

Complexity Theory

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4 Oracles and the polynomial hierarchy

4.1 Oracle Turing machines

Definition 4.1. A deterministic (respectively nondeterministic) *oracle Turing machine* is a Turing machine with a designated oracle tape and three special states ? (query), Y (yes) and N (no).

A configuration C of an Oracle Turing machine with k working tapes and a distinguished oracle tape is a tuple

$$C = (q, p_0, \dots, p_k, w_0, \dots, w_k),$$

where

- q is the state of the Turing machine,
- p_0, \dots, p_k are the head positions (p_0 is the head position of the oracle tape), and
- w_0, \dots, w_k are the head inscriptions (w_0 is the inscription of the oracle tape).

The computation (respectively the computation tree) of an oracle Turing machine depends on a previously defined oracle set $A \subseteq \Sigma^*$ (where Σ is the alphabet of M). The successor configurations of a configuration C are defined as usual for $q \neq ?$ while the successor configuration C' for $q = ?$ is defined as :

$$C' = \begin{cases} (Y, 0, p_1, \dots, p_k, \varepsilon, w_1, \dots, w_k) & \text{if } w_0 \in A \\ (N, 0, p_1, \dots, p_k, \varepsilon, w_1, \dots, w_k) & \text{if } w_0 \notin A \end{cases}$$

where ε is the empty word. The oracle therefore determines whether or not w_0 (the inscription of the oracle tape) is in A . The machine consequently enters the corresponding state (Y or N) and the inscription of the oracle tape is erased.

Definition 4.2. Let M be an Oracle Turing machine and $A \subseteq \Sigma^*$ be some oracle set. Then the accepted language is

$$L(M^A) := \{x : M \text{ accepts the input } x \text{ with oracle } A\}.$$

Based on the oracle set A , we define the following complexity classes:

- (i) $P^A := \{L : \text{there is a deterministic Oracle TM } M \text{ that decides } L \text{ using oracle } A \text{ in polynomial time}\}.$
- (ii) $NP^A := \{L : \text{there is a nondeterministic Oracle TM } M \text{ that decides } L \text{ using oracle } A \text{ in polynomial time}\}.$

Let \mathcal{C} be some class of languages, e.g., a complexity class. Then

$$P^{\mathcal{C}} = \bigcup_{A \in \mathcal{C}} P^A \quad \text{and} \quad NP^{\mathcal{C}} = \bigcup_{A \in \mathcal{C}} NP^A.$$

Example 4.3.

- (a) Let $B \in NP$. Then $B \in P^{\text{SAT}}$. Since SAT is NP-hard, there is a polynomially computable function f with $x \in B \iff f(x) \in \text{SAT}$. The following oracle algorithm then decides B :

Input: x
 Compute $f(x)$
 Query the oracle whether $f(x) \in \text{SAT}$
if Y **then** accept
if N **then** reject

- (b) It is likely that $NP \subsetneq P^{\text{SAT}}$ as every $B \in \text{coNP}$ is in P^{SAT} . One can use the preceding algorithm and interchange the behaviour for the answers Yes and No.
- (c) Let $B := \{(G, k) : G \text{ a graph, } \omega(G) = k\}$ where $\omega(G)$ is the maximal number of nodes of cliques in G . Reminder: The problem $\text{CLIQUE} := \{(G, k) : k \leq \omega(G)\}$ is NP-complete. It is straightforward to see that $B \in P^{\text{CLIQUE}}$:

Input: G, k
 Query the oracle whether $(G, k) \in \text{CLIQUE}$
if N **then** reject **else**
 Query the oracle whether $(G, k + 1) \in \text{CLIQUE}$
 if N **then** accept
 if Y **then** reject
endif

4.2 The polynomial hierarchy

Definition 4.4. We define the complexity classes Σ_k^p , Π_k^p , and Δ_k^p for all $k \in \mathbb{N}$:

- $\Sigma_0^p := \Pi_0^p := \Delta_0^p := P$
- $\Sigma_{k+1}^p := \text{NP}^{\Sigma_k^p}$
- $\Pi_k^p := \text{co}\Sigma_k^p = \{\bar{A} : A \in \Sigma_k^p\}$
- $\Delta_{k+1}^p := P^{\Sigma_k^p}$

Theorem 4.5. The classes Σ_k^p , Π_k^p , and Δ_k^p have the following elementary properties:

- (i) $\Delta_1^p = P$.
- (ii) $\Sigma_1^p = \text{NP}$, $\Pi_1^p = \text{coNP}$.
- (iii) $\Sigma_{k+1}^p = \text{NP}^{\Pi_k^p} = \text{NP}^{\Delta_{k+1}^p}$.
- (iv) $P^{\Delta_k^p} = \Delta_k^p$.

Proof. (i) $\Delta_1^p = P^P = P$.

(ii) $\Sigma_1^p = \text{NP}^P = \text{NP}$, $\Pi_1^p = \text{co}\Sigma_1^p = \text{coNP}$.

(iii) Let $B \in \Sigma_{k+1}^p$, $B = L(M^A)$ and $A \in \Sigma_k^p$. Further, let M' be the machine obtained from M by interchanging the states Y and N . Obviously, $B = L(M'^{\bar{A}})$ and therefore $B \in \text{NP}^{\Pi_k^p}$. In addition, $\text{NP}^{\Delta_{k+1}^p} = \text{NP}^{P^{\Sigma_k^p}} = \text{NP}^{\Sigma_k^p} = \Sigma_{k+1}^p$ holds.

(iv) $k = 0$: $P^{\Delta_0^p} = P^P = P = \Delta_0^p$.

$k > 0$: $P^{\Delta_k^p} = P^{P^{\Sigma_{k-1}^p}} = P^{\Sigma_{k-1}^p} = \Delta_k^p$. Q.E.D.

4.2 The polynomial hierarchy

Theorem 4.6. For all k , $\Sigma_k^p \cup \Pi_k^p \subseteq \Delta_{k+1}^p \subseteq \Sigma_{k+1}^p \cap \Pi_{k+1}^p$.

Proof. For $k = 0$, the theorem states $P \subseteq P \subseteq NP \cap coNP$. This is obviously true.

For $k > 0$:

- $\Sigma_k^p \subseteq P^{\Sigma_k^p}$ and therefore also $\Pi_k^p \subseteq P^{\Sigma_k^p}$ because $P^{\Sigma_k^p}$ is closed under complement. Hence, $\Sigma_k^p \cup \Pi_k^p \subseteq \Delta_{k+1}^p$.
- $\Delta_{k+1}^p = P^{\Sigma_k^p} = coP^{\Sigma_k^p} \subseteq coNP^{\Sigma_k^p} = \Pi_{k+1}^p$.
 $\Delta_{k+1}^p = P^{\Sigma_k^p} \subseteq NP^{\Sigma_k^p} = \Sigma_{k+1}^p$. Therefore, $\Delta_{k+1}^p \subseteq \Sigma_{k+1}^p \cap \Pi_{k+1}^p$.
Q.E.D.

Theorem 4.7. If there is a k such that $\Sigma_{k+1}^p = \Sigma_k^p$, then $\Sigma_{k+i}^p = \Pi_{k+i}^p = \Sigma_k^p$ for all $i > 0$.

Proof. For $i = 1$, $\Sigma_{k+1}^p = \Sigma_{k+1}^p = \Sigma_k^p$ by assumption. By induction hypothesis, assume $\Sigma_{k+i}^p = \Sigma_k^p$. Then,

$$\Sigma_{k+i+1}^p = NP^{\Sigma_{k+i}^p} = NP^{\Sigma_k^p} = \Sigma_{k+1}^p = \Sigma_k^p,$$

and therefore also

$$\Pi_{k+i}^p \subseteq \Sigma_{k+i+1}^p = \Sigma_k^p \quad \text{for all } i.$$

In particular, $\Pi_k^p \subseteq \Sigma_k^p$ holds. It remains to show that $\Sigma_k^p \subseteq \Pi_k^p$. If $B \in \Sigma_k^p$, then $\bar{B} \in \Pi_k^p \subseteq \Sigma_k^p$ and, hence, $B \in \Pi_k^p$.
Q.E.D.

Corollary 4.8. If there is a $k > 0$ with $\Sigma_k^p \neq P$, then $P \neq NP$.

Definition 4.9. $PH := \bigcup_{k \in \mathbb{N}} \Sigma_k^p$ is called the *polynomial hierarchy*.

In case $\Sigma_{k+i}^p = \Sigma_k^p$, we say that the polynomial hierarchy collapses at level k .

Theorem 4.10. $PH \subseteq PSPACE$.

Proof. By induction over k we show that $\Sigma_k^p \subseteq PSPACE$ for all $k \in \mathbb{N}$:

$$\begin{aligned} \Sigma_0^p &= P \subseteq PSPACE \\ \Sigma_{k+1}^p &= NP^{\Sigma_k^p} \subseteq NP^{PSPACE} \subseteq PSPACE^{PSPACE} = PSPACE. \end{aligned}$$

Here, $\text{PSPACE}^{\text{PSPACE}}$ is the class of languages that can be decided by a deterministic polynomially-space bounded Oracle Turing machine with some oracle in PSPACE . When we speak about space complexity, we also count the space used on the oracle tape. Q.E.D.

If $\text{PH} = \text{PSPACE}$, the polynomial hierarchy collapses:

Theorem 4.11. If $\text{PH} = \text{PSPACE}$, there is some k with $\text{PH} = \Sigma_k^P$.

Proof. If $\text{PH} = \text{PSPACE}$, then $\text{QBF} \in \text{PH}$ holds. Consequently, there is some k such that $\text{QBF} \in \Sigma_k^P$. For each $A \in \text{PSPACE}$, we have $A \leq_m^P \text{QBF}$, i.e., $A \in \Sigma_k^P$. Thus, we obtain $\text{PH} = \Sigma_k^P$. Q.E.D.

It is assumed that $\Sigma_k^P \subsetneq \Sigma_{k+1}^P$ for all k , the polynomial hierarchy is strict and therefore, $\text{PH} \subsetneq \text{PSPACE}$.

4.2.1 Additions

There are two natural complete problems for Σ_k^P and Π_k^P :

$$\Sigma_k\text{-QBF} = \{ \psi = (\exists \overline{X_1})(\forall \overline{X_2}) \dots (Q_k \overline{X_k}) \varphi : \varphi \text{ quantifier free,} \\ \psi \text{ true} \}$$

Here, Q_k is the universal quantifier if k is even and the existential quantifier otherwise. $\Pi_k\text{-QBF}$ is defined analogously but the formulae begin with universal quantifiers. The problem $\Sigma_k\text{-QBF}$ is Σ_k^P -complete and, analogously, $\Pi_k\text{-QBF}$ is Π_k^P -complete. This generalises the NP-completeness of SAT .

Recall the definition of NP. A problem $A \in \text{NP}$ if, and only if, there is some $B \in \text{P}$ and some polynomial $p(n)$ such that $A = \{x : \exists^p y (x \# y \in B)\}$ where $\exists^p y$ is an abbreviation for $\exists y : |y| \leq p(|x|)$. We can generalise this definition to obtain a characterisation of Σ_k^P and Π_k^P as follows:

- $A \in \Sigma_k^P$ if, and only if, there is some $B \in \text{P}$ and some polynomial $p(n)$ such that

$$A = \{x : (\exists^p y_1)(\forall^p y_2)(\exists^p y_3) \dots (Q_k^p y_k) x \# y_1 \# y_2 \# \dots \# y_k \in B\}.$$

4.3 Relativisations

Here, Q_k is the existential quantifier if k is odd and the universal quantifier otherwise.

- Π_k^p can be characterised analogously, using formulae that begin with universal quantifiers.

4.3 Relativisations

To approach the $P = NP$ question, it is interesting to see whether there are oracles A, B such that

- $P^A = NP^A$
- $P^B \neq NP^B$.

The first question is easy to answer since $P^{\text{QBF}} = NP^{\text{QBF}} = PSPACE$. The answer to the second question is not as straightforward.

Theorem 4.12. There is an oracle B such that $P^B \neq NP^B$.

Proof. Just as Turing machines, polynomially time-bounded oracle Turing machines can also be enumerated recursively (exercise). We choose one such recursive enumeration $\{M_i : i \in \mathbb{N}\}$ of deterministic polynomial oracle Turing machines such that the following holds for all oracles A :

- (1) $P^A = \{L(M_i^A) : i \in \mathbb{N}\}$.
- (2) There is a sequence $\{p_i(n) : i \in \mathbb{N}\}$ of polynomials with:
 - (i) M_i is p_i -time bounded,
 - (ii) $p_i(n) \leq p_{i+1}(n)$ for all i, n .

Any given sequence $\{q_i(n) : i \in \mathbb{N}\}$ of time bounds for $\{M_i : i \in \mathbb{N}\}$ can be modified to such a sequence $p_{i+1}(n)$ by setting:

- $p_0(n) := q_0(n)$,
- $p_{i+1}(n) := \max(p_i(n), q_{i+1}(n))$.

For every $C \subseteq \{0, 1\}^*$, let $S(C) = \{0^n : \text{there is an } x \in C \text{ with } |x| = n\}$.

We obviously have:

Lemma 4.13. $S(B) \in NP^B$ for all B .

The goal now is to find some B such that $S(B) \notin P^B$. This B is constructed as follows: At the beginning, initialise $B_0 := \emptyset$ and $k_0 := 0$. For $n > 0$, construct B_n, k_n as follows:

- Set k_n so it is the smallest integer with $2^{k_n} > p_n(k_n)$ and $k_n > p_{n-1}(k_{n-1})$.
- If $0^{k_n} \in L(M_n^{B_{n-1}})$, then set $B_n := B_{n-1}$. Otherwise, let $w(n)$ be the lexicographically first word in $\{0, 1\}^{k(n)}$ for which the oracle is not queried during the computation of $M_n^{B_{n-1}}$ on input 0^{k_n} . Such a word exists since $p_n(k_n) < 2^{k_n}$. Set $B_n = B_{n-1} \cup \{w(n)\}$.

Set $B := \bigcup_{n \in \mathbb{N}} B_n$.

Lemma 4.14. $0^{k_n} \in L(M_n^B) \iff 0^{k_n} \in L(M_n^{B_{n-1}})$ for all n .

The oracle is never queried for $w \in B \setminus B_{n-1}$ during the computation of M_n^B on 0^{k_n} :

- for $w = w(n)$ by construction and
- for $w = w(m)$ with $m > n$ because $|w(m)| = k(m) > p_n(k_n)$.

Lemma 4.15. $S(B) \notin P^B$.

Otherwise, there would be some $n \in \mathbb{N}$ with $S(B) = L(M_n^B)$. However, this cannot be the case since, by definition, $0^{k_n} \in S(B)$ if, and only if, there is some w such that $|w| = k_n$ and $w \in B$. By construction, this is the case if, and only if, $0^{k_n} \notin L(M_n^{B_{n-1}})$. This follows from the fact that a word of length k_n is added to B if, and only if, $0^{k_n} \notin L(M_n^{B_{n-1}})$. By Lemma 4.14, $0^{k_n} \notin L(M_n^{B_{n-1}})$ is equivalent to $0^{k_n} \notin L(M_n^B)$, and therefore, $S(B) \neq L(M_n^B)$. Q.E.D.

The problem whether $\mathcal{C}_1 = \mathcal{C}_2$ remains open for many pairs of complexity classes \mathcal{C}_1 and \mathcal{C}_2 . In most cases, there are oracles A and B such that:

$$\mathcal{C}_1^A = \mathcal{C}_2^A \quad \text{and} \quad \mathcal{C}_1^B \neq \mathcal{C}_2^B.$$

It has further been shown that for almost all oracles D : $\mathcal{C}_1^D \neq \mathcal{C}_2^D$. Contradictory relativisations of this kind show that, with respect to the

4.3 Relativisations

problem $\mathcal{C}_1 \neq \mathcal{C}_2$, proof techniques that 'relativise' (i.e., techniques that are independent of oracles) fail.