Lehr- und Forschungsgebiet
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Algorithmic Model Theory — Assignment 7

Due: Tuesday, 26 November, 10:30

Exercise 1 4 Points

Prove or disprove that for all τ-structures $\mathfrak{A} \subseteq \mathfrak{B}$ with respective Gaifman-graphs $\mathcal{G}(\mathfrak{A}), \mathcal{G}(\mathfrak{B})$ it holds that $\mathcal{G}(\mathfrak{A}) \subseteq \mathcal{G}(\mathfrak{B})$.

Exercise 2 10 Points

In this exercise we want to apply Hanf's Theorem to show that connectivity of graphs cannot be defined in *existential monadic second-order logic (EMSO)*, that is in the logic consisting of all Σ_1^1-sentences of the form

$$\exists P_1 \cdots \exists P_k \vartheta, \vartheta \in \text{FO},$$

where the existential quantifiers over the second-order variables P_i are restricted to *monadic* predicates P_i.

(a) Show that EMSO can define the class of all (finite) graphs which are *not* connected.

(b) Show that EMSO cannot define the class of all (finite) graphs which are connected.
- Assume that some sentence $\psi = \exists P_1 \cdots \exists P_k \vartheta \in \text{EMSO}$ defines this class.
- For $n \geq 1$, consider a *connected* graph $\mathcal{G}_n$ consisting of a directed cycle of lenght n, i.e. $\mathcal{G}_n \models \psi$.
- Think of the predicates P_i as colours of the nodes of this cycle and determine the number of isomorpishm types of r-neighbourhoods (to get the appropriate value of r apply Hanf's Theorem to ϑ).
- Choose the parameter n large enough such that at least two nodes on the (coloured) cycle have disjoint and isomorphic r-neighbourhoods. Use these two nodes to construct a new graph consisting of two disjoint cycles which is a model of ψ.

Exercise 3 10 Points

We consider the class $\mathcal{K}_d$ of (undirected, finite) graphs $G = (V, E)$ with degree $\leq d$ for some constant d. We want to apply Gaifman's Theorem to show that for each fixed first-order sentence $\varphi \in \text{FO}(\{E\})$, the model-checking problem $G \models \varphi$ for graphs $G \in \mathcal{K}_d$ is decidable in linear time.

- Explain why it suffices to solve this problem for basic local sentences
$$\exists x_1 \cdots \exists x_\ell (\bigwedge_{i \neq j} d(x_i, x_j) > 2r \wedge \bigwedge_i \psi^r(x_i)).$$
- Show that one can compute in linear time, given a graph $G = (V, E) \in \mathcal{K}_d$, the set P_ψ of elements $v \in V$ such that $G \models \psi^r(v)$.

- Show that one can decide in linear time, given a graph $G = (V, E) \in \mathcal{K}_d$, whether the set P_ψ (as defined above) contains an r-scattered tuple of length ℓ.

  Hint: Use the fact that the $2r$-neighbourhood of a maximal r-scattered tuple of P_ψ elements covers P_ψ.

Exercise 4 6 Points

Remember that an (undirected) graph $G = (V, E)$ is k-connected if the removal of any set of at most $k - 1$ edges does not disconnect the graph. Show that for all $k \geq 2$ there is no sentence $\psi_k \in \mathrm{FO}(E)$ auch that for all $(k-1)$-connected graphs G:

$$G \models \psi_k \quad \Leftrightarrow \quad G \text{ is } k\text{-connected.}$$

(That is FO cannot axiomatise k-connectivity inside the class of $(k-1)$-connected graphs.)

Hint: