(1) *Find the Nash equilibria of the following strategic game*

		Player 2	
		L	R
Player 1	T	$(2,2)$	$(0,0)$
	B	$(0,0)$	$(1,1)$

Solution:

(T,L) Neither player can increase its payoff by choosing a different action, so this action profile is a Nash equilibrium.

(T,R) Player 1 can increase his payoff from 0 to 1 by choosing the action B rather than the action T. Thus this action profile is not a Nash equilibrium.

(B,L) Player 1 can increase his payoff from 0 to 2 by choosing the action T rather than the action B. Thus this action profile is not a Nash equilibrium.

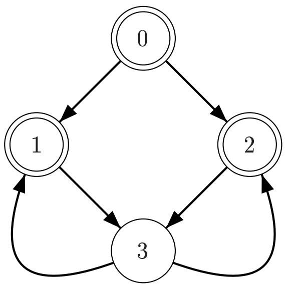
(B,R) Neither player can increase its payoff by choosing a different action, so this action profile is a Nash equilibrium.

We conclude that the game has two Nash equilibria, (T,L) and (B,R).

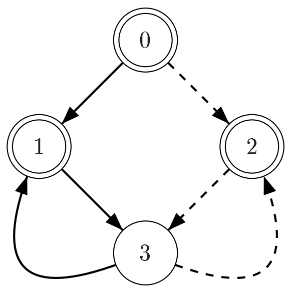
(2) We consider the game of geography played on a directed game graph $G = (V_0, V_1, E)$, from an initial node v.

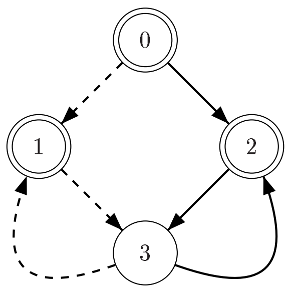
- *Does the following hold: whoever has a winning strategy in* $\mathbb{G}\mathrm{eo}\,(G\ ,\ i)$ *set, also has a memoryless winning strategy ?*

Solution: The answer is no. In the following graph where vertices for 0 have only one circle, while vertices for 1 have two, player 0 has a w.s. from vertex 0 (namely go to vertex 1 if player 1 went through vertex 2, and go to vertex 2 if player 1 went through vertex 1.



But player 0 does not have any memoryless vinning strategy from 0. Since a memoryless strategy would yield one of the two following subgraphs where clearly 1 has a w.s. from vertex 0:



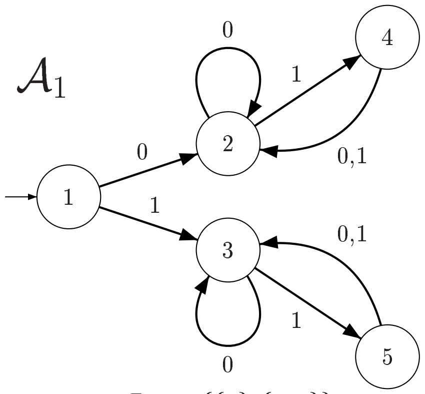


(3) Consider the following four deterministic Muller automata $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and $L(\mathcal{A}_1), L(\mathcal{A}_2), L(\mathcal{A}_3), L(\mathcal{A}_4)$, the languages they recognize. Find the right integers i_1, i_2, i_3, i_4 such that:

$$L(\mathcal{A}_{i_1}) <_W L(\mathcal{A}_{i_2}) <_W L(\mathcal{A}_{i_3}) <_W L(\mathcal{A}_{i_4})$$

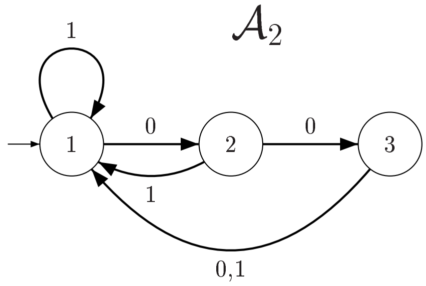
where $<_W$ is the strict Wadge ordering: $A <_W B \iff_{def} A \leq_W B$ but $B \nleq_W A$.

$\mathcal{A}_1$



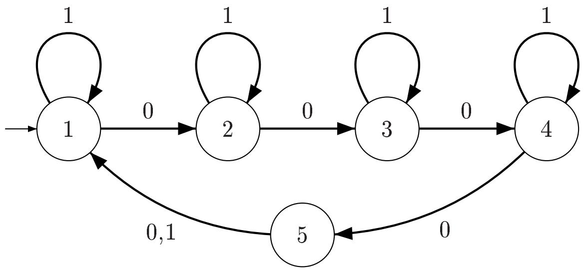
$\mathcal{F}_{A_1} = \{\{2\}, \{3, 5\}\}$

$\mathcal{A}_2$



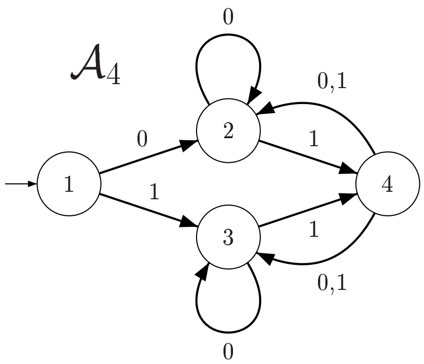
$\mathcal{F}_{A_2} = \{\{1, 2\}\}$

$\mathcal{A}_3$



$\mathcal{F}_{A_3} = \{\{1, 2, 3, 4, 5\}\}$

$\mathcal{A}_4$



$\mathcal{F}_{A_4} = \{\{4\}\}$

Solution: By playing the differennt underlying Wadge games, one comes to the conclusion:

$$\mathcal{A}_4 \;<_W\; \mathcal{A}_3 \;<_W\; \mathcal{A}_1 \;<_W\; \mathcal{A}_2$$

(4) Let $G = (V_0, V_1, E, C, \phi)$ be an arena for a parity games G is called 0-*solitaire* (resp. 1-*solitaire*) if each vertex from V_1 (resp. V_0) has only one successor.

1-solitaire-PARITY-GAMES $= \{(G, v) :\ G$ is 1-solitaire, player **0** has a w.s. from vertex $v\}$.

- *Show that 1-solitaire-PARITY-GAMES* $\in$ **PTIME**.
  (*hint*: recall the proof that PARITY-GAMES $\in$ **NP**).

Solution: Since G is 0-*solitaire*, player 0 has no choice and player 1 wins the game from vertex v iff there is a path from v going to some loop whose maximal priority (color) is odd.

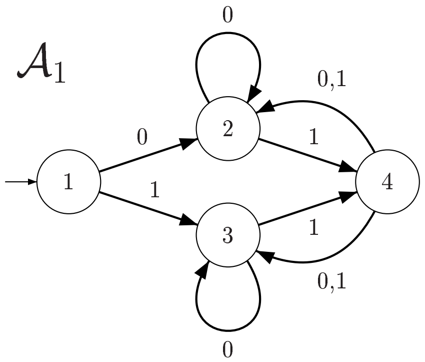
Recall first that REACHABILITY $\in$ **PTIME**. Given G' any subgraph of (V_0, V_1, E), we write $v \underset{G'}{\longmapsto} w$ to say that there is a path from v to w in G'. We also write $G_{\leq p}$ to denote the subgraph induced by $\phi^{-1}\{0, \ldots, p\}$.

One can define the following deterministic algorithm to determine whether player 0 has a w.s. from v inside G or not:

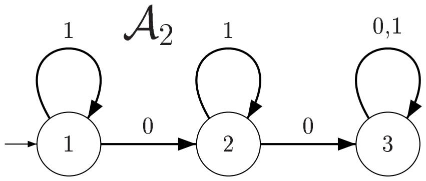
Input : G, v
for each v' s.t. $\phi(v')$ is odd **do**
  for each $v \in G_{\leq \phi(v')}$ **do**
    if $\left(v \xrightarrow[G_{\leq \phi(v')}]{} v' \right.$ **and** $v' \xrightarrow[G_{\leq \phi(v')}]{} v$ **and** $\left. v \xrightarrow[G]{} v' \right)$
      reject
accept .

This algorithm clearly works in polynomial time.

(5) *For the following 2 deterministic Büchi automata $\mathcal{A}_i$, answer the questions $L(\mathcal{A}_1) \in \mathbf{\Sigma}_1^\circ$?, $L(\mathcal{A}_1) \in \mathbf{\Pi}_1^\circ$?, $L(\mathcal{A}_2) \in \mathbf{\Sigma}_1^\circ$?, $L(\mathcal{A}_2) \in \mathbf{\Pi}_1^\circ$?* where $L(\mathcal{A}_i)$ stands for the language accepted by $\mathcal{A}_i$.



$\mathcal{F}_{A_1} = \{2, 3, 4\}$



$\mathcal{F}_{A_2} = \{1, 2\}$

Solution: we set $\Sigma = \{0, 1\}$. We see that $L\mathcal{A}_1) = \Sigma^\omega$ and $L\mathcal{A}_2) = \Sigma^\omega \setminus (1^* \cdot 0 \cdot 1^* \cdot 0) \cdot \Sigma^\omega$.
Hence $L\mathcal{A}_1)$ is both open (belongs to $\mathbf{\Sigma}_1^\circ$) and closed (belongs to $\mathbf{\Pi}_1^\circ$). While $L\mathcal{A}_2)$ is closed (since it is the complement of $(1^* \cdot 0 \cdot 1^* \cdot 0) \cdot \Sigma^\omega$ which is an open set), but not open for it cannot be written of the form $L\mathcal{A}_2) = W \cdot \Sigma^\omega$ for some set of finite words W (notice for example that $L\mathcal{A}_2)$ contains no infinite word with infinitely many 0, while any set of the form $W \cdot \Sigma^\omega$ does).

(6) We recall that WF is not Borel, where $WF \subseteq \{0, 1\}^\omega$ id defined by

$$x = x_1 x_2 \ldots \in WF \iff T_x \begin{cases} \text{is a tree over } \{0,1\} \ (a_1 a_2 \ldots a_k \in T_x \longrightarrow \forall q < k \ \ a_1 a_2 \ldots a_q \in T_x) \\ \text{is well founded (there is no infinite branch)} \end{cases}$$

where given any $x \in \{0,1\}^\omega$, T_x is the subset of $\{0,1\}^*$ defined by:

$$a_1 a_2 \dots a_k \in T_x \iff x_{2^{(a_1+1)}.3^{(a_2+1)}\dots P_k^{(a_k+1)}} = 1$$

where P_i is the i^{th} prime number ($P_1 = 2, P_2 = 3, P_3 = 5, P_4 = 7, P_5 = 11, \dots$).

(a) *Describe an algorithm for a nondeterministic Turing machine reading infinite words with Büchi acceptance conditions that recognizes $IF = \{0,1\}^\omega \setminus WF$.*

(b) *Is the set IF Borel?*

Solution: We design an algorithm for a nondeterministic Turing machine reading infinite words with one single accepting state q_A. The idea of the algorithm is first to check whether the set T_x coded by the infinite word x is a tree or not (which means it is closed under prefix or not). And then to recursively guess an infinite branch, going through the accepting state each time one succeeds in extending the branch..

Input : x

guess $b_0 b_1 \dots b_k \in \{0,1\}^*$ and $q < k$

  if $x_{2^{(b_1+1)}.3^{(b_2+1)}\dots P_k^{(b_k+1)}} = 1$ **and if** $x_{2^{(b_1+1)}.3^{(b_2+1)}\dots P_q^{(b_q+1)}} = 0$ **then** reach q_a infinitely often

  else do

    $i := 1$

    repeat guess $a_i \in \{0,1\}$

      if $x_{2^{(a_1+1)}.3^{(a_2+1)}\dots P_i^{(a_i+1)}} = 0$ **reject**

      else do

        reach q_a once

        $i := i+1$

(7) Given $\mathcal{U}$ any free ultrafilter over $\mathbb{N}$, let $A_\mathcal{U} \subseteq \mathbb{N}^\omega$ be the following set:

$$x = x_0 x_1 \dots \in A_\mathcal{U} \iff \begin{cases} x \text{ is strictly increasing} \quad \wedge \quad ([0, x_0] \cup \bigcup_{i\in\mathbb{N}}]x_{2i+1}, x_{2i+2}]) \in \mathcal{U} \\ \qquad\qquad \text{or} \\ \exists i \in \mathbb{N} \ \ (x_{2i+1} \geq x_{2i+2} \quad \wedge \quad (\forall j < 2i+1 \ \ x_j < x_{j+1})). \end{cases}$$

- *Is the set $A_\mathcal{U}$ Borel?*

Solution: This is precisely the set introduced in the lecture notes (see p. 32), in order to prove that there exists a Gale-Stewart game that is not determined. Thus we know both that $\mathbb{G}(A_\mathcal{U})$ is not determined, and whenever A is Borel, then the game $\mathbb{G}(A)$ is determined (Martin's theorem - see p. 37). We conclude that $A_\mathcal{U}$ is not Borel.

(8) Let $G = (V_0, V_1, E, C, \phi)$ be an arena for a parity games, we define the following relation between vertices:

$$u \leq_r v \iff_{def} v \text{ is reachable from u}$$

$$u <_r v \iff_{def} u \leq_r v \ \wedge v \not\leq_r u$$

$$u \equiv_r v \iff_{def} u \leq_r v \ \wedge v \leq_r u$$

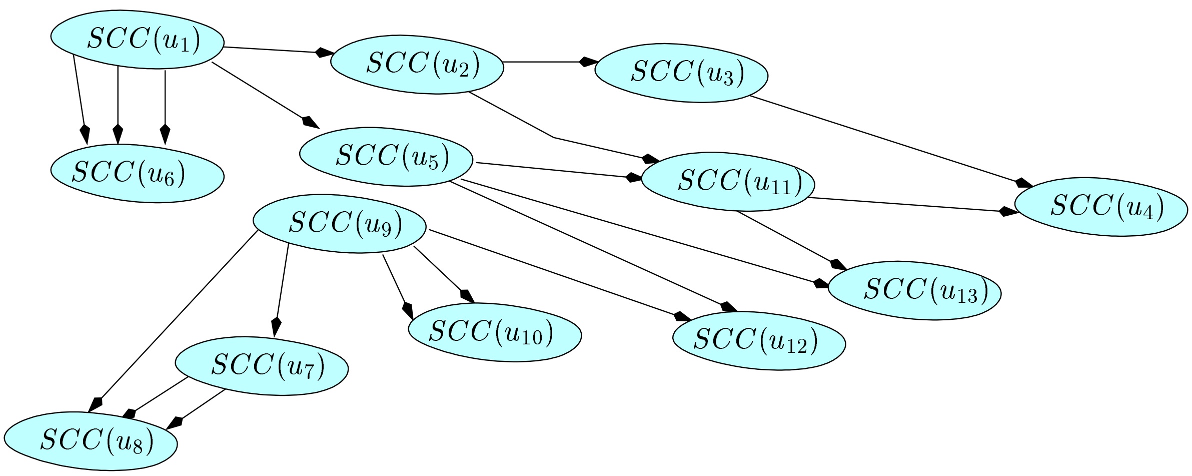
Given a vertex u, the Strongly Connected Component of u is

$$SCC(u) = \{v \in V_0 \cup V_1 : \ u \equiv_r v\}.$$

- *We assume that for any vertex $u \in V_0 \cup V_1$ the parity game on the arena restricted to $SCC(u)$ can be solved in polynomial time. Show that the whole parity game on G can also be solved in polynomial time.*

  Hint: *Notice that given two vertices u and v, either $SCC(u) = SCC(v)$ or $SCC(u) \cap SCC(v) = \emptyset$. Use an induction on the number of different Stongly Connected Components ($card(\{SCC(u) : u \in V_0 \cup V_1\})$) and consider $SCC(u)$ for some u that satisfies $u <_r v$ fails for all vertices v, as for example $SCC(u_4)$ in the picture below.*

Figure 1: The SCC in G



Solution: The proof goes by induction on the number of different Stongly Connected Components. If there is only one, the hypothesis takes care of everything. So we assume there are $n+1$ different Stongly Connected Components. First, remark that one can find in polynomial time such vertex u such that $u <_r v$ fails for all vertices v. One can then also in polynomial time define the set $SCC(u)$ and the subarena $G[SCC(u)]$. By hypothesis, still in polynomial time, one gets a partition of $SCC(u)$ into $W^0_{SCC(u)}$ and $W^1_{SCC(u)}$, the winning set for respectively 0 and 1.

Now we consider $\{u_0, u_1, \dots, u_k\}$ the set of all vertices in $V_0 \setminus SCC(u)$ such that there exists an edge from u_i to some vertex inside $SCC(u)$, and symmetrically, $\{v_0, v_1, \dots, v_l\}$ the set of all vertices in $V_1 \setminus SCC(u)$ such that there exists an edge from v_i to some vertex inside $SCC(u)$.

We remark that the subarena $G[(V_0 \cup V_1) \setminus SCC(u)]$ contains precisely n different Stongly Connected Components. We build a new arena $G' = (V'_0, V'_1, E', C', \phi')$:

(a) $V'_0 = (V_0 \setminus SCC(u)) \cup \{v'_0, v'_1, \dots, v'_l\}$ where $v'_0, v'_1, \dots, v'_l$ are new vertices.

(b) $V'_1 = (V_1 \setminus SCC(u)) \cup \{u'_0, u'_1, \dots, u'_k\}$ where $u'_0, u'_1, \dots, u'_k$ are new vertices.

(c) E' is just E restricted to vertices among $(V_0 \cup V_1) \setminus SCC(u)$ *plus* an edge from v_i to v'_i and one backwards from v'_i to v_i, for each $i \in \{1, 2, \dots, l\}$ *plus* an edge from u_i to u'_i and one backwards from u'_i to u_i, for each $i \in \{1, 2, \dots, k\}$.

(d) C' contains C and the new priorities (colors that are numbers) are indicated in the definition of ϕ' below:

(e) ϕ' is exactly ϕ for vertices among $(V_0 \cup V_1) \setminus SCC(u)$. For the *new ones*, we proceed as follow.

  (1) for each new vertex u'_i:

    (A) if $u_i E \cap W^0_{SCC(u)} \neq \emptyset$, then $\phi'(u'_i)$ =any even number greater that $\phi(u_i)$ (in such a way that the loop $u_i \longleftrightarrow u'_i$ is winning for player 0)

    (B) if $u_i E \cap W^0_{SCC(u)} = \emptyset$, then $\phi'(u'_i)$ =any odd number greater that $\phi(u_i)$ (in such a way that the loop $u_i \longleftrightarrow u'_i$ is losing for player 0)

(2) for each new vertex v_i':

(A) if $v_iE \cap W^1_{SCC(u)} \neq \emptyset$, then $\phi'(v_i')$ =any odd number greater that $\phi(v_i)$ (in such a way that the loop $v_i \longleftrightarrow v_i'$ is winning for player 1)

(B) if $v_iE \cap W^1_{SCC(u)} = \emptyset$, then $\phi'(v_i')$ =any even number greater that $\phi(v_i)$ (in such a way that the loop $v_i \longleftrightarrow v_i'$ is losing for player 0)

Clearly, all this can be done in polynomial time. To get the result it is enough to notice

(a) G' has exactly n different Stongly Connected Components,

(b) $W^0_{G'}$, $W^1_{G'}$ the winning sets in G' for respectively player 0 and 1 are computable in polynomial time (this is the hypothesis).

(c) W^0_G, W^1_G the winning sets in G for respectively player 0 and 1 satisfy:

(1) $W^0_G = W^0_{G'}\left(\smallsetminus \{u_0', u_1', \ldots, u_k', v_0', v_1', \ldots, v_l'\}\right) \cup W^0_{SCC(u)}$,

(2) $W^1_G = W^1_{G'}\left(\smallsetminus \{u_0', u_1', \ldots, u_k', v_0', v_1', \ldots, v_l'\}\right) \cup W^1_{SCC(u)}$.