Exercise 1  
4 Points  
Prove or disprove that for all $\tau$-structures $\mathfrak{A} \subseteq \mathfrak{B}$ with respective Gaifman-graphs $G(\mathfrak{A}), G(\mathfrak{B})$ it holds that $G(\mathfrak{A}) \subseteq G(\mathfrak{B})$.

Exercise 2  
10 Points  
In this exercise we want to apply Hanf’s Theorem to show that connectivity of graphs cannot be defined in existential monadic second-order logic (EMSO), that is in the logic consisting of all $\Sigma^1_1$-sentences of the form

$$\exists P_1 \cdots \exists P_k \vartheta, \vartheta \in \text{FO},$$

where the existential quantifiers over the second-order variables $P_i$ are restricted to monadic predicates $P_i$.

(a) Show that EMSO can define the class of all (finite) graphs which are not connected.

(b) Show that EMSO cannot define the class of all (finite) graphs which are connected.
   - Assume that some sentence $\psi = \exists P_1 \cdots \exists P_k \vartheta \in \text{EMSO}$ defines this class.
   - For $n \geq 1$, consider a connected graph $G_n$ consisting of a directed cycle of length $n$, i.e. $G_n \models \psi$.
   - Think of the predicates $P_i$ as colours of the nodes of this cycle and determine the number of isomorphism types of $r$-neighbourhoods (to get the appropriate value of $r$ apply Hanf’s Theorem to $\vartheta$).
   - Choose the parameter $n$ large enough such that at least two nodes on the (coloured) cycle have disjoint and isomorphic $r$-neighbourhoods. Use these two nodes to construct a new graph consisting of two disjoint cycles which is a model of $\psi$.

Exercise 3  
10 Points  
We consider the class $K_d$ of (undirected, finite) graphs $G = (V,E)$ with degree $\leq d$ for some constant $d$. We want to apply Gaifman’s Theorem to show that for each fixed first-order sentence $\varphi \in \text{FO}(\{E\})$, the model-checking problem $G \models \varphi$ for graphs $G \in K_d$ is decidable in linear time.

- Explain why it suffices to solve this problem for basic local sentences

$$\exists x_1 \cdots \exists x_\ell (\bigwedge_{i \neq j} d(x_i, x_j) > 2r \land \bigwedge_i \psi^r(x_i)).$$

- Show that one can compute in linear time, given a graph $G = (V,E) \in K_d$, the set $P_\psi$ of elements $v \in V$ such that $G \models \psi^r(v)$.

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Show that one can decide in linear time, given a graph $G = (V, E) \in \mathcal{K}_d$, whether the set $P_\psi$ (as defined above) contains an $r$-scattered tuple of length $\ell$.

Hint: Use the fact that the $2r$-neighbourhood of a maximal $r$-scattered tuple of $P_\psi$ elements covers $P_\psi$.

Exercise 4

Remember that an (undirected) graph $G = (V, E)$ is $k$-connected if the removal of any set of at most $k - 1$ edges does not disconnect the graph. Show that for all $k \geq 2$ there is no sentence $\psi_k \in \text{FO}(E)$ such that for all $(k - 1)$-connected graphs $G$:

\[ G \models \psi_k \iff G \text{ is } k\text{-connected}. \]

(That is FO cannot axiomatise $k$-connectivity inside the class of $(k - 1)$-connected graphs.)

Hint: