right answer by drawing an input x uniformly at random. Clearly, if $f(x) = 1$, you can be sure that f is balanced. On the other hand, if f is balanced, then the probability that $f(x) = 0$ for k inputs, chosen uniformly at random, is $1/2^k$, which converges to 0 exponentially fast. However, unless you query more than 2^{n-1} many inputs or get the answer that $f(x) = 1$, you cannot be sure of your answer.

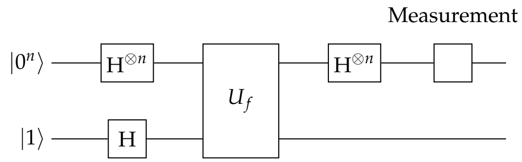
Suppose now that you may query a QGA on $n + 1$ qubits for computing the function U_f defined by[1]

$$U_f|x\rangle|j\rangle = |x\rangle|f(x) \oplus j\rangle.$$

Clearly, QGAs are more expensive than classical circuits, so let us say that each application of U_f costs 500 €. Can you get the correct answer and still make money in this case?

Surprisingly, the answer is *yes* since there exists a QGA that decides whether f is balanced with just one application of U_f:

[1]Note that U_f has to be unitary.



Let us examine what the circuit does: First, the vector $|0^n\rangle \otimes |1\rangle$ is mapped by $\mathrm{H}^{\otimes n+1}$ to

$$\frac{1}{\sqrt{2^{n+1}}} \sum_{x\in\{0,1\}^n} |x\rangle \otimes (|0\rangle - |1\rangle)\,.$$

Second, the QGA for U_f is applied to this vector, which yields the vector

$$\begin{aligned}
&\frac{1}{\sqrt{2^{n+1}}} \sum_{x\in\{0,1\}^n} \left(|x\rangle \otimes (-1)^{f(x)}(|0\rangle - |1\rangle)\right) \\
&= \left(\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}}\right) \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}} \\
&= \underbrace{\left(\sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}}\right)}_{=:|\psi_f\rangle} \otimes \mathrm{H}\,|1\rangle
\end{aligned}$$

To see what is the result of $\mathrm{H}^{\otimes n}\,|\psi_f\rangle$, note that for $x \in \{0,1\}$, we can write $\mathrm{H}\,|x\rangle$ as follows:

$$\begin{aligned}
\mathrm{H}\,|x\rangle &= \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^x|1\rangle\right) \\
&= \frac{1}{\sqrt{2}} \sum_{z\in\{0,1\}} (-1)^{xz}|z\rangle\,.
\end{aligned}$$

Analogously, for $x = x_1 \cdots x_n \in \{0,1\}^n$, we have

$$\begin{aligned}
\mathrm{H}^{\otimes n}\,|x\rangle &= \frac{1}{\sqrt{2^n}} \sum_{z=z_1\cdots z_n\in\{0,1\}^n} (-1)^{x_1z_1+\cdots+x_nz_n}|z\rangle \\
&= \frac{1}{\sqrt{2^n}} \sum_{z\in\{0,1\}^n} (-1)^{x\cdot z}|z\rangle\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{H}^{\otimes n}\,|\psi_f\rangle &= \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} (-1)^{f(x)}\,\mathrm{H}^{\otimes n}\,|x\rangle \\
&= \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \sum_{z\in\{0,1\}^n} (-1)^{f(x)+x\cdot z}|z\rangle \\
&= \frac{1}{2^n} \sum_{z\in\{0,1\}^n} \sum_{x\in\{0,1\}^n} (-1)^{f(x)+x\cdot z}|z\rangle\,.
\end{aligned}$$

In particular, the amplitude of the basis vector $|0^n\rangle$ in $\mathrm{H}^{\otimes n}\,|\psi_f\rangle$ is $\frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)}$. If $f \equiv 0$, then this amplitude is equal to 1 and, with probability 1, the final measurement yields $|0^n\rangle$. On the other hand, if f is balanced, then the amplitude of $|0^n\rangle$ is 0 and, with probability 1, the final measurement yields a basis vector different from $|0^n\rangle$.

3.2 Grover's search algorithm

While the Deutsch-Jozsa algorithm arguably solves an artificial problem, Grover's algorithm solves a canonical search problem: This time, the task is to find, given an arbitrary Boolean function $f : \{0,1\}^n \to \{0,1\}$, an input x with $f(x) = 1$ (or to determine that there is no such input). Classically, there is no better way than to test each input, which requires 2^n queries to f in the worst case. Grover showed that if one has access to a QGA for computing the function

$$U_f : H_{2^{n+1}} \to H_{2^{n+1}} |x\rangle \otimes |j\rangle \mapsto |x\rangle \otimes |f(x) \oplus j\rangle,$$

then one can build a quantum algorithm that finds an x with $f(x) = 1$ in time $\mathrm{O}(\sqrt{2^n})$.

Our first approach is to apply a Hadamard transformation to $|0^n\rangle$ to obtain a superposition of all inputs and then to apply U_f on $\mathrm{H}^{\otimes n}\,|0^n\rangle \otimes |0\rangle$. The resulting vector is

$$\psi := \frac{1}{\sqrt{2^n}} \sum_{x\in\{0,1\}^n} |x\rangle \otimes |f(x)\rangle.$$

Can we measure $|\psi\rangle$ to find an input x with $f(x) = 1$? For each x with $f(x) = 1$, the amplitude of $|x1\rangle$ in $|\psi\rangle$ is $\frac{1}{\sqrt{2^n}}$. Hence, if for instance there is only one such x, then a measurement of ψ will very likely not find this x. The idea of the algorithm is to apply a transformation on $|\psi\rangle$ that makes the amplitudes of the basis vectors $|x1\rangle$ much larger while making those of $|x0\rangle$ smaller. After this transformation, with high probability a measurement of the last results in a basis vector of the form $|x1\rangle$, i.e. $f(x) = 1$. If the measurement fails and we obtain a vector $|x0\rangle$, we just repeat the process.

It turns out that this idea can be made to work using a modified approach, where we apply U_f not to $\mathrm{H}^{\otimes n}|0^n\rangle \otimes |0\rangle$, but to $\mathrm{H}^{\otimes n}|0^n\rangle \otimes \mathrm{H}|1\rangle$. As in the Deutsch-Jozsa algorithm, the resulting vector is $|\psi_f\rangle \otimes \mathrm{H}|1\rangle$, where

$$|\psi_f\rangle = \sum_{x\in\{0,1\}^n} \frac{(-1)^{f(x)}|x\rangle}{\sqrt{2^n}}.$$

Let V_f the transformation on the first n qubits defined by U_f, œ

$$V_f|x\rangle = (-1)^{f(x)}|x\rangle.$$

For $|\psi\rangle = \sum_x a_x|x\rangle$, we have

$$V_f|\psi\rangle = \sum_{x\colon f(x)=0} a_x|x\rangle - \sum_{x\colon f(x)=1} a_x|x\rangle.$$

For $|\psi\rangle = \sum_x a_x|x\rangle$, let $A := 2^{-n}\sum_x a_x$ the *average amplitude*. Consider the transformation D that maps $|\psi\rangle$ to the vector $\sum_x(2A - a_x)|x\rangle$. The corresponding matrix is

$$D = \begin{pmatrix} \frac{2}{2^n} & \frac{2}{2^n} & \cdots & \frac{2}{2^n} \\ \frac{2}{2^n} & \frac{2}{2^n} - 1 & & \frac{2}{2^n} \\ \vdots & & \ddots & \vdots \\ \frac{2}{2^n} & \frac{2}{2^n} & \cdots & \frac{2}{2^n} - 1 \end{pmatrix}.$$

To see this, consider a basis vector $|y\rangle = \sum_x \delta_{xy}|x\rangle$ (where $\delta_{xy} = 1$ if

$x = y$ and $\delta_{xy} = 0$ otherwise). The average amplitude of $|y\rangle$ is $A = \frac{1}{2^n}$. Hence, $D|y\rangle = \left(\frac{2}{2^n} - 1\right)|y\rangle + \sum_{x\neq y}\frac{2}{2^n}|x\rangle$.

Lemma 3.1. $D = \mathrm{H}^{\otimes n} \cdot R_n \cdot \mathrm{H}^{\otimes n}$ with

$$R_n = \begin{pmatrix} 1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{pmatrix}.$$

Note that R_n can be implemented using $\mathrm{O}(n)$ simple gates.

Proof. Consider the matrix

$$R' = R_n + \mathrm{I}_n = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}.$$

We claim that

$$\mathrm{H}^{\otimes n} \cdot R'_n \cdot \mathrm{H}^{\otimes n} = \frac{2}{2^n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

i.e. $\mathrm{H}^{\otimes n} \cdot R'_n \cdot \mathrm{H}^{\otimes n}|x\rangle = \frac{2}{2^n}\sum_y |y\rangle$ for all $x \in \{0,1\}^n$:

$$\begin{aligned} |x\rangle &\xmapsto{\mathrm{H}^{\otimes n}} \frac{1}{\sqrt{2^n}}\sum_z (-1)^{x\cdot z}|z\rangle \\ &\xmapsto{R'_n} \frac{1}{\sqrt{2^n}}\sum_z (-1)^{x\cdot z}R'_n|z\rangle = \frac{2}{\sqrt{2^n}}|0\rangle \\ &\xmapsto{\mathrm{H}^{\otimes n}} \frac{2}{2^n}\sum_y |y\rangle. \end{aligned}$$

Finally,

$$\begin{aligned} \mathrm{H}^{\otimes n} \cdot R_n \cdot \mathrm{H}^{\otimes n} &= \mathrm{H}^{\otimes n}(R'_n - \mathrm{I}_n)\,\mathrm{H}^{\otimes n} \\ &= \mathrm{H}^{\otimes n} \cdot R'_n\,\mathrm{H}^{\otimes n} - \mathrm{H}^{\otimes n} \cdot \mathrm{I}_n \cdot \mathrm{H}^{\otimes n} \\ &= \mathrm{H}^{\otimes n} \cdot R'_n\,\mathrm{H}^{\otimes n} - \mathrm{I}_n \\ &= D. \end{aligned}$$

Q.E.D.

For a given function $f\colon \{0,1\}^n \to \{0,1\}$, Grover's search algorithm iterates the *Grover operator* $G := D \cdot V_f$ *sufficiently often* on input $\mathrm{H}^{\otimes n}|0\rangle$ in order to magnify the amplitudes of the basis vectors $|x\rangle$ with $f(x) = 1$. But what do we mean by *sufficiently often*?

Consider the sets $T = \{x\colon f(x) = 1\}$ and $F = \{x\colon f(x) = 0\}$. After r iterations of G, the resulting vector will be of the form $|\psi_r\rangle = t_r \sum_{x\in T}|x\rangle + f_r \sum_{x\in F}|x\rangle$ with average amplitude $A_r = \frac{1}{2^n}(t_r|T| + f_r(2^n - |T|))$. Now,

$$\begin{aligned}
|\psi_{r+1}\rangle &= G|\psi_r\rangle \\
&= DV_f\Big(t_r \sum_{x\in T}|x\rangle + f_r \sum_{x\in F}|x\rangle\Big) \\
&= D\Big(-t_r \sum_{x\in T}|x\rangle + f_r \sum_{x\in F}|x\rangle\Big) \\
&= (2A_r + t_r)\sum_{x\in T}|x\rangle + (2A_r - f_r)\sum_{x\in F}|x\rangle.
\end{aligned}$$

Hence,

$$\begin{aligned}
t_{r+1} &= 2A + t_r = \Big(1 - \frac{2|T|}{2^n}\Big)t_r + \Big(2 - \frac{2|T|}{2^n}\Big)f_r; \\
f_{r+1} &= 2A - f_r = -\frac{2|T|}{2^n}t_r + \Big(1 - \frac{2|T|}{2^n}\Big)f_r.
\end{aligned}$$

This means that the coefficients t_r and f_r satisfy the following recursion:

$$\begin{pmatrix} t_{r+1} \\ f_{r+1} \end{pmatrix} = \begin{pmatrix} 1-\delta & 2-\delta \\ -\delta & 1-\delta \end{pmatrix}\begin{pmatrix} t_r \\ f_r \end{pmatrix}, \tag{3.1}$$

where $\delta = \frac{2|T|}{2^n}$.

To compute the effect of the iterated application of G on $\mathrm{H}^{\otimes n}|0^n\rangle$, we have to solve (3.1) under the initial condition $t_0 = f_0 = \frac{1}{\sqrt{2^n}}$. Since G is unitary, we have $\|G|\psi\rangle\| = \|\psi\|$, i.e. $|T|t_r^2 + (2^n - |T|)f_r^2 = 1$ for all $r \in \mathbb{N}$. Hence, there exist ϑ_r such that $t_r = \frac{1}{\sqrt{|T|}}\sin\vartheta_r$ and $f_r = \frac{1}{\sqrt{2^n - |T|}}\cos\vartheta_r$.

The Grover operator G can be interpreted geometrically as a rota-

tion in the 2-dimensional space that is generated by the vectors

$$\begin{aligned}
|\varphi^+\rangle &= \frac{1}{\sqrt{|T|}}\sum_{x\in T}|x\rangle, \\
|\varphi^-\rangle &= \frac{1}{\sqrt{2^n - |T|}}\sum_{x\in F}|x\rangle.
\end{aligned}$$

We have

$$\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2^n}}\sum_x |x\rangle \\
&= \sqrt{\frac{|T|}{2^n}}|\varphi^+\rangle + \sqrt{\frac{2^n - |T|}{2^n}}|\varphi^-\rangle \\
&= \sin\vartheta_0\,|\varphi^+\rangle + \cos\vartheta_0\,|\varphi^-\rangle.
\end{aligned}$$

Now, the Grover operator applied first performs a reflection across $|\varphi^-\rangle$ followed by a reflection across $|\psi_0\rangle$. The resulting operation is a rotation by $2\vartheta_0$ towards $|\varphi^+\rangle$. Hence, $\vartheta_r = (2r+1)\vartheta_0$ for all $r \in \mathbb{N}$.

In order for the final measurement to yield $|x\rangle$ with $x \in T$, we need that $\vartheta_r \approx \frac{\pi}{2}$ (so that $|\psi_r\rangle$ is close to $|\varphi^+\rangle$). Solving the equation $(2r+1)\vartheta_0 = \frac{\pi}{2}$, we obtain $r = \frac{\pi}{4\vartheta_0} - \frac{1}{2}$. Hence, for $\vartheta_0 \approx \sin\vartheta_0 = \sqrt{\frac{|T|}{2^n}}$, we can expect that $r = \lfloor \frac{\pi}{4}\sqrt{\frac{2^n}{|T|}} \rfloor$ iterations suffice to find a solution with high probability. More precisely, we have the following theorem.

Theorem 3.2. Let $f\colon \{0,1\}^n \to \{0,1\}$ and $m := |\{x\colon f(x) = 1\}|$ such that $0 < m \le \frac{3}{4}\cdot 2^n$, and let $\vartheta_0 < \frac{\pi}{3}$ such that $\sin\vartheta_0 = \frac{m}{2^n}$. After $\lfloor \frac{\pi}{4\vartheta_0} \rfloor$ iterations of G on $|\psi_0\rangle = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}|x\rangle$, a measurement of the resulting vector yields a basis vector $|x\rangle$ such that $f(x) = 1$ with probability $\ge \frac{1}{4}$.

Proof. For $|\psi_r\rangle = \sin(2r+1)\vartheta_0\,|\varphi^+\rangle + \cos(2r+1)\vartheta_0\,|\varphi^-\rangle$, we denote by $p(r) := \sin^2(2r+1)\vartheta_0$ the probability of a projection onto $|\varphi^+\rangle$. (This is precisely the probability with which a measurement of $|\psi_r\rangle$ results in a basis vector $|x\rangle$ such that $f(x) = 1$.) Let $\delta \in (0, \frac{1}{2}]$ such that $\lfloor \frac{\pi}{4\vartheta_0} \rfloor = \frac{\pi}{4\vartheta_0} - \frac{1}{2} + \delta$. Since $|2\delta\vartheta_0| \le |\vartheta_0| \le \frac{\pi}{3}$, we have

$$p\Big(\Big\lfloor \frac{\pi}{4\vartheta_0} \Big\rfloor\Big) = \sin^2\Big(\Big\lfloor \frac{\pi}{4\vartheta_0} \Big\rfloor\Big)\vartheta_0$$

$$= \sin^2\left(\frac{\pi}{2} + 2\delta\vartheta_0\right)$$
$$\geq \sin^2\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{1}{4}. \qquad \text{Q.E.D.}$$

Finally, we can state Grover's search algorithm. Given a QGA for the operator V_f defined by $V_f|x\rangle = (-1)^{f(x)}|x\rangle$ and for *known* $m := |\{x\colon f(x) = 1\}|$, the algorithm determines an input x such that $f(x) = 1$ by the following procedure:

if $m \geq \frac{3}{4} \cdot 2^n$ **then**
  $|\psi\rangle := \mathrm{H}^{\otimes n}\,|0^n\rangle$
else
  $r := \lfloor \frac{\pi}{4\vartheta_0} \rfloor$ for $0 \leq \vartheta_0 \leq \frac{\pi}{3}$ with $\sin^2\vartheta_0 = \frac{m}{2^n}$
  $|\psi\rangle := G^r\,\mathrm{H}^{\otimes n}\,|0^n\rangle$
end if
measure $|\psi\rangle$ to obtain a basis vector $|x\rangle$
output x

If $m \geq \frac{3}{4} \cdot 2^n$, the algorithm finds x such that $f(x) = 1$ with probability $\geq \frac{3}{4}$ since $|\psi\rangle$ is a uniform superposition of all basis vectors. Otherwise, Theorem 3.2 applies, and the algorithm finds x such that $f(x) = 1$ with probability $\geq \frac{1}{4}$.

For $m = 1$ and for large n, we have $\lfloor \frac{\pi}{4\vartheta_0} \rfloor \approx \frac{\pi}{4}\sqrt{2^n}$ (since $\sin^2\vartheta_0 \approx \vartheta_0^2 = \frac{1}{2^n}$)). Hence, in this case, $\mathrm{O}(\sqrt{2^n})$ calls to V_f suffice to find an input x such that $f(x) = 1$ with probability $\geq \frac{1}{4}$, whereas classical randomised algorithms need to evaluate f at $\mathrm{O}(2^n)$ points to find such an x with the same probability of success.

Another interesting special case is when one fourth of the inputs are positive instances, i.e. if $m = \frac{1}{4} \cdot 2^n$. Recall that after r iterations of G the resulting state is

$$|\psi_r\rangle = \sin(2r+1)\vartheta_0\,|\varphi^+\rangle + \cos(2r+1)\vartheta_0\,|\varphi^-\rangle.$$

For $m = \frac{1}{4} \cdot 2^n$, we have $\sin^2\vartheta_0 = \frac{1}{4}$, and therefore $\vartheta_0 = \frac{\pi}{6}$. After *one* iteration of G, the resulting state is $|\psi_1\rangle = \sin\frac{\pi}{2}\,|\varphi^+\rangle + \cos\frac{\pi}{2}\,|\varphi^-\rangle = |\varphi^+\rangle$ and a measurement will *surely* result in a basis vector x such that $f(x) = 1$.

In typical applications, the number m of positive instances is *not* known. How can we modify the algorithm such that it also finds a solution with good probability in this case?

Lemma 3.3. For all $\alpha \in \mathbb{R}$ and all $m \in \mathbb{N}$:

$$\sum_{r=0}^{m-1} \cos(2r+1)\alpha = \frac{\sin 2m\alpha}{2\sin\alpha}.$$

In particular, $\sin 2\alpha = 2\sin\alpha\cos\alpha$, and $\cos 2\alpha = 1 - 2\sin^2\alpha$.

We can now state Grover's search algorithm for *unknown* m:

choose $x \in \{0,1\}^n$ uniformly at random
if $f(x) = 1$ **then**
  output x
else
  choose $r \in \{0, 1, \ldots, \lfloor\sqrt{2^n}\rfloor\}$ uniformly at random
  $|\psi\rangle := G^r\,\mathrm{H}^{\otimes n}\,|0^n\rangle$
  measure $|\psi\rangle$ to obtain a basis vector $|x\rangle$
  output x
end if

Clearly, if $m \geq \frac{3}{4} \cdot 2^n$, then the algorithm returns x such that $f(x) = 1$ with probability $\geq \frac{3}{4}$. Hence, assume now that $m < \frac{3}{4} \cdot 2^n$, and set $t := \lfloor\sqrt{2^n}\rfloor + 1$. What is the probability that the algorithm outputs a *good* x? We have already seen that the probability of finding a good x after r iterations of G is $\sin^2(2r+1)\vartheta_0$. Now, since r is chosen uniformly at random from $\{0, 1, \ldots, t-1\}$, the probability that the algorithm outputs a good x is

$$\frac{1}{t}\sum_{r=0}^{t-1} \sin^2(2r+1)\vartheta_0$$
$$= \frac{1}{2t}\sum_{r=0}^{t-1} \big(1 - \cos(2r+1)2\vartheta_0\big) \qquad (\text{since } \sin^2\alpha = (1-\cos 2\alpha)/2)$$
$$= \frac{1}{2} - \frac{1}{2t}\sum_{r=0}^{t-1} \cos(2r+1)2\vartheta_0$$

$$= \frac{1}{2} - \frac{\sin 4t\vartheta_0}{4t \sin 2\vartheta_0} \qquad \text{(by Lemma 3.3).}$$

For $0 < m \leq \frac{3}{4} \cdot 2^n$ and $t = \lfloor \sqrt{2^n} \rfloor + 1$, we have

$$\begin{aligned}
\sin 2\vartheta_0 &= 2 \sin \vartheta_0 \cos \vartheta_0 \\
&= 2\sqrt{\frac{m}{2^n}} \cdot \sqrt{\frac{2^n - m}{2^n}} \\
&\geq 2\sqrt{\frac{m}{2^n}} \cdot \sqrt{\frac{1}{4}} = \sqrt{\frac{m}{2^n}} \\
&\geq \sqrt{\frac{1}{2^n}}
\end{aligned}$$

and therefore

$$t \geq \frac{1}{\sin 2\vartheta_0}.$$

Hence, the algorithm finds a good x with probability

$$\frac{1}{2} - \frac{\sin 4t\vartheta_0}{4t \sin 2\vartheta_0} \geq \frac{1}{2} - \frac{\sin 4t\vartheta_0}{4} \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

To sum up, we have the following theorem.

Theorem 3.4 (Grover)**.** Given a function $f : \{0,1\}^n \to \{0,1\}$, $f \not\equiv 0$, and a QGA for $V_f : H_{2^n} \to H_{2^n} : |x\rangle \mapsto (-1)^{f(x)}|x\rangle$, there exists a quantum algorithm that finds an x such that $f(x) = 1$ with probability $\geq \frac{1}{4}$ by evaluating V_f at most $\mathrm{O}(\sqrt{2^n})$ times.

3.3 Fourier transformation

In the following, let $(G, +)$ be an abelian group, and let $\mathbb{C}^* = (\mathbb{C} \setminus \{0\}, \cdot)$. A *character* of $(G, +)$ is a homomorphism $\chi : (G, +) \to \mathbb{C}^*$. For two characters χ_1, χ_2, their product $\chi_1 \cdot \chi_2$, defined by

$$\chi_1 \cdot \chi_2 \colon (G, +) \to \mathbb{C}^* \colon g \mapsto \chi_1(g) \cdot \chi_2(g)$$

is also a character. In fact the set of characters of $(G, +)$ together with this operations forms a new group, called the *dual group* and denoted by $(\hat{G}, \cdot)$.

Lemma 3.5. Let $(G, +)$ be a finite abelian group with n elements. Then $\chi(g)^n = 1$ for all $g \in G$, i.e. $\chi(g)$ is an nth root of unity. Hence, $\chi(g) = \mathrm{e}^{2\mathrm{i}\pi k/n}$ for some $k \in \{0, 1, \ldots, n-1\}$.

Proof. For $m \in \mathbb{N}$ and $g \in G$, let

$$m \cdot g := \underbrace{g + \cdots + g}_{m \text{ times}}.$$

The set $\{0, g, 2 \cdot g, \ldots\}$ forms a subgroup of $(G, +)$. Let

$$k = \min\{m > 0 \colon m \cdot g = 0\}$$

be the order of this subgroup. Since the order of a subgroup divides the order of the group, we have $n \cdot g = \frac{n}{k} \cdot k \cdot g = \frac{n}{k} \cdot 0 = 0$. Hence, $\chi(g)^n = \chi(n \cdot g) = \chi(0) = 1$. Q.E.D.

Example 3.6. Consider the cyclic group $(\mathbb{Z}_n, +)$, where $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$, with addition modulo n. For each $y \in \mathbb{Z}_n$, define

$$\chi_y \colon \mathbb{Z}_n \to \mathbb{C}^* \colon x \mapsto \mathrm{e}^{2\pi\mathrm{i}\frac{xy}{n}}.$$

We claim that χ_y is a character of $(\mathbb{Z}_n, +)$, i.e. a group homomorphism from $(\mathbb{Z}_n, +)$ to $(\mathbb{C}^*, \cdot)$. Let $x, x' \in \mathbb{Z}_n$. We have:

$$\begin{aligned}
\chi_y(x + x') &= \mathrm{e}^{2\pi\mathrm{i}\frac{x+x'}{n}} \\
&= \mathrm{e}^{2\pi\mathrm{i}\frac{xy}{n}} \mathrm{e}^{2\pi\mathrm{i}\frac{x'y}{n}} \\
&= \chi_y(x) \cdot \chi_y(x')
\end{aligned}$$

Now consider $y \neq y' \in \mathbb{Z}_n$. We have

$$\chi_y(1) = \mathrm{e}^{2\pi\mathrm{i}\frac{y}{n}} \neq \mathrm{e}^{2\pi\mathrm{i}\frac{y'}{n}} = \chi_{y'}(1).$$

Hence, also $\chi_y \neq \chi_{y'}$. On the other hand, let χ be a character of

$(\mathbb{Z}_n, +)$. By Lemma 3.5, $\chi(1) = \mathrm{e}^{2\mathrm{i}\pi y/n}$ for some $y \in \mathbb{Z}_n$. But then $\chi = \chi_y$. Finally, note that $\chi_y \cdot \chi_{y'} = \chi_{y+y'}$. Hence, the mapping $\mathbb{Z}_n \to \hat{\mathbb{Z}}_n \colon y \mapsto \chi_y$ is an isomorphism between $(\mathbb{Z}_n, +)$ and the dual group $(\hat{\mathbb{Z}}_n, \cdot)$, i.e. $(\mathbb{Z}_n, +) \cong (\hat{\mathbb{Z}}_n, \cdot)$.

More generally, we have the following theorem.

Theorem 3.7. Let $(G, +)$ be a finite abelian group. Then $(G, +) \cong (\hat{G}, \cdot)$.

Proof. Every abelian group is (isomorphic to) a *direct sum* (or a direct product if the group operation is understood as multiplication) of cyclic groups:

$$(G, +) = (\mathbb{Z}_{n_1}, +) \oplus \cdots \oplus (\mathbb{Z}_{n_k}, +).$$

We already know that $(\mathbb{Z}_n, +) \cong (\hat{\mathbb{Z}}_n, \cdot)$ and therefore also

$$(G, +) \cong (\hat{\mathbb{Z}}_{n_1}, \cdot) \times \cdots \times (\hat{\mathbb{Z}}_{n_k}, \cdot).$$

To establish that $(G, +) \cong (\hat{G}, \cdot)$, it remains to show that there exists an isomorphism

$$\varphi : (\hat{\mathbb{Z}}_{n_1}, \cdot) \times \cdots \times (\hat{\mathbb{Z}}_{n_k}, \cdot) \to (\hat{G}, \cdot).$$

For each $g \in G$ there exists a unique decomposition into its components: $g = g_1 + \cdots + g_k$ with $g_i \in \mathbb{Z}_{n_i}$. For $\chi_1 \in \hat{\mathbb{Z}}_{n_1}, \ldots, \chi_k \in \hat{\mathbb{Z}}_{n_k}$, we define $(\varphi(\chi_1, \ldots, \chi_k))(g) := \chi_1(g_1) \cdots \chi_k(g_k)$. Clearly, φ is a homomorphism. It remains to show that φ is a bijection.

Let us first prove that φ is injective: Let $(\chi_1, \ldots, \chi_k) \neq (\chi'_1, \ldots, \chi'_k)$, $\chi = \varphi(\chi_1, \ldots, \chi_k)$, and $\chi' = \varphi(\chi'_1, \ldots, \chi'_k)$. There exists i with $\chi_i \neq \chi'_i$; in particular, there exists $g_i \in \mathbb{Z}_{n_i}$ with $\chi_i(g_i) \neq \chi'_i(g_i)$. We have $\chi(g_i) = \chi_i(g_i) \neq \chi'_i(g_i) = \chi'(g_i)$ and therefore also $\chi \neq \chi'$.

It remains to prove that φ is surjective: Let $\chi \in \hat{G}$. For each $i = 1, \ldots, k$, χ induces a character $\chi_i \in \hat{\mathbb{Z}}_{n_i}$ by setting $\chi_i(g_i) = \chi(g_i)$ for all $g_i \in \mathbb{Z}_{n_i}$. For all $g \in G$, we have:

$$\begin{aligned}\chi(g) &= \chi(g_1 + \cdots + g_k) \\ &= \chi(g_1) \cdots \chi(g_k) \\ &= \chi_1(g_1) \cdots \chi_k(g_k) \\ &= (\varphi(\chi_1, \ldots, \chi_k))(g)\end{aligned}$$

Hence, $\chi = \varphi(\chi_1, \ldots, \chi_k)$. Q.E.D.

Example 3.8. Consider the m-fold direct sum of $(\mathbb{Z}_2, +)$,

$$(\mathbb{Z}_2^m, +) = \underbrace{(\mathbb{Z}_2, +) \oplus \cdots \oplus (\mathbb{Z}_2, +)}_{m \text{ times}}.$$

We already know that $(\mathbb{Z}_2, +)$ has two characters, namely $\chi_0 \colon x \mapsto 1$ and $\chi_1 \colon x \mapsto \mathrm{e}^{\pi \mathrm{i} x} = (-1)^x$. The characters of $(\mathbb{Z}_2^m, +)$ are of the form

$$\chi_y \colon x = x_1 \ldots x_m \mapsto (-1)^{x \cdot y} = (-1)^{x_1 y_1 + \cdots + x_m y_m},$$

where $y = y_1 \ldots y_m \in \{0, 1\}^m$.

The set of all functions $f \colon G \to \mathbb{C}$ from a finite abelian group $(G, +)$ to $\mathbb{C}$ naturally forms a vector space V over $\mathbb{C}$. If $G = \{g_1, \ldots, g_n\}$, then this vector space is isomorphic to $\mathbb{C}^n$, where the isomorphisms maps a function f to the tuple $(f(g_1), \ldots, f(g_n))$, and the functions e_i defined by

$$e_i(g_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases}$$

form a basis of V. The vector space V can be equipped with an inner product by setting

$$\langle f \mid f' \rangle := \sum_{i=1}^{n} f(g_i)^* \cdot f'(g_i).$$

As usual, this inner product gives rise to a norm $\|\cdot\|$ on V, namely $\|f\| = \sqrt{\langle f \mid f \rangle}$. Since $\langle e_i \mid e_i \rangle = 1$ and $\langle e_i \mid e_j \rangle = 0$ for $i \neq j$, the set $\{e_1, \ldots, e_n\}$ is, in fact, an orthonormal basis of V. The characters of $(G, +)$ give rise to a different orthonormal basis of V. For $\hat{G} = \{\chi_1, \ldots, \chi_k\}$, set $B_i := \frac{1}{\sqrt{n}} \chi_i$ for all $i = 1, \ldots, n$.

Theorem 3.9. Let $(G, +)$ be a finite abelian group with characters $\chi_1, \ldots, \chi_n$, and let $B_i := 1/\sqrt{n} \cdot \chi_i$ for all $i = 1, \ldots, n$. The vectors $B_1, \ldots, B_n$ form an orthonormal basis of $V = \mathbb{C}^G$, called the *Fourier basis*.

Proof. Since $|\{B_1 \ldots, B_n\}| = |\{e_1, \ldots, e_n\}|$, it suffices to show that

$$\langle \chi_i \,|\, \chi_j \rangle = \begin{cases} n & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

For each $g \in G$ and for all $\chi \in \hat{G}$, by Lemma 3.5, we have $\chi(g)^n = 1$ and therefore $|\chi(g)| = 1$. Hence, $\chi(g)^* \cdot \chi(g) = |\chi(g)|^2 = 1$ and $\chi(g)^* = \chi(g)^{-1}$. We have:

$$\begin{aligned} \langle \chi_i \,|\, \chi_j \rangle &= \sum_{k=1}^{n} \chi_i(g_k)^* \cdot \chi_j(g_k) \\ &= \sum_{k=1}^{n} \chi_i(g_k)^{-1} \cdot \chi_j(g_k) \\ &= \sum_{k=1}^{n} (\chi_i^{-1} \cdot \chi_j)(g_k). \end{aligned}$$

For $i = j$, we have $\chi^{-}1 \cdot \chi_j = 1$ (the trivial character) and therefore $\langle \chi_i \,|\, \chi_j \rangle = n$. For $i \neq j$, consider the character $\chi := \chi_i^{-1} \cdot \chi_j$. Since $\chi_i \neq \chi_j$, we have $\chi \neq 1$, i.e. there exists $g \in G$ with $\chi(g) \neq 1$. Consider the mapping $h_g \colon G \to G \colon g' \mapsto g' + g$. Since G is finite, this mapping is not only injective, but also surjective. Hence,

$$\begin{aligned} \langle \chi_i \,|\, \chi_j \rangle &= \sum_{k=1}^{n} \chi(g_k) \\ &= \sum_{k=1}^{n} \chi(g + g_k) \\ &= \chi(g) \sum_{k=1}^{n} \chi(g_k) \\ &= \chi(g) \cdot \langle \chi_i \,|\, \chi_j \rangle. \end{aligned}$$

Since $\chi(g) \neq 1$, we must have $\langle \chi_i \,|\, \chi_j \rangle = 0$. Q.E.D.

Let $G = \{g_1, \ldots, g_n\}$, $\hat{G} = \{\chi_1, \ldots, \chi_n\}$, and consider the matrix $X = (\chi_j(g_i))_{1 \leq i,j \leq n}$ and its conjugate transpose $X^* = ((\chi_i(g_j)^*))_{1 \leq i,j \leq n}$. We claim that $X^* \cdot X = n \cdot \mathrm{I}$. To see this, consider the entry at position i, j:

$$\begin{aligned} (X^* \cdot X)_{ij} &= \sum_{k=1}^{n} X^*_{ik} \cdot X_{kj} \\ &= \sum_{k=1}^{n} \chi_i(g_k)^* \cdot \chi_j(g_k) \\ &= \langle \chi_i \,|\, \chi_j \rangle \\ &= \begin{cases} n & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that also $X \cdot X^* = n \cdot \mathrm{I}$, i.e.

$$\sum_{k=1}^{n} \chi_k(g_i) \cdot \chi_k(g_j)^* = \begin{cases} n & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Corollary 3.10. Let $(G, +)$ be a finite abelian group, $g \in G$ and $\chi \in \hat{G}$.

(a) $\sum_{k=1}^{n} \chi(g_k) = \begin{cases} n & \text{if } \chi = 1, \\ 0 & \text{otherwise.} \end{cases}$

(b) $\sum_{k=1}^{n} \chi_k(g) = \begin{cases} n & \text{if } g = 0, \\ 0 & \text{otherwise.} \end{cases}$

Proof. To prove (a), note that

$$\sum_{k=1}^{n} \chi(g_k) = \langle 1 \,|\, \chi \rangle = \begin{cases} n & \text{if } \chi = 1, \\ 0 & \text{otherwise.} \end{cases}$$

To prove (b), it suffices to apply (3.2) with $g_i = g$ and $g_j = 0$:

$$\sum_{k=1}^{n} \chi_k(g) = \sum_{k=1}^{n} \chi_k(g) \cdot \chi_k(0)^* = \begin{cases} n & \text{if } g = 0, \\ 0 & \text{otherwise.} \end{cases}$$ Q.E.D.

Example 3.11. For $G = \mathbb{Z}_n$, the characters are the mappings χ_y, $y \in \mathbb{Z}_n$, with $\chi_y(x) = \mathrm{e}^{2\pi \mathrm{i} xy/n}$. Hence,

$$\sum_{y \in \mathbb{Z}_n} \mathrm{e}^{2\pi \mathrm{i} \frac{xy}{n}} = \begin{cases} n & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $G = \mathbb{Z}_2^m$, the characters are the mappings χ_y, $y \in \mathbb{Z}_2^m$, with $\chi_y(x) = (-1)^{x \cdot y}$. Hence,

$$\sum_{y \in \mathbb{Z}_n} (-1)^{x \cdot y} = \begin{cases} 2^m & \text{if } x = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we can define the Fourier transformation. By Theorem 3.9, the vectors $B_i = 1/\sqrt{n} \cdot \chi_i$ form a basis of $\mathbb{C}^G$. The discrete Fourier transform of f is the function $\hat{f}$ that maps the elements of G to the coefficients in the unique representation of f according to this basis.

Definition 3.12. Let $(G, +)$ be a finite abelian group with elements $g_1, \ldots, g_n$, and let $B_1, \ldots, B_n$ be the Fourier basis of $\mathbb{C}^G$. Given a function $f = \hat{f}_1 \cdot B_1 + \cdots + \hat{f}_n \cdot B_n \in \mathbb{C}^G$, its *discrete Fourier transform (DFT)* is the function $\hat{f} \colon G \to \mathbb{C} : g_i \to \hat{f}_i$.

How can we compute the DFT of a given function f? It turns out that $\hat{f}$ can be computed via the conjugate transpose of the matrix $X = (\chi_j(g_i))_{1 \le i,j \le n}$ as defined above.

Theorem 3.13. Let $(G, +)$ be a finite abelian group with elements $g_1, \ldots, g_n$ and characters $\chi_1, \ldots, \chi_n$, and let $X = (\chi_j(g_i))_{1 \le i,j \le n}$. With respect to the standard basis, for any function $f \colon G \to \mathbb{C}$, we have $\hat{f} = 1/\sqrt{n} \cdot X^* \cdot f$, i.e.

$$\begin{pmatrix} \hat{f}(g_1) \\ \hat{f}(g_2) \\ \vdots \\ \hat{f}(g_n) \end{pmatrix} = \frac{1}{\sqrt{n}} \cdot \begin{pmatrix} \chi_1(g_1)^* & \cdots & \chi_1(g_n)^* \\ \chi_2(g_1)^* & \cdots & \chi_2(g_n)^* \\ \vdots & & \vdots \\ \chi_n(g_1)^* & \cdots & \chi_n(g_n)^* \end{pmatrix} \begin{pmatrix} f(g_1) \\ f(g_2) \\ \vdots \\ f(g_n) \end{pmatrix}.$$

Proof. Since $\{B_1, \ldots, B_n\}$ is an orthonormal basis, we have

$$\langle B_i \mid f \rangle = \sum_{j=1}^{n} \langle B_i \mid \hat{f}_j \cdot B_j \rangle = \sum_{j=1}^{n} \hat{f}_j \cdot \langle B_i \mid B_j \rangle = \hat{f}_i$$

and therefore

$$\hat{f}(g_i) = \hat{f}_i = \langle B_i \mid f \rangle = \langle 1/\sqrt{n} \cdot \chi_i \mid f \rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_i(g_k)^* \cdot f(g_k).$$

Q.E.D.

Corollary 3.14 (Parseval's theorem)**.** Let $f \colon G \to \mathbb{C}$ and $\hat{f}$ the DFT of f. Then $\|\hat{f}\| = \|f\|$.

Proof. Since $X^* \cdot X = n \cdot \mathrm{I}$, the matrix $1/\sqrt{n} \cdot X^*$ is unitary. Hence, $\|\hat{f}\| = \|1/\sqrt{n} \cdot X^* \cdot f\| = \|f\|$. Q.E.D.

The mapping $f \mapsto 1/\sqrt{n} \cdot X \cdot f$ (wrt. the standard basis) is called the *inverse Fourier transform*.

Example 3.15. For $G = \mathbb{Z}_n$ the characters are χ_y, $y \in \mathbb{Z}_n$, with $\chi_y(x) = \mathrm{e}^{2\pi \mathrm{i} xy/n}$. Hence, the Fourier transform of $f \colon \mathbb{Z}_n \to \mathbb{C}$ is

$$\hat{f} \colon \mathbb{Z}_n \to \mathbb{C} \colon x \mapsto \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} \mathrm{e}^{-2\pi \mathrm{i} xy/n} f(y),$$

and its inverse Fourier transform is the function

$$\tilde{f} \colon \mathbb{Z}_n \to \mathbb{C} \colon x \mapsto \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} \mathrm{e}^{2\pi \mathrm{i} xy/n} f(y).$$

For $G = \mathbb{Z}_2^m$ the characters are χ_y, $y \in \mathbb{Z}_2^m$, with $\chi_y(x) = (-1)^{x \cdot y}$. The Fourier transform of $f \colon \mathbb{Z}_2^m \to \mathbb{C}$ is

$$\hat{f} \colon \mathbb{Z}_2^m \to \mathbb{C} \colon x \mapsto \frac{1}{\sqrt{2^m}} \sum_{y \in \mathbb{Z}_2^m} (-1)^{x \cdot y} f(y).$$

The same function is also the inverse Fourier transform of f.

3.4 Quantum Fourier transformation

Let $(G, +)$ be a finite abelian group with elements $g_1, \ldots, g_n$ and characters $\chi_1, \ldots, \chi_k$, and consider the n-dimensional Hilbert space with basis $\{|g_1\rangle, \ldots, |g_1\rangle\}$. Every state $|\psi\rangle$ of H_G can be described by the function $f\colon G \to \mathbb{C}$ with $|\psi\rangle = \sum_{g \in G} f(g) \cdot |g\rangle$, i.e. $f(g) = \langle g \mid \psi\rangle$.

Definition 3.16. Let $(G, +)$ be a finite abelian group; $G = \{g_1, \ldots, g_n\}$ and $\hat{G} = \{\chi_1, \ldots, \chi_k\}$. The mapping

$$\mathrm{QFT}\colon H_G \to H_G\colon \sum_{i=1}^{n} f(g_i) \cdot |g_i\rangle \mapsto \sum_{i=1}^{n} \hat{f}(g_i) \cdot |g_i\rangle$$

is called the *quantum Fourier transformation (QFT)*. In particular,

$$\mathrm{QFT}\,|g\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \chi_k(g)^* \cdot |g_k\rangle$$

for all $g \in G$.

Lemma 3.17. QFT is a unitary transformation.

Proof. Follows from Corollary 3.14. Q.E.D.

How can we implement QFT by a QGA with elementary gates? To do this, we will follow a bottom-up process. Let $G = \{g_1, \ldots, g_m\}$ and $G' = \{g_1, \ldots, g'_n\}$ with dual groups $\hat{G} = \{\chi_1, \ldots, \chi_m\}$ and $\hat{G}' = \{\chi'_1, \ldots, \chi'_n\}$. From G and G' we can build a new group $G \oplus G' = \{g + g'\colon g \in G, g' \in G'\}$, the direct sum of G and G'. (Formally, the domain of $G \oplus G'$ is the cartesian product of G and G', and addition is applied componentwise). The corresponding Hilbert space is $H_{G\oplus G'} = H_G \otimes H_{G'}$ with basis vectors $|g\rangle \otimes |g'\rangle$, $g \in G$, $g' \in G'$.

By Theorem 3.7, the dual group of $G \oplus G'$ is isomorphic to $\hat{G} \times \hat{G}'$. Hence, the characters of $G \oplus G'$ are χ_{ij}, $1 \le i \le m$, $1 \le j \le n$, with $\chi_{ij}(g + g') = \chi_i(g) \cdot \chi'_j(g')$ for all $g \in G$ and all $g' \in G'$.

How does QFT behave on $H_{G\oplus G'}$? For a basis vector $|g_i\rangle|g'_j\rangle = |g_i\rangle \otimes |g'_j\rangle$, we have

$$\begin{aligned}
\mathrm{QFT}\,|g_i\rangle|g'_j\rangle &= \frac{1}{\sqrt{mn}} \sum_{k=1}^{m} \sum_{l=1}^{n} \chi_{ij}(g_k + g'_l)^* \cdot |g_k\rangle|g'_l\rangle \\
&= \frac{1}{\sqrt{mn}} \sum_{k=1}^{m} \sum_{l=1}^{n} \left(\chi_i(g_k)^*|g_k\rangle \otimes \chi_j(g'_l)^*|g'_l\rangle\right) \\
&= \Big(\frac{1}{\sqrt{m}} \sum_{k=1}^{m} \chi_i(g_k)^*|g_k\rangle\Big) \otimes \Big(\frac{1}{\sqrt{n}} \sum_{l=1}^{n} \chi_j(g'_l)^*|g_l\rangle\Big) \\
&= \mathrm{QFT}\,|g_i\rangle \otimes \mathrm{QFT}\,|g'_j\rangle
\end{aligned}$$

Example 3.18. Consider the group $G = \mathbb{Z}_2{}^m$ (the m-fold direct product of $\mathbb{Z}_2$). Then QFT on the Hilbert space H_G is equivalent to $\mathrm{H}^{\otimes m}$ since for all $x = x_1 \ldots x_m \in \{0,1\}^m$ we have

$$\begin{aligned}
\mathrm{H}^{\otimes m}\,|x\rangle &= \bigotimes_{i=1}^{m} \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{x_i}|1\rangle) \\
&= \frac{1}{\sqrt{2^m}} \sum_{y_1\ldots y_m \in \{0,1\}^m} (-1)^{x_1y_1 + \cdots + x_my_m} \cdot |y\rangle \\
&= \frac{1}{\sqrt{2^m}} \sum_{y \in \{0,1\}^m} (-1)^{x\cdot y} \cdot |y\rangle \\
&= \mathrm{QFT}\,|x\rangle.
\end{aligned}$$

We are interested in QFT for the group $G = \mathbb{Z}_n$, $n \in \mathbb{N}$. For this group, we have $\mathrm{QFT}\,|x\rangle = \sum_{y=0}^{n-1} \mathrm{e}^{-2\pi \mathrm{i} xy/n} \cdot |y\rangle$ for all $x \in \{0, \ldots, n-1\}$. If $n = p \cdot q$ with $\gcd(p, q) = 1$, then $\mathbb{Z}_n \cong \mathbb{Z}_p \times \mathbb{Z}_q$, and QFT on $\mathbb{Z}_n$ can be composed from QFT on $\mathbb{Z}_p$ and QFT on $\mathbb{Z}_q$. However, in most applications no factorisation of n is known, or $n = 2^m$ and no two factors are relatively prime.

For $G = \mathbb{Z}_{2^m}$, instead of QFT, let us look at the inverse QFT. For $x = \sum_{i=0}^{m-1} x_i \cdot 2^i \in \mathbb{Z}_{2^m}$, we identify the basis vector $|x\rangle$ in H_G with the corresponding basis vector in H_{2^m}, i.e. $|x\rangle = |x_{m-1} \ldots x_0\rangle$. On H_{2^m}, the inverse QFT on G corresponds to the transformation

$$\mathrm{IQFT}_m\colon H_{2^m} \to H_{2^m}\colon |x\rangle \mapsto \frac{1}{\sqrt{2^m}} \sum_{y \in \mathbb{Z}_{2^m}} \mathrm{e}^{2\pi\mathrm{i}\cdot xy/2^m} \cdot |y\rangle.$$

Lemma 3.19. $\mathrm{IQFT}_m\,|x\rangle$ is decomposable for all $x \in \mathbb{Z}_{2^m}$ and all $m > 0$:

$$\sum_{y \in \mathbb{Z}_{2^m}} \mathrm{e}^{2\pi\mathrm{i}\cdot xy/2^m} \cdot |y\rangle = \bigotimes_{l=0}^{m-1} \big(|0\rangle + \mathrm{e}^{\pi\mathrm{i}\cdot x/2^l} \cdot |1\rangle\big).$$

Proof. The proof is by induction on m. For $m = 1$, the statement is trivial. Hence, let $m > 1$ and assume that IQFT_{m-1} is decomposable. For all $x \in \mathbb{Z}_{2^m}$, we have:

$$\begin{aligned}
&\sum_{y \in \mathbb{Z}_{2^m}} \mathrm{e}^{2\pi\mathrm{i}\cdot xy/2^m} \cdot |y\rangle \\
&= \sum_{z \in \mathbb{Z}_{2^{m-1}}} \Big(\mathrm{e}^{2\pi\mathrm{i}\cdot x\cdot 2z/2^m} \cdot |z0\rangle + \mathrm{e}^{2\pi\mathrm{i}\cdot x(2z+1)/2^m} \cdot |z1\rangle\Big) \\
&= \sum_{z \in \mathbb{Z}_{2^{m-1}}} \Big(\mathrm{e}^{2\pi\mathrm{i}\cdot xz/2^{m-1}}|z0\rangle + \mathrm{e}^{2\pi\mathrm{i}\cdot xz/2^{m-1}} \mathrm{e}^{2\pi\mathrm{i}\cdot x/2^m}|z1\rangle\Big) \\
&= \Big(\sum_{z \in \mathbb{Z}_{2^{m-1}}} \mathrm{e}^{2\pi\mathrm{i}\cdot xz/2^{m-1}} \cdot |z\rangle\Big) \otimes \big(|0\rangle + \mathrm{e}^{2\pi\mathrm{i}\cdot x/2^m} \cdot |1\rangle\big) \\
&= \bigotimes_{l=0}^{m-2} \big(|0\rangle + \mathrm{e}^{\pi\mathrm{i}\cdot x/2^l}|1\rangle\big) \otimes \big(|0\rangle + \mathrm{e}^{\pi\mathrm{i}\cdot x/2^{m-1}} \cdot |1\rangle\big) \\
&= \bigotimes_{l=0}^{m-1} \big(|0\rangle + \mathrm{e}^{\pi\mathrm{i}\cdot [x]/2^l} \cdot |1\rangle\big).
\end{aligned}$$

Q.E.D.

Let $x = \sum_{i=0}^{2^m} x_i \cdot 2^i \in \mathbb{Z}_{2^m}$ and consider the operation of IQFT_m on the lth qubit:

$$|x_l\rangle \mapsto \frac{1}{\sqrt{2}}\big(|0\rangle + \mathrm{e}^{\pi\mathrm{i}\cdot x/2^l} \cdot |1\rangle\big).$$

We have

$$\mathrm{e}^{\pi\mathrm{i}\cdot x/2^l} = \prod_{k=0}^{m-1} \mathrm{e}^{\pi\mathrm{i}\cdot x_k/2^{l-k}} = \prod_{k=0}^{l} \mathrm{e}^{\pi\mathrm{i}\cdot x_k/2^{l-k}} = (-1)^{x_l} \prod_{\substack{k<l\\ x_k=1}} \mathrm{e}^{\pi\mathrm{i}/2^{l-k}}.$$

Hence, IQFT_m operates on the lth qubit like a Hadamard transformation, followed by a phase shift that depends on the qubits $|x_k\rangle$ for $k < l$. Formally, for $j \in \mathbb{N}$ define

$$R_j = \begin{pmatrix} 1 & 0 \\ 0 & e^{\pi\mathrm{i}/2^j} \end{pmatrix}.$$

In particular, $R_1 = \mathrm{S}$ and $R_2 = \mathrm{T}$. Then

$$\mathrm{IQFT}_m\,|x\rangle = \bigotimes_{l=0}^{m-1} \Big(\prod_{\substack{k<l\\ x_k=1}} R_{l-k}\Big) \mathrm{H}\,|x_l\rangle$$

for all $x \in \{0,1\}^m$. It follows that we can implement IQFT_m using $\mathrm{O}(m^2)$ Hadamard and controlled R_j gates.

Theorem 3.20. For all $m > 0$, IQFT_m can be implemented using $\mathrm{O}(m^2)$ Hadamard and controlled R_j gates, $j = 1, \ldots, m-1$.

QFT AND PERIODICAL FUNCTIONS. Let $f\colon \mathbb{Z}_n \to \mathbb{C}$ be a function with period $p \in \mathbb{Z}_n$, i.e. $f(m+p) = f(m)$ for all $m \in \mathbb{Z}_n$. For all $x \in \mathbb{Z}_n$, we have

$$\begin{aligned}
\hat{f}(x) &= \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} \mathrm{e}^{-2\pi\mathrm{i}xy/n} f(y) \\
&= \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} \mathrm{e}^{-2\pi\mathrm{i}xy/n} f(y+p) \\
&= \mathrm{e}^{2\pi\mathrm{i}xp/n} \cdot \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} \mathrm{e}^{-2\pi\mathrm{i}x(y+p)/n} f(y+p) \\
&= \mathrm{e}^{2\pi\mathrm{i}xp/n} \cdot \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}_n} \mathrm{e}^{-2\pi\mathrm{i}xy/n} f(y) \\
&= \mathrm{e}^{2\pi\mathrm{i}xp/n} \cdot \hat{f}(x)
\end{aligned}$$

Hence, if $\hat{f}(x) \neq 0$, then $\mathrm{e}^{2\pi\mathrm{i}xp/n} = 1$ and therefore $n \mid xp$.

We conclude that the Fourier transform of a function with period p can only take non-zero values on arguments x of the form $x = k \cdot n/p$.

3.5 Shor's factorisation algorithm

We can finally turn to Shor's algorithm for factoring a composite number n, i.e. the task to find given n numbers $p, q < n$ such that $n = p \cdot q$. The general idea in almost all good factorisation algorithms is to find numbers $b, c < n$ such that

$$b^2 \equiv c^2 \pmod{n}, \tag{3.3}$$
$$b \not\equiv \pm c \pmod{n}. \tag{3.4}$$

We then have $(b+c)(b-c) \equiv 0 \pmod{n}$, but $b + c \not\equiv 0 \pmod{n}$ and $b - c \not\equiv 0 \pmod{n}$. Hence, $b + c$ contains a factor of n, which can be extracted by computing $\gcd(b+c, n)$ in polynomial time, e.g. using Euklid's algorithm.

Shor's algorithm computes

$$r := \mathrm{ord}_n(a) = \min\{k > 0\colon a^k = 1 \pmod{n}\}$$

for a randomly chosen $a < n$ with $\gcd(a, n) = 1$. If we are lucky, then r is even and $a^{r/2} \not\equiv -1 \pmod{n}$. In this case, $b = a^{r/2}$ and $c = 1$ satisfy (3.3) and (3.4).

What is the probability that we are lucky? We can assume without loss of generality that n is neither even nor a prime power because it is easy to decide whether $n = 2^l \cdot m$ or $n = a^k$ and to compute suitable numbers l, m or a, k if so.

Lemma 3.21. Let $n \in \mathbb{N}$ be neither even nor a prime power, and let $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n\colon \gcd(a, n) = 1\}$. Then

$$\Pr_{a \in \mathbb{Z}_n^*}[\mathrm{ord}_n(a) \text{ is even and } a^{\mathrm{ord}_n(a)/2} \not\equiv -1 \pmod{n}] \geq \frac{9}{16}.$$

To prove this lemma, we need to make a small digression into number theory.

3.5.1 *Number theory in a nutshell*

For $n \in \mathbb{N}$, let $\mathbb{Z}_n^*$ the set of all $a \in \mathbb{Z}_n$ with $\gcd(a, n) = 1$; we denote by $\varphi(n)$ the cardinality of $\mathbb{Z}_n^*$. When equipped with multiplication mod n, the set $\mathbb{Z}_n^*$ forms an abelian group.

For prime numbers p, we have $\mathbb{Z}_p^* = \{1, 2, \ldots, p-1\}$ and $\varphi(p) = p - 1$. In this case, the group $(\mathbb{Z}_p^*, \cdot)$ is is isomorphic to the cyclic group $(\mathbb{Z}_{p-1}, +)$. More generally, if $n = p^k$ is a prime power, then

$$\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n\colon a \neq 0, p, 2p, \ldots, (p^{k-1} - 1)p\}$$

and $\varphi(n) = p^k - p^{k-1} = p^{k-1}(p-1)$.

Theorem 3.22. Let $n = p^k$ for a prime $p > 2$ and $k \geq 1$. Then the group $(\mathbb{Z}_n^*, \cdot)$ is cyclic.

Proof. We prove that there exists an element $b \in \mathbb{Z}_n^*$ with $\mathrm{ord}_n(b) = \varphi(n) = p^{k-1}(p-1)$. We prove this by establishing the following three facts:

(1) there exists $b \in \mathbb{Z}_n^*$ with $\mathrm{ord}_n(b) = p - 1$;
(2) $\mathrm{ord}_n(1+p) = p^{k-1}$;
(3) if $(G, \cdot)$ is an abelian group and $g, h \in G$ with $\mathrm{ord}_G(g)$ and $\mathrm{ord}_G(h)$ being relatively prime, then $\mathrm{ord}_G(g \cdot h) = \mathrm{ord}_G(g) \cdot \mathrm{ord}_G(h)$.

It follows that $\mathrm{ord}_n(b \cdot (1+p)) = \varphi(n)$.

We start by proving (1). Consider the natural homomorphism

$$f\colon \mathbb{Z}_n^* \to \mathbb{Z}_p^*\colon a \mapsto a \pmod{p}.$$

Since $\mathbb{Z}_p^*$ is cyclic and f is surjective, there exists $a \in \mathbb{Z}_n^*$ with $\mathrm{ord}_p(f(a)) = p - 1$. Let $r := \mathrm{ord}_n(a)$. Since $a^r \equiv 1 \pmod{p}^k$, we have $f(a)^r = q \pmod{p}$ and therefore $r = l(p-1)$ for some $l \in \mathbb{N}$. Set $b := a^l$. We have $b^{p-1} = a^r \equiv 1 \mod n$. On the other hand, whenever $b^s \equiv 1 \pmod{n}$, then $(p-1) \mid s$ because if $b^s \equiv 1 \pmod{n}$, then also $a^{l \cdot s} \equiv 1 \mod n$ and therefore $r = l(p-1) \mid l \cdot s$. Hence, $\mathrm{ord}_n(b) = p - 1$.

To prove (2), we first prove that for all $m > 0$ we have $(1+p)^{p^m} = 1 + \lambda p^{m+1}$ for some $\lambda \in \mathbb{N}$ such that $p \nmid \lambda$. We prove this by induction over m. For $m = 1$, we have

$$\begin{aligned}(1+p)^p &= \sum_{i=0}^{p} \binom{p}{i} \cdot p^i \\ &= 1 + p^2 + \sum_{i=3}^{p} \binom{p}{i} \cdot p^i \qquad \text{(since } p > 2\text{)} \\ &= 1 + p^2 + p^3 \cdot \underbrace{\sum_{i=3}^{p} \binom{p}{i} \cdot p^{i-3}}_{l} \\ &= 1 + p^2(1 + l \cdot p),\end{aligned}$$

which proves the statement since $(1 + l \cdot p) \nmid p$.

Now let $m > 1$ and assume that the statement holds for $m - 1$. We have:

$$\begin{aligned}(1+p)^{p^m} &= (1+p)^{p^{m-1}\cdot p} \\ &= (1 + \lambda \cdot p^m)^p \\ &= \sum_{i=0}^{p} \binom{p}{i} \lambda^i p^{mi} \\ &= 1 + \lambda p^{m+1} + \sum_{i=2}^{p} \binom{p}{i} \lambda^i p^{mi} \\ &= 1 + \lambda p^{m+1} + p^{m+2} \cdot \underbrace{\sum_{i=2}^{p} \binom{p}{i} \lambda^i p^{m(i-1)-2}}_{l} \\ &= 1 + p^{m+1}(\lambda + lp).\end{aligned}$$

Since $\lambda \nmid p$, we also have $(\lambda + lp) \nmid p$, which proves the statement.

It follows that there exist $\lambda_1, \lambda_2 \in \mathbb{N}$ with $p \nmid \lambda_1$ and $p \nmid \lambda_2$ such that

$$\begin{aligned}(1+p)^{p^{k-1}} &= 1 + \lambda_1 \cdot p^k \quad \equiv 1 \pmod{n}; \\ (1+p)^{p^{k-2}} &= 1 + \lambda_2 \cdot p^{k-1} \not\equiv 1 \pmod{n}.\end{aligned}$$

Hence, $\text{ord}_n(1+p) \mid p^{k-1}$ but $\text{ord}_n(1+p) \nmid p^{k-2}$. Thus, $\text{ord}_n(1+p) = p^{k-1}$.

It remains to prove (3). Let $r = \text{ord}_G(g)$ and $s = \text{ord}_G(h)$ with $\gcd(r,s) = 1$. Clearly, $(gh)^{rs} = 1$ and therefore $\text{ord}_G(gh) \mid rs$. On the other hand, assume that $(gh)^t = 1$. We have $1^r = (gh)^{ts} = g^{ts} \cdot h^{ts} = g^{ts} \cdot 1^t = g^{ts}$ and therefore $r \mid ts$. Since $\gcd(r,s) = 1$, this implies $r \mid t$, and an analogous argument shows that $s \mid t$. Hence, also $rs \mid t$, which proves that $\text{ord}_G(gh) = rs$. Q.E.D.

Remark 3.23. Theorem 3.22 does not hold for $p = 2$. For instance, we have $\mathbb{Z}_8^* = \{1,3,5,7\}$ with $3^2 \equiv 5^2 \equiv 7^2 \equiv 1 \pmod{n}$. Hence, the group $(\mathbb{Z}_8^*, \cdot)$ is isomorphic to $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$, the *Klein four-group*.

Let n be an odd prime power, i.e. $n = p^e$ for some prime $p > 2$. Since $\mathbb{Z}_n^*$ is cyclic, there exists a generator g of this group, i.e. $\mathbb{Z}_n^* = \{g, g^2, \ldots, g^{\varphi(n)}\}$. Moreover, $\varphi(n) = \varphi(p^e) = p^{e-1}(p-1) = 2^d \cdot u$ for $d \geq 1$ and an odd number u.

Lemma 3.24. Let $n = p^e$, $p > 2$, $\varphi(n) = 2^d \cdot u$ with $2 \nmid u$, and let g be a generator of $\mathbb{Z}_n^*$. Then $i \in \mathbb{N}$ is odd if and only if $2^d \mid \text{ord}_n(g^i)$.

Proof. ($\Rightarrow$) Let $i \in \mathbb{N}$ be odd. We have $g^{i \cdot \text{ord}_n(g^i)} \equiv 1 \pmod{n}$ and therefore $\varphi(n) \mid i \cdot \text{ord}_n(g^i)$. Since $\varphi(n) = 2^d \cdot u$ and i is odd, this implies that $2^d \mid \text{ord}_n(g^i)$.

($\Leftarrow$) Let $i \in \mathbb{N}$ be even. We have $g^{i \cdot \varphi(n)/2} = g^{\varphi(n) \cdot i/2} \equiv 1 \pmod{n}$ and therefore $\text{ord}_n(g^i) \mid \varphi(n)/2$. Since $2^d \nmid \varphi(n)/2$, this implies that $2^d \nmid \text{ord}_n(g^i)$. Q.E.D.

Corollary 3.25. Let $n = p^e$, $p > 2$, and $\varphi(n) = 2^d \cdot u$ with $2 \nmid u$. Then

$$\Pr_{a \in \mathbb{Z}_n^*}[2^d \mid \text{ord}_n(a)] = \frac{1}{2}.$$

Finally, we can prove Lemma 3.21.

Proof (of Lemma 3.21). Let $n \in \mathbb{N}$ be neither even nor a prime power. Hence, $n = p_1^{e_1} \cdots p_r^{e_k}$, $k > 1$ for primes $p_i > 2$ such that $p_i \neq p_j$ for

$i \neq j$. The *Chinese remainder theorem* tells us that the mapping

$$\mathbb{Z}_n^* \to \mathbb{Z}_{p_1^{e_1}}^* \times \cdots \times \mathbb{Z}_{p_k^{e_k}} : a \to (a \bmod p_1^{e_1}, \ldots, a \bmod p_k^{e_k})$$

is an isomorphism. In particular, we have

$$\varphi(n) = \prod_{i=1}^{k} \varphi(p_i^{e_i}) = \prod_{i=1}^{k} p_i^{e_i - 1}(p_i - 1).$$

Moreover, for $a \in \mathbb{Z}_n^*$ we have $\mathrm{ord}_n(a) = \gcd(\mathrm{ord}_{p_1^{e_1}}(a), \ldots, \mathrm{ord}_{p_k^{e_k}}(a))$ because, by the Chinese remainder theorem, $a^r \equiv 1 \pmod{n}$ is equivalent to $a^r \equiv 1 \pmod{p_i^{e_i}}$ for all i, and the latter holds if and only if $\mathrm{ord}_{p_i^{e_i}}(a) \mid r$.

By the Chinese remainder theorem, a random choice of $a \in \mathbb{Z}_n^*$ corresponds to a random choice of $a_1, \ldots, a_k$ with $a_i \in \mathbb{Z}_{p_i^{e_i}}$. For $a \in \mathbb{Z}_n^*$, let $r_i = \mathrm{ord}_{p_i^{e_i}}(a)$. Then $\mathrm{ord}_n(a) = \gcd(r_1, \ldots, r_k)$ is odd if and only if each r_i is odd. It follows from Corollary 3.25 that $\Pr_{a \in \mathbb{Z}_n^*}[r_i \text{ is odd}] \leq \frac{1}{2}$ and $\Pr_{a \in \mathbb{Z}_n^*}[\mathrm{ord}_n(a) \text{ is odd}] \leq \frac{1}{2^k}$.

Assume now that $r = \mathrm{ord}_n(a)$. If $a^{r/2} \equiv -1 \pmod{n}$, then $n \,/\, a^{r/2} + 1$. But then also $p_i^{e_i} \mid a^{r/2} + 1$ and therefore $a^{r/2} \equiv -1 \pmod{p_i^{e_i}}$ for all $i = 1, \ldots, k$. Since $a^{r_i} \equiv 1 \pmod{p_i^{e_i}}$ and $p_i > 2$, this implies that $r_i \nmid \frac{r}{2}$ for all i. For $r = 2^d \cdot u$ (where u is odd), this means that $2^d \mid r_i$ for all $i = 1, \ldots, k$. Hence,

$$\begin{aligned}
&\Pr_{a \in \mathbb{Z}_n^*}[a^{\mathrm{ord}_n(a)/2} \equiv -1 \pmod{n} \mid \mathrm{ord}_n(a) \text{ is even}] \\
&\leq \Pr_{a \in \mathbb{Z}_n^*}[2^d \mid \mathrm{ord}_{p_i^{e_i}}(a) \text{ for all } i] \\
&= \frac{1}{2^k},
\end{aligned}$$

where the last equality follows from Corollary 3.25. Finally,

$$\begin{aligned}
&\Pr_{a \in \mathbb{Z}_n^*}[2 \mid \mathrm{ord}_n(a) \text{ and } a^{\mathrm{ord}_n(a)/2} \not\equiv -1 \pmod{n}] \\
&= \Pr_{a \in \mathbb{Z}_n^*}[2 \mid \mathrm{ord}_n(a)] \cdot \Pr_{a \in \mathbb{Z}_n^*}[a^{\mathrm{ord}_n(a)/2} \not\equiv -1 \pmod{n} \mid 2 \mid \mathrm{ord}_n(a)] \\
&\geq (1 - \frac{1}{2^k}) \cdot (1 - \frac{1}{2^k})
\end{aligned}$$

$$\geq \frac{3}{4} \cdot \frac{3}{4} \geq \frac{9}{16}$$

Q.E.D.

3.5.2 Factoring and QFT

To sum up, we can reduce factoring to the problem of computing, given a number $n \in \mathbb{N}$ that is neither odd nor a prime power, the order $\mathrm{ord}_n(a)$ of $a \in \mathbb{Z}_n^*$. The number $r = \mathrm{ord}_n(a)$ is the period of the function

$$f : \mathbb{Z} \to \mathbb{Z}_n : x \mapsto a^x \bmod n$$

since $f(x + r) \equiv a^{x+r} \equiv a^x \cdot a^r \equiv a^x \pmod{n}$. We can use QFT to determine this period! However, QGAs only operate on the Hadamard space H_{2^m}. Hence, we choose a sufficiently large number $m \in \mathbb{N}$ such that the period of f occurs in $\mathbb{Z}_{2^m}$: in fact, we can always take the unique number m such that $n^2 \leq 2^m < 2n^2$.

We can now give an informal description of Shor's algorithm. First, after having randomly chosen $a < n$, the algorithm computes the quantum state

$$|\psi\rangle = \frac{1}{\sqrt{2^m}} \sum_{x \in \mathbb{Z}_{2^m}} |x\rangle |a^x \bmod n\rangle \in H_{2^{m+k}},$$

where $2^k \leq n < 2^{k+1}$. Note that the function $x \mapsto a^x \bmod n$ is computable in polynomial time (by a classical circuit) and thus also by a QGA since for $x = \sum_{i=0}^{m-1} x_i \cdot 2^i$ we have $a^x \equiv \prod_{i:\, x_i = 1} a_i \pmod{n}$ where $a_0 = a$ and $a_{i+1} = a_i{}^2 \bmod n$ for all $i < m$.

Since $x \mapsto a^x \bmod n$ has period $r = \mathrm{ord}_n(a)$, we have

$$|\psi\rangle = \frac{1}{\sqrt{2^m}} \sum_{l=0}^{r-1} \sum_{q=0}^{s_l} |qr + l\rangle |a^l \bmod n\rangle,$$

where $s_l = \max\{s \in \mathbb{N} : sr + l < 2^m\}$.

The next step of the algorithm is to apply IQFT_m to the first m qubits of $|\psi\rangle$. The resulting state is

$$
\begin{aligned}
|\varphi\rangle &= \frac{1}{\sqrt{2^m}} \sum_{l=0}^{r-1} \sum_{q=0}^{s_l} \frac{1}{\sqrt{2^m}} \sum_{y \in \mathbb{Z}_{2^m}} \mathrm{e}^{2\pi\mathrm{i}\cdot y(qr+l)/2^m} |y\rangle |a^l \bmod n\rangle \\
&= \frac{1}{2^m} \sum_{l=0}^{r-1} \sum_{y=0}^{2^m-1} \mathrm{e}^{2\pi\mathrm{i}\cdot yl/2^m} \sum_{q=0}^{s_l} \mathrm{e}^{2\pi\mathrm{i}\cdot yr\cdot q/2^m} |y\rangle |a^l \bmod n\rangle
\end{aligned}
$$

Finally, the algorithm performs a measurement on the first m qubits of $|\varphi\rangle$, which yields $y \in \mathbb{Z}_{2^m}$. Then, with some luck, $y \approx k \cdot 2^m / r$ and $\gcd(k, r) = 1$. The number r can then be extracted using the method of *continued fractions* (see below).

Example 3.26. Let $n = 15$ and $a = 7$. In this case, it suffices to choose $m = 4$ (as opposed to $m = 8$). Hence,

$$
\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{16}} \sum_{x=0}^{15} |x\rangle |7^x \bmod 15\rangle \\
&= \frac{1}{4} \big(|0\rangle|1\rangle + |1\rangle|7\rangle + |2\rangle|4\rangle + \cdots + |15\rangle|13\rangle \big) \\
&= \frac{1}{4} \Big(\big(|0\rangle + |4\rangle + |8\rangle + |12\rangle\big)|1\rangle \\
&\qquad + \big(|1\rangle + |5\rangle + |9\rangle + |13\rangle\big)|7\rangle \\
&\qquad + \big(|2\rangle + |6\rangle + |1\rangle 0 + |1\rangle 4\big)|4\rangle \\
&\qquad + \big(|3\rangle + |7\rangle + |11\rangle + |15\rangle\big)|13\rangle \Big) \\
&= \sum_{j=0}^{4} \Big(\sum_{y=0}^{15} f_j(y)|y\rangle \Big) |7^j \bmod 15\rangle,
\end{aligned}
$$

where

$$
f_j(y) = \begin{cases} \frac{1}{4} & \text{if } y \equiv j \pmod 4 \\ 0 & \text{otherwise.} \end{cases}
$$

Each f_j has period 4. Hence, $\hat{f}_j(x) \neq 0$ only for $x \in \{0, 4, 8, 12\}$. For $k = 0, 1, 2, 3$, we have

$$
\begin{aligned}
\hat{f}_j(4k) &= \frac{1}{4} \sum_{y=0}^{15} \mathrm{e}^{2\pi\mathrm{i}\cdot 4k\cdot y/16} \cdot f_j(y) \\
&= \frac{1}{4} \sum_{l=0}^{3} \mathrm{e}^{2\pi\mathrm{i}\cdot 4k(4l+j)/16} \cdot \frac{1}{4} \\
&= \frac{1}{16} \sum_{l=0}^{3} \mathrm{e}^{2\pi\mathrm{i}\cdot 4k(4l+j)/16} \\
&= \frac{1}{16} \cdot \mathrm{e}^{\pi\mathrm{i}\cdot kj/2} \sum_{l=0}^{3} \mathrm{e}^{2\pi\mathrm{i}\cdot kl} \\
&= \frac{1}{16} \cdot \mathrm{e}^{\pi\mathrm{i}\cdot kj/2} \sum_{l=0}^{3} 1 \\
&= \frac{1}{4} \cdot \mathrm{e}^{\pi\mathrm{i}\cdot kj}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|\varphi\rangle = \frac{1}{4} \Big(&\big(|0\rangle + |4\rangle + |8\rangle + |12\rangle\big)|1\rangle \\
&+ \big(|0\rangle + \mathrm{i}|4\rangle - |8\rangle - \mathrm{i}|12\rangle\big)|7\rangle \\
&+ \big(|0\rangle - |4\rangle + |8\rangle - |12\rangle\big)|4\rangle \\
&+ \big(|0\rangle - \mathrm{i}|4\rangle - |8\rangle + \mathrm{i}|12\rangle\big)|13\rangle \Big).
\end{aligned}
$$

With probability $\frac{1}{4}$ each, a measurement of the first m qubits of $|\varphi\rangle$ yields $|0\rangle$, $|4\rangle$, $|8\rangle$ or $|12\rangle$, with probability $\frac{1}{4}$ each. From $|0\rangle$ and $|8\rangle$, the period $4 = \mathrm{ord}_{15}(7)$ cannot be extracted. However, for $y = 4, 12$ we have $y = 4k$ with $\gcd(k, 4) = 1$, and the period can be extracted.

The period $r = 4$ is even and $7^{r/2} = 7^2 - 4 \not\equiv -1 \pmod{15}$. Hence, $3 = 4 - 1$ and $5 = 4 + 1$ are identified as factors of 15.

The probability that a measurement of the first m qubits of $|\varphi\rangle$ returns $y \in \mathbb{Z}_{2^m}$ is

$$
\begin{aligned}
\Pr[y] &= \frac{1}{2^{2m}} \sum_{l=0}^{r-1} \Big| \mathrm{e}^{2\pi\mathrm{i}\cdot yl/2^m} \sum_{q=0}^{s_l} \mathrm{e}^{2\pi\mathrm{i}\cdot yrq/2^m} \Big|^2 \\
&= \frac{1}{2^{2m}} \sum_{l=0}^{r-1} \Big| \sum_{q=0}^{s_l} \mathrm{e}^{2\pi\mathrm{i}\cdot yrq/2^m} \Big|^2.
\end{aligned}
$$

If $r \mid 2^m$, i.e. for $r = 2^s$ with $s \leq m$, we know that $\Pr[y] \neq 0$ only if $k = \cdot 2^m/r$. Moreover, all these y occur with probability $1/r$ because $s_l = 2^{m-s} = 1$ for all $l < r$ and

$$
\begin{aligned}
\Pr[y] &= \frac{r}{2^{2m}} \Big| \sum_{q=0}^{2^{m-s}-1} \mathrm{e}^{2\pi \mathrm{i} \cdot yq/2^{m-s}} \Big|^2 \\
&= \frac{r}{2^{2m}} \Big| \sum_{q=0}^{2^{m-s}-1} \chi_q(y) \Big|^2 \\
&= \begin{cases} \frac{r}{2^{2m}} |2^{m-s}|^2 & \text{if } y \equiv 0 \pmod{2}^{m-s}, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{r}{2^{2m}} \cdot \frac{2^{2m}}{r^2} = \frac{1}{r} & \text{if } y = k \cdot 2^m/r, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}
$$

However, in general, we cannot assume that $r \mid 2^m$. For $l < r$, consider the summand $\sum_{q=0}^{s_l} |qr + l\rangle |a^l \bmod n\rangle$ of $|\psi\rangle$. This summand can be written as $\sum_{y \in \mathbb{Z}_{2^m}} f_l(y) |y\rangle |a^l \bmod n\rangle$, where

$$
f_l(y) = \begin{cases} 1 & \text{if } y \equiv l \pmod{r} \\ 0 & \text{otherwise.} \end{cases}
$$

Since $r \nmid 2^m$, the function $f_l \colon \mathbb{Z}_{2^m} \to \mathbb{C}$ is not exactly periodic. Hence, the Fourier transformation and subsequent measurement does not necessarily yield $y = k \cdot 2^m/r$. However, with high probability, it yields a $y \in \mathbb{Z}_{2^m}$ that is sufficiently close to such an element.

Lemma 3.27. Let $|\varphi\rangle$ be the quantum state obtained by Shor's algorithm on input $n \geq 100$ after applying IQFT_m. For all $k < r = \mathrm{ord}_n(a)$, a measurement of the first m qubits of $|\varphi\rangle$ yields the unique $y \in \mathbb{Z}_{2^m}$ such that $|y - k \cdot 2^m/r| \leq 1/2$ with probability $\geq 2/5r$.

Proof. By an elementary, but long calculation. Q.E.D.

It follows from Lemma 3.27 that a measurement of the first m qubits of $|\varphi\rangle$ yields $y \in \mathbb{Z}_{2^m}$ such that $|y - k \cdot 2^m/r| \leq 1/2$ for *some* $k \in \{0, \ldots, r-1\}$ with probability $\geq 2/5$. The probability that $\gcd(k, r) = 1$ for a randomly chosen $k \in \{0, \ldots, r-1\}$ is $\varphi(r)/r$.

Lemma 3.28. For all $r \geq 19$,

$$
\frac{\varphi(r)}{r} \geq \frac{1}{4 \log \log r}.
$$

Corollary 3.29. Let $|\varphi\rangle$ be the quantum state obtained by Shor's algorithm on input $n \geq 100$ after applying IQFT_m. A measurement of the first m qubits of $|\varphi\rangle$ yields an element $y \in \mathbb{Z}_{2^m}$ such that $|y - k \cdot 2^m/r| \leq 1/2$ for some $k < r$ with $\gcd(k, r) = 1$ with probability $\geq 1/(10 \log \log n)$.

For the obtained y with $|y - k \cdot 2^m/r| \leq 1/2$, it holds that

$$
\Big| \frac{y}{2^m} - \frac{k}{r} \Big| \leq \frac{1}{2 \cdot 2^m} \leq \frac{1}{2n^2} < \frac{1}{2r^2}.
$$

(Recall that m was chosen in a way such that $n^2 \leq 2^m$.)

It remains to show that we can extract r from y and 2^m efficiently. For this task, we will use the method of continued fractions, and we will prove that 1. we can compute all *convergents* of the continued fraction representation for a rational number x efficiently, and 2. if $x \in \mathbb{Q}$ and p and q are relatively prime such that $|x - p/q| \leq 1/2q^2$, then p/q is a convergent of the continued fraction representation for x.

3.5.3 *Continued fractions*

Every number $\alpha \in \mathbb{R}$ can be represented as a continued fraction

$$
[a_0, a_1, \ldots] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}},
$$

where $a_0 \in \mathbb{Z}$ and $a_n \in \mathbb{N} \setminus \{0\}$ for all $n > 0$. If α is irrational, then α has unique continued fraction representation, which is infinite. Rational numbers, on the other hand, have a two different finite continued fraction representations.

Example 3.30. Consider the rational number $x = \frac{31}{13}$. We have

$$\begin{aligned}
x &= 2 + \frac{5}{13} = 2 + \cfrac{1}{\frac{13}{5}} \\
&= 2 + \cfrac{1}{2 + \frac{3}{5}} = 2 + \cfrac{1}{2 + \cfrac{1}{\frac{5}{3}}} \\
&= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \frac{2}{3}}} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{\frac{3}{2}}}} \\
&= 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{2}}}} = 2 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \frac{1}{1}}}}} \\
&= [2, 2, 1, 1, 2] = [2, 2, 1, 1, 1, 1]
\end{aligned}$$

We will show that a continued fraction representation of a rational number p/q with $p, q < 2^n$ can be computed using Euklid's algorithm in $\mathrm{O}(n)$ basic steps. Note that we can form the expression

$$[a_0, a_1, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\begin{array}{c}\vdots\\ \hline a_{n-1} + \frac{1}{a_n}\end{array}}}}$$

for arbitrary numbers $a_0, a_1, \ldots, a_n \in \mathbb{R}_{>0}$. For $\alpha = [a_0, \ldots, a_n]$ and $j \leq n$, we call $[a_0, \ldots, a_j]$ the *jth convergent* of α.

Theorem 3.31. For $\alpha = [a_0, \ldots, a_n] \in \mathbb{R}$, we have $[a_0, \ldots, a_j] = p_j/q_j$ for all $j \leq n$, where

$$\begin{aligned}
p_0 &= a_0, & q_0 &= 1, \qquad (3.5)\\
p_1 &= 1 + a_0 \cdot a_1, & q_1 &= a_1, \qquad (3.6)\\
p_{j+2} &= a_{j+2} \cdot p_{j+1} + p_j, & q_{j+2} &= a_{j+2} \cdot q_{j+1} + q_j. \qquad (3.7)
\end{aligned}$$

Proof. We have

$$[a_0] = \frac{a_0}{1} = \frac{p_0}{q_0}$$

and

$$[a_0, a_1] = a_0 + \frac{1}{a_1} = \frac{a_0 \cdot a_1 + 1}{a_1} = \frac{p_1}{q_2},$$

which proves (3.5) and (3.6). We prove (3.7) by induction over j: We have

$$\begin{aligned}
[a_0, a_1, a_2] &= a_0 + \cfrac{1}{a_1 + \frac{1}{a_2}} \\
&= \frac{a_0 \cdot a_1 \cdot a_2 + a_0 + a_2}{a_1 \cdot a_2 + 1} \\
&= \frac{a_2(1 + a_0 \cdot a_1) + a_0}{a_2 \cdot a_1 + 1} \\
&= \frac{a_2 \cdot p_1 + p_0}{a_2 \cdot q_1 + q_0} = \frac{p_2}{q_2},
\end{aligned}$$

which establishes the base case. Now let $0 \leq j \leq n - 3$ and assume that p_{j+2} and q_{j+2} satisfy (3.7). Then

$$\begin{aligned}
[a_0, \ldots, a_{j+3}] &= [a_0, \ldots, a_{j+1}, a_{j+2} + 1/a_{j+3}] \\
&= \frac{(a_{j+2} = \frac{1}{a_{j+3}})p_{j+1} = p_j}{(a_{j+2} + \frac{1}{a_{j+3}})q_{j+1} + q_j} \\
&= \frac{a_{j+3}(a_{j+2} \cdot p_{j+1} + p_j) + p_{j+1}}{a_{j+3}(a_{j+2} \cdot q_{j+1} + q_j) + q_{j+1}} \\
&= \frac{a_{j+3} \cdot p_{j+2} + p_{j+1}}{a_{j+3} \cdot q_{j+2} + q_{j+1}} = \frac{p_{j+3}}{q_{j+3}},
\end{aligned}$$

which proves (3.7) for j replaced by $j + 1$. Q.E.D.

Corollary 3.32. For $\alpha = [a_0, \dots, a_n] \in \mathbb{R}$ such that $[a_0, \dots, a_j] = p_j/q_j$ for $j \leq n$, we have $p_{j-1} \cdot q_j - p_j \cdot q_{j-1} = (-1)^j$ for all $j \geq 1$.

It follows from Corollary 3.32 that $\gcd(p_j, q_j) = 1$ if $a_j \in \mathbb{N} \setminus \{0\}$ for all j. Hence, Euklid's algorithm can be used to obtain p_{j+1} and q_{j+1}. Moreover, by the definition of p_j, q_j, we have $p_0 < p_1 < \cdots < p_n$ and $q_0 < q_1 < \cdots < q_n$. More precisely,

$$p_{j+2} = a_{j+2} \cdot p_{j+1} + p_j \geq 2p_j$$

and analogously $q_{j+2} \geq 2q_j$. Hence, $p_n, q_n \geq 2^{\lfloor n/2 \rfloor}$.

This proves that any rational number p/q with $p, q < 2^n$ has a continued fraction representation $[a_0, \dots, a_m]$ with $m \leq 2n$.

Theorem 3.33. Let $p \in \mathbb{Z}$, $q \in \mathbb{N} \setminus \{0\}$ and $x \in \mathbb{Q}$ such that $\gcd(p, q) = 1$ and $|p/q - x| \leq 1/2q^2$. Then p/q is a convergent of the continued fraction representation for x.

Proof. Consider the continued fraction representation $[a_0, \dots, a_n]$ of p/q with convergents $p_1/q_1, \dots, p_n/q_n = p/q$. Since $[a_0, \dots, a_n] = [a_0, \dots, a_{n-1}, a_n - 1, 1]$, we can assume without loss of generality that n is even. Let $\delta \in \mathbb{R}$ be defined by the equation

$$x = \frac{p_n}{q_n} + \frac{\delta}{2 {q_n}^2}.$$

Since $|p/q - x| \leq 1/2q^2$ we have $|\delta| < 1$. Without loss of generality, $\delta > 0$. Set

$$\lambda := \frac{2}{\delta} \cdot (p_{n-1} \cdot q_n - p_n \cdot q_{n-1}) - \frac{q_{n-1}}{q_n}.$$

We have

$$\begin{aligned}
\lambda p_n + p_{n-1} &= \frac{2 \cdot p_n \cdot q_n \cdot (p_{n-1} \cdot q_n - p_n \cdot q_{n-1})}{\delta \cdot q_n} \\
&\quad - \frac{\delta \cdot q_{n-1} \cdot p_n + \delta \cdot q_n \cdot p_{n-1}}{\delta \cdot q_n} \\
&= \frac{(2 \cdot p_n \cdot q_n + \delta)(p_{n-1} \cdot q_n - p_n \cdot q_{n-1})}{\delta \cdot q_n}
\end{aligned}$$

and

$$\begin{aligned}
\lambda \cdot q_n + q_{n-1} &= \frac{2 \cdot q_n^2 (p_{n-1} \cdot q_n - p_n \cdot q_{n-1})}{\delta \cdot q_n} - q_{n-1} + q_{n-1} \\
&= \frac{2 \cdot q_n^2 (p_{n-1} \cdot q_n - p_n \cdot q_{n-1})}{\delta \cdot q_n}.
\end{aligned}$$

Hence,

$$\frac{\lambda p_n + p_{n-1}}{\lambda q_n + q_{n-1}} = \frac{2 \cdot p_n \cdot q_n + \delta}{2 q_n^2} = \frac{p_n}{q_n} + \frac{\delta}{2 q_n^2} = x.$$

By Theorem 3.31, this implies that $x = [a_0, \dots, a_n, \lambda]$. Since n is even, $p_{n-1} \cdot q_n - p_n \cdot q_{n-1} = 1$. Hence,

$$\lambda = \frac{2}{\delta} - \frac{q_{n-1}}{q_n} > 2 - 1 = 1.$$

Since λ is a rational number > 1, λ has a finite continued fraction representation $\lambda = [b_0, \dots, b_m]$ with $b_0 \geq 1$. Hence $x = [a_0, \dots, a_n, b_0, \dots, b_m]$ is a continued fraction representation of x with convergent p/q. Q.E.D.

3.5.4 Complexity

Shor's algorithm is summarised as Algorithm 3.1. To evaluate the time complexity and success probability of Shor's algorithm, let $k = \lfloor \log n \rfloor + 1$ the length of the binary representation of n. Hence, $m \leq 2k$.

Steps 1–2 of Shor's algorithm can be performed in time $O(k^3)$ and produce either a factor of n or confirm that n is neither even nor a prime power. Step 3 can also be performed in time $O(k^3)$ and produces either a factor of n or a randomly chosen element $a \in \mathbb{Z}_n^*$. As we have shown, Step 4 can be implemented by a QGA with $O(k^3)$ gates on 1 or 2 qubits. Step 5 also takes time $O(k^3)$ and succeeds with probability $\Omega(1/\log k)$ (see Corollary 3.29). Finally, Step 6 takes time $O(k^3)$ as well and succeeds with probability $\geq \frac{9}{16}$ (by Lemma 3.21).

Theorem 3.34. Shor's algorithm computes, given a composite number $n \in \mathbb{N}$, a non-trivial factor of n with probability $\geq 9/(160 \log\log n)$.

Algorithm 3.1. Shor's factorisation algorithm

input $n \in \mathbb{N}$ composite
1. **if** n is even **then output** 2 **end.**
2. **if** $n = a^k$ for some $a \in \mathbb{N}$, $k \geq 2$ **then output** a **end.**
3. **randomly choose** $a \in \{1, 2, \ldots, n-1\}$
   $d := \gcd(a, n)$
   if $d > 1$ **then output** d **end.**
4. **compute** $m \in \mathbb{N}$ such that $n^2 \leq 2^m < 2n^2$
   $|\varphi\rangle := \frac{1}{2^m} \sum_{l=0}^{r-1} \sum_{y=0}^{2^m-1} \mathrm{e}^{2\pi \mathrm{i} \cdot yl/2^m} \sum_{q=0}^{s_l} \mathrm{e}^{2\pi \mathrm{i} \cdot yrq/2^m} |y\rangle |a^l \bmod n\rangle$
   measure first m qubits of $|\varphi\rangle$ to obtain $y \in \mathbb{Z}_{2^m}$
5. **compute** convergents p_j/q_j of $y/2^m$
   $i := \min\{j \colon a^{q_j} \equiv 1 \pmod n\} \cup \{\infty\}$
   if $i = \infty$ **then output** ? **end else** $r := q_i$
6. **if** a^r is odd or $a^{r/2} \equiv -1 \pmod n$ **then**
   output ?
   else
   $d := \gcd(n, a^{r/2} - 1)$; **output** d

The algorithm can be implemented using $\mathrm{O}(\log n^3)$ classical operations and $\mathrm{O}(\log n^3)$ elementary quantum gates.

By repeating the algorithm $\log n$ times, we are able to find a factor with very high probability.