Lehr- und Forschungsgebiet
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Algorithmic Model Theory — Assignment 11

Due: Tuesday, 7 January, 10:30

Exercise 1 10 Points

A graph $\mathcal{G} = (V, E^{\mathfrak{A}})$ encodes an $(V \times V)$-matrix $M^{\mathcal{G}}$ over $\mathbb{F}_2$ which is

$$M^{\mathfrak{A}}(a,b) = \begin{cases} 0, & \text{if } (a,b) \notin E \\ 1, & \text{if } (a,b) \in E. \end{cases}$$

In other words, $M^{\mathcal{G}}$ is just the adjacency matrix of the graph $\mathcal{G}$. In the same way, every FPC-formula $\varphi(x,y)$ defines an $(V \times V)$-matrix $\varphi^{\mathcal{G}}$ over $\mathbb{F}_2$ in the graph $\mathcal{G}$. We want to show that matrix multiplication is definable in FPC.

(a) Construct a formula $\varphi(x,y) \in \text{FPC}$ such that for any graph $\mathcal{G}$ it holds $\varphi^{\mathcal{G}} = (M^{\mathcal{G}})^2$.

(b) Construct a formula $\varphi(x,y) \in \text{FPC}$ such that for any graph $\mathcal{G}$ it holds $\varphi^{\mathcal{G}} = (M^{\mathcal{G}})^{2^{|V|}}$.

Exercise 2 6 Points

Let $\mathfrak{A}$ be a finite τ-structure. We make the following convention: we interpret numerical tuples $\bar{\nu} = (\nu_{k-1}, \dots, \nu_1, \nu_0) \in \{0, \dots, |A|-1\}^k$ as numbers in $|A|$-adic representation, i.e. we associate the value $\sum_{i=0}^{k-1} \nu_i |A|^i$ to each tuple $\bar{\nu} \in \{0, \dots, |A|-1\}^k$.

Show that the expressive power of FPC does not increase if we allow counting quantifiers of higher arity, i.e. formulas $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1}) \leq (\nu_{k-1}, \dots, \nu_0)$ where in a structure $\mathfrak{A}$ the value of $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1})$ is the number of tuples $\bar{a}$ such that $\mathfrak{A} \models \varphi(\bar{a})$ (with respect to the encoding introduced above). For simplicity, you may only consider the case $k = 2$.

Exercise 3 8 Points

We encode linear equation systems over the finite field $\mathbb{F}_2$ as relational structures $\mathfrak{A}$ over the signature $\tau = \{E, V, R_0, R_1\}$ where the intended meaning of the relations is as follows.

- E, V are unary predicates which partition the universe into equations and variables, and
- the equation $e \in E$ corresponds to the linear equation $\sum_{v \in V : R_i(e,v)} v = i$.

(a) Construct an FO(τ)-sentence φ such that $\mathfrak{A} \models \varphi$ if, and only if, $\mathfrak{A}$ encodes a linear equation over $\mathbb{F}_2$ in the described way.

(b) For any fixed finite field $\mathbb{F}$, generalise the above encoding for linear equation systems over $\mathbb{F}$.

Exercise 4 6 Points

Recall the encoding of linear equation systems over $\mathbb{F}_2$ as relational structures from Exercise 2. Here we want to reduce bipartiteness of undirected graphs to the solvability of linear equation systems over $\mathbb{F}_2$.

Construct FO($\{F\}$)-formulae $\psi_E(x,y), \psi_V(x,y)$ and $\psi_{R_i}(x,y,x',y')$ such that for any (finite, undirected) graph $\mathcal{G} = (W, F)$ the $\{E, V, R\}$-structure $\mathcal{G}^\psi = (W^2, E^\psi, V^\psi, R_0^\psi, R_1^\psi)$ where

- $E^\psi = \{(w, w') : \mathcal{G} \models \psi_E(w, w')\}$, $V^\psi = \{(w, w') : \mathcal{G} \models \psi_V(w, w')\}$ and
- $R_i^\psi = \{((u, u'), (w, w')) : \mathcal{G} \models \psi_{R_i}(u, u', w, w')\}$,

encodes a linear equation system over $\mathbb{F}_2$ which has a solution if, and only if, $\mathcal{G}$ is bipartite.