Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
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WS 2019/20

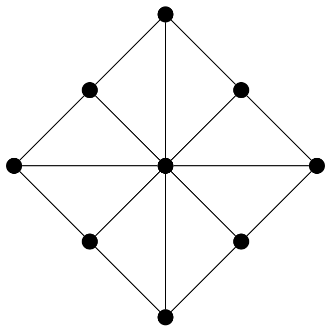
Algorithmic Model Theory — Assignment 12

Due: Tuesday, 14 January, 10:30

Exercise 1

8 Points

For $k \in \mathbb{N}$, the *k-wheel* W_k is the undirected graph which consists of a cycle with k vertices and an extra vertex in the middle which is connected to every vertex on the cycle. For instance, this is the graph W_8:

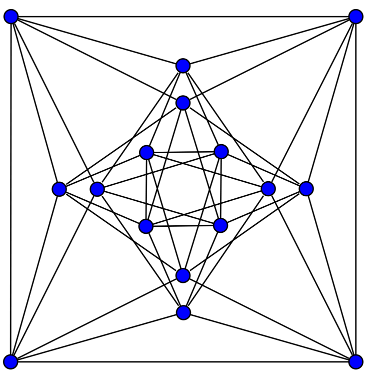


Show that for all $k \geq 3$, the graphs W_k have treewidth 3.

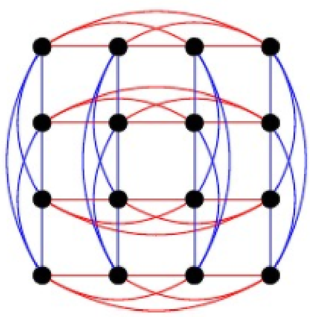
Exercise 2

8 Points

Consider these two undirected graphs (they are also uncoloured; the colours are just for optical reasons).



The graph G.



The graph H.

(a) Show that $G \equiv^{C^3_{\infty\omega}} H$ by describing a winning strategy for Duplicator in the bijective 3-pebble game.

(b) Does the winner change when there is one pebble more, i.e. in the bijective 4-pebble game on (G, H)? Why (not)?

Hint: The graphs are less complicated than they look. Each of them is symmetric in the sense that every vertex can be mapped to every other vertex by an automorphism. Hence, the relevant structure of G and H can already be seen by picking an arbitrary vertex and looking at the structure of its neighbourhood.

Exercise 3 14 Points

In this exercise, we consider the CFI-construction on *ordered* graphs. This works as follows: Let $G = (V, E, <)$ be an undirected graph equipped with a linear order $<$ on V. The CFI-graph $X(G)$ is an $\{E, <\}$-structure whose universe and edge relation are defined as in the lecture notes. The relation $<^{X(G)}$ is no longer a total order; it is defined as follows:

$$<^{X(G)} := \{(x, y) \mid (v, w) \in <^G, x \in Z(v), y \in Z(w)\}.$$

Here, $Z(v)$ is the set of nodes in the gadget of v, as in the lecture notes.

(a) Let $G = (V, E, <)$ be a linearly ordered graph and $X(G)$ the corresponding CFI-graph. Let $\pi \in Aut(X(G))$ be an automorphism of $X(G)$. Show that $\pi(Z(v)) = Z(v)$ for every $v \in V(G)$. Further, show that for every $v \in V(G)$ and $w \in vE$, $\pi(\{a_{vw}, b_{vw}\}) = \{a_{vw}, b_{vw}\}$.

(b) We consider the same setting as in (a). For $\pi \in Aut(X(G))$ and $e = \{v, w\} \in E(G)$ we say that π flips the edge e if $\pi(a_{vw}) = b_{vw}$ and $\pi(a_{wv}) = b_{wv}$. Show that π is uniquely determined by the set $F \subseteq E(G)$ of edges that it flips: That means, if π flips exactly the edges in F, then there is no $\pi' \in Aut(X(G)) \setminus \{\pi\}$ that also flips F.

(c) Suppose $G = (V, E, <)$ is a linearly ordered forest (i.e. acyclic). Show that $X(G)$ has no non-trivial automorphisms.

Hint: We have established that we can describe every automorphism π by its set F of flipped edges. Fix any vertex $v \in V(G)$ and consider $\delta_F(v) := |F \cap vE|$. Given that π must map the inner vertices of the gadget $Z(v)$ to each other, what property does the number $\delta_F(v)$ necessarily have?

(d) Now let $(G_n)_{n\in\mathbb{N}}$ be a family of linearly ordered graphs such that for all $n \in \mathbb{N}$, G_n contains a clique of size $\geq n$. Show that the size of the automorphism group of the corresponding CFI-graphs grows at least exponentially in n, i.e. $|Aut(X(G_n))| \in \Omega(2^n)$.