Complexity Theory and Quantum Computing — Assignment 8

Due: Monday, December 21, 12:00

Exercise 1

Prove the following claims:

(a) $\text{NP}^{\text{BPP}} \subseteq \text{BPP}^{\text{NP}}$.

*Hint:* Show that the language consisting of all words $x \# r$ such that $x$ is accepted by a polynomial NTM $M$ using a BPP oracle with random bit-sequence $r$ is in NP. Consider, for a fixed sequence of random bits, the number of oracle queries for which the BPP oracle gives the wrong answer.

(b) $\text{BPP}^{\text{BPP}} = \text{BPP}$.

*Hint:* Consider a suitable simulation of the oracle machine.

(c) $\text{NP} \subseteq \text{BPP}$ implies $\Sigma_2^p \subseteq \text{BPP}$. Show that this implies $\text{PH} \subseteq \text{BPP}$.

Exercise 2

A non-deterministic Turing machine $M$ is called *standardised* if it has, for each configuration, exactly two possible successor configurations, and if there is a polynomial $p$ such that each computation path has a length of exactly $p(n)$ (where $n$ is the length of the input). Hence, the computation tree of such a machine $M$ on an input $x$ of length $n$ is a full binary tree of depth $n$ with $2^{p(n)}$ leaves. Each leaf is labelled by 0 or 1 to denote a rejecting or accepting state, respectively. If you read the labels from left to right, you obtain the word $M(x) \in \{0, 1\}^{2^{p(n)}}$.

We introduce the following uniform model for defining complexity classes: Let $A, R \subseteq \{0, 1\}^*$ be a pair of disjoint so-called leaf languages. Then $(A, R)$ defines the complexity class $C[A, R]$ of those languages for which there exists a standardised non-deterministic Turing machine $M$ such that $x \in L$ if, and only if, $M(x) \in A$ and $x \not\in L$ if, and only if, $M(x) \in R$.

(a) Specify pairs $(A, R)$ of leaf-languages such that $C[A, R]$ corresponds to the following complexity classes: $\text{P}$, $\text{NP}$, $\text{NP} \cap \text{coNP}$, $\text{RP}$, $\text{coRP}$, $\text{ZPP}$, $\text{BPP}$, $\Sigma_2^p$, $\text{PSPACE}$.

*Hint:* Use the characterisation of classes in the Polynomial Hierarchy by alternating Turing machines.

(b) For which of these classes can $A$ and $R$ be chosen such that $R = \overline{A}$?

(c) Prove that $C[A, R] = \text{PSPACE}$ if $A$ is $\text{NLOGSPACE}$-complete.
Exercise 3

Prove the following closure properties of probabilistic complexity classes:

(a) PP, BPP, RP and ZPP are closed under polynomial-time reductions.

(b) BPP and RP are closed under union and intersection.

(c) PP is closed under complement and symmetric difference.

*Hint:* Eliminate the asymmetry in the definition concerning the acceptance probability of $1/2$. Show that, for every probabilistic TM $M$, there exists a PTM $M'$ such that, for all inputs $x$, $\Pr[M' \text{ acc. } x] \neq 1/2$ and $\Pr[M' \text{ acc. } x] > 1/2$ if, and only if, $\Pr[M \text{ acc. } x] > 1/2$.  

http://www.logic.rwth-aachen.de/Teaching/KTQC-WS09/