Algorithmic Model Theory
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3 LFP and Infinitary Logics

One of the distinguishing features of finite model theory compared with other branches of logic is the eminent role of various kinds of fixed-point logics. Fixed-point logics extend a basic logical formalism (such as first-order logic, conjunctive queries, or propositional modal logic) by a constructor for expressing *fixed points of relational operators*.

What do we mean by a *relational operator*? Note that any formula $\psi(R, \overline{x})$ of vocabulary $\tau \cup \{R\}$ where R is a relational symbol of arity k and $\overline{x}$ is a k-tuple of variables that are free in ψ can be viewed as defining, for every τ-structure $\mathfrak{A}$, an update operator $F_\psi : \mathcal{P}(A^k) \to \mathcal{P}(A^k)$ on the class of k-ary relations on A, namely

$$F_\psi : R \mapsto \{\overline{a} : (\mathfrak{A}, R) \models \psi(R, \overline{a})\}.$$

A fixed point of F_ψ is a relation R for which $F_\psi(R) = R$. In general, a fixed point of F_ψ need not exist, or there may exist many of them. However, if R happens to occur only positively in ψ, then the operator F_ψ is monotone, and in that case there exists a *least* relation $R \subseteq A^k$ such that $F_\psi(R) = R$. The most influential fixed-point formalisms in logic are concerned with least (and greatest) fixed points, so we shall discuss these first. We start by reviewing the necessary mathematical foundations and we also show how least fixed-point logic is related to infinitary first-order logic.

3.1 Ordinals

The standard basic notion used in mathematics is the notion of a set, and all mathematical theorems follow from *the axioms of set theory*. The standard set of axioms is known as *Zermelo-Fraenkel Set Theory ZF*. These axioms guarantee, for instance, the existence of an empty set, an infinite set, the power set of any set, and that no set is a member of itself (i.e. $\forall x\ \neg x \in x$). It is common in mathematics to extend ZF by *the axiom of choice AC* and to denote the resulting set of axioms by ZFC.

In particular, the notion of *numbers* can be formalised by sets. The standard way to do this is to start with the empty set, i.e. let $0 = \emptyset$, and proceed by induction, defining $n + 1 = n \cup \{n\}$. Here are the first few numbers in this representation:

- $0 = \emptyset$,
- $1 = \{\emptyset\}$,
- $2 = \{\emptyset, \{\emptyset\}\}$,
- $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

In this way we can construct all natural numbers. Observe that for each such number n (viewed as a set) it holds that

$$m \in n \implies m \subseteq n.$$

In particular, the relation $\in$ is *transitive* in such sets, i.e. if $k \in m$ and $m \in n$ then $k \in n$. We use this property of sets to define a more general class of numbers.

Definition 3.1. A set α is *an ordinal number* if $\in$ is transitive in α.

Besides natural numbers, what other ordinal numbers are there? The smallest example is $\omega = \bigcup_{n \in \mathbb{N}} n$, the union of all natural numbers. Indeed, it is easy to check that the union of ordinals is always an ordinal as well (as long as it is a set).

What is the next ordinal number after ω? We can again apply the $+1$ operation in the same way as for natural numbers, so

$$\omega + 1 = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, \{0, 1, \ldots\}\}.$$

But does it make sense to say that $\omega + 1$ is the *next* ordinal, or, to put it more generally: is there an order on ordinals? In fact both, each ordinal as a set and all ordinals as a class, are well-ordered, i.e. the following holds:

- for any two ordinal numbers α and β either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$;

- there exists no infinite descending sequence of ordinals

$$\alpha_0 \supsetneq \alpha_1 \supsetneq \alpha_3 \supsetneq \cdots ;$$

- each ordinal α is well-ordered by $\in$.

Ordinals are intimately connected to well-orders, in fact any structure $(A, <)$ where $<$ is a well-order is isomorphic to some ordinal α. To get an intuition on how ordinals look like, consider the following examples of countable ordinals: $\omega + 1, \omega + \omega$, $\omega^2, \omega^3, \omega^\omega$.

The well-order of ordinals allows to define and prove the principle of *transfinite induction*. This principle states that the class of *all ordinals* is generated from $\emptyset$ by taking the successor ($+1$) and the union on limit steps, as shown on the examples before. Specifically, for each ordinal α it holds that either

- there exists an ordinal $\beta < \alpha$ such that $\alpha = \beta + 1 = \beta \cup \{\beta\}$, or
- there exist ordinals $\beta_\gamma < \alpha$ such that $\alpha = \bigcup_\gamma \beta_\gamma$.

Besides ordinals, we sometimes need cardinal numbers Cn which formalise the notion of cardinalities of sets. A *cardinal number* $\kappa \in \mathrm{Cn}$ is *a smallest ordinal number*, i.e. κ is an ordinal number with which no strictly smaller ordinal number can be put into bijection. For example, every natural number and ω itself are cardinal numbers, but $\omega^2 \notin \mathrm{Cn}$. We denote the class of infinite cardinal numbers by Cn^∞.

3.2 Some Fixed-Point Theory

There is a well-developed mathematical theory of fixed points of monotone operators on complete lattices. A *complete lattice* is a partial order $(A, \leq)$ such that each set $X \subseteq A$ has a supremum (a least upper bound) and an infimum (a greatest lower bound). Here we are interested mainly in power set lattices $(\mathcal{P}(A^k), \subseteq)$ (where A is the universe of a structure), and later in product lattices $(\mathcal{P}(B_1) \times \cdots \times \mathcal{P}(B_m), \subseteq)$. For simplicity, we shall describe the basic facts of fixed-point theory for lattices $(\mathcal{P}(B), \subseteq)$, where B is an arbitrary (finite or infinite) set.

Definition 3.2. Let $F : \mathcal{P}(B) \to \mathcal{P}(B)$ be an operator.

(1) $X \subseteq B$ is a *fixed point* of F if $F(X) = X$.

(2) A *least fixed point* or a *greatest fixed point* of F is a fixed point X of F such that $X \subseteq Y$ or $Y \subseteq X$, respectively, for each fixed point Y of F.
(3) F is *monotone*, if $X \subseteq Y \Longrightarrow F(X) \subseteq F(Y)$ for all $X, Y \subseteq B$.

Theorem 3.3 (Knaster and Tarski)**.** Every monotone operator $F : \mathcal{P}(B) \to \mathcal{P}(B)$ has a *least fixed point* $\mathrm{lfp}(F)$ and a *greatest fixed point* $\mathrm{gfp}(F)$. Further, these fixed points may be written as

$$\begin{aligned} \mathrm{lfp}(F) &= \bigcap\{X : F(X) = X\} = \bigcap\{X : F(X) \subseteq X\} \\ \mathrm{gfp}(F) &= \bigcup\{X : F(X) = X\} = \bigcup\{X : F(X) \supseteq X\}. \end{aligned}$$

Proof. Let $S = \{X \subseteq B : \ F(X) \subseteq X\}$ and $Y = \bigcap S$. We first show that Y is a fixed point of F.

$F(Y) \subseteq Y$**.** Clearly, $Y \subseteq X$ for all $X \in S$. As F is monotone, it follows that $F(Y) \subseteq F(X) \subseteq X$. Hence $F(Y) \subseteq \bigcap S = Y$.

$Y \subseteq F(Y)$**.** As $F(Y) \subseteq Y$, we have $F(F(Y)) \subseteq F(Y)$, and hence $F(Y) \in S$. Thus $Y = \bigcap S \subseteq F(Y)$.

By definition, Y is contained in all X such that $F(X) \subseteq X$. In particular Y is contained in all fixed points of F. Hence Y is the least fixed point of F.

The argument for the greatest fixed point is analogous. Q.E.D.

Least fixed points can also be constructed inductively. We call an operator $F : \mathcal{P}(B) \to \mathcal{P}(B)$ *inductive* if the sequence of its *stages* X^α (where α is an ordinal), defined by

$$\begin{aligned} X^0 &:= \emptyset, \\ X^{\alpha+1} &:= F(X^\alpha), \text{ and} \\ X^\lambda &:= \bigcup_{\alpha<\lambda} X^\alpha \text{ for limit ordinals } \lambda, \end{aligned}$$

is increasing, i.e. if $X^\beta \subseteq X^\alpha$ for all $\beta < \alpha$. Obviously, monotone operators are inductive. The sequence of stages of an inductive operator eventually reaches a fixed point, which we denote by X^∞. The least ordinal β for which $X^\beta = X^{\beta+1} = X^\infty$ is called $\mathrm{cl}(F)$, the *closure ordinal* of F.

Lemma 3.4. For every inductive operator $F : \mathcal{P}(B) \to \mathcal{P}(B)$, $|\mathrm{cl}(F)| \leq |B|$.

Proof. Let $|B|^+$ denote the smallest cardinal greater than $|B|$. Suppose that the claim is false for F. Then for each $\alpha < |B|^+$ there exists an element $x_\alpha \in X^{\alpha+1} - X^\alpha$. The set $\{x_\alpha : \alpha < |B|^+\}$ is a subset of B of cardinality $|B|^+ > |B|$, which is impossible. Q.E.D.

Proposition 3.5. For monotone operators, the inductively constructed fixed point coincides with the least fixed point, i.e. $X^\infty = \mathrm{lfp}(F)$.

Proof. As X^∞ is a fixed point, $\mathrm{lfp}(X) \subseteq X^\infty$. For the converse, we show by induction that $X^\alpha \subseteq \mathrm{lfp}(F)$ for all α. As $\mathrm{lfp}(F) = \bigcap\{Z : F(Z) \subseteq Z\}$, it suffices to show that X^α is contained in all Z for which $F(Z) \subseteq Z$.

For $\alpha = 0$, this is trivial. By monotonicity and the induction hypothesis, we have $X^{\alpha+1} = F(X^\alpha) \subseteq F(Z) \subseteq Z$. For limit ordinals λ with $X^\alpha \subseteq Z$ for all $\alpha < \lambda$ we also have $X^\lambda = \bigcup_{\alpha<\lambda} \subseteq Z$. Q.E.D.

The greatest fixed point can be constructed by a dual induction, starting with $Y^0 = B$, by setting $Y^{\alpha+1} := F(Y^\alpha)$ and $Y^\lambda = \bigcap_{\alpha<\lambda} Y^\alpha$ for limit ordinals. The *decreasing* sequence of these stages then eventually converges to the greatest fixed point $Y^\infty = \mathrm{gfp}(F)$.

The least and greatest fixed points are dual to each other. For every monotone operator F, the dual operator $F^d : X \mapsto \overline{F(\overline{X})}$ (where $\overline{X}$ denotes the complement of X) is also monotone, and we have that

$$\mathrm{lfp}(F) = \overline{\mathrm{gfp}(F^d)} \text{ and } \mathrm{gfp}(F) = \overline{\mathrm{lfp}(F^d)}.$$

Everything said so far holds for operators on arbitrary (finite or infinite) power set lattices. In *finite model theory*, we consider operators $F : \mathcal{P}(A^k) \to \mathcal{P}(A^k)$ for finite A only. In this case the inductive constructions will reach the least or greatest fixed point in a polynomial number of steps. As a consequence, these fixed points can be constructed efficiently.

Lemma 3.6. Let $F : \mathcal{P}(A^k) \to \mathcal{P}(A^k)$ be a monotone operator on a finite set A. If F is computable in polynomial time (with respect to $|A|$), then so are the fixed points $\mathrm{lfp}(F)$ and $\mathrm{gfp}(F)$.

3.3 Least Fixed-Point Logic

LFP is the logic obtained by adding least and greatest fixed points to first-order logic.

Definition 3.7. *Least fixed-point logic* (LFP) is defined by adding to the syntax of first-order logic the following *least fixed-point formation rule*: If $\psi(R, \overline{x})$ is a formula of vocabulary $\tau \cup \{R\}$ with only positive occurrences of R, if $\overline{x}$ is a tuple of variables, and if $\overline{t}$ is a tuple of terms (such that the lengths of $\overline{x}$ and $\overline{t}$ match the arity of R), then

$$[\mathrm{lfp}\, R\overline{x}\,.\,\psi](\overline{t}) \text{ and } [\mathrm{gfp}\, R\overline{x}\,.\,\psi](\overline{t})$$

are formulae of vocabulary τ. The free first-order variables of these formulae are those in $(\mathrm{free}(\psi) \setminus \{x : x \text{ in } \overline{x}\}) \cup \mathrm{free}(\overline{t})$.

Semantics. For any τ-structure $\mathfrak{A}$ providing interpretations for all free variables in the formula, we have that $\mathfrak{A} \models [\mathrm{lfp}\, R\overline{x}\,.\,\psi](\overline{t})$ if $\overline{t}^{\mathfrak{A}}$ (the tuple of elements of $\mathfrak{A}$ interpreting $\overline{t}$) is contained in $\mathrm{lfp}(F_\psi)$, where F_ψ is the update operator defined by ψ on $\mathfrak{A}$. The semantics for greatest fixed point operators are defined analogously.

Example 3.8. Here is a fixed-point formula that defines the transitive closure of the binary predicate E:

$$\mathrm{TC}(u, v) := [\mathrm{lfp}\, Txy\,.\,Exy \lor \exists z(Exz \land Tzy)](u, v).$$

Note that in a formula $[\mathrm{lfp}\, R\overline{x}\,.\,\varphi](\overline{t})$, there may be free variables in φ additional to those in $\overline{x}$, and these remain free in the fixed-point formula. They are often called *parameters* of the fixed-point formula. For instance, the transitive closure can also be defined by the formula

$$\varphi(u, v) := [\mathrm{lfp}\, Ty\,.\,Euy \lor \exists x(Tx \land Exy)](v)$$

which has u as a parameter.

It can be shown that every LFP-formula is equivalent to one without parameters (at the cost of increasing the arity of the fixed-point variables). The proof is left to the reader.

Example 3.9. Let $\varphi := \forall y(y < x \to Ry)$ and let $(A, <)$ be a partial order.

The formula $[\mathrm{lfp}\, Rx \,.\, \varphi](x)$ then defines the well-founded part of $<$. The closure ordinal of F_φ on $(A,<)$ is the length of the longest well-founded initial segment of $<$, and $(A,<) \models \forall x[\mathrm{lfp}\, Rx \,.\, \varphi](x)$ if, and only if, $(A,<)$ is well-founded.

Example 3.10. The LFP-sentence

$$\psi := \forall y \exists z Fyz \land \forall y[\mathrm{lfp}\, Ry \,.\, \forall x(Fxy \to Rx)](y)$$

is an infinity axiom, i.e. it is satisfiable but does not have a finite model.

Example 3.11. The GAME query asks, given a finite game $\mathcal{G} = (V, V_0, V_1, E)$, to compute the set of winning positions for Player 0. The GAME query is LFP-definable, by use of $[\mathrm{lfp}\, Wx \,.\, \varphi](x)$ with

$$\varphi(W,x) := (V_0x \land \exists y(Exy \land Wy)) \lor (V_1x \land \forall y(Exy \to Wy)).$$

The GAME query plays an important role for LFP. It can be shown that every LFP-definable property of finite structures can be reduced to GAME by a quantifier-free interpretation. Hence GAME is complete for LFP via this notion of reduction, and thus a natural candidate if one is trying to separate a weaker logic from LFP.

Example 3.12. Maximal bisimulation B on a Kripke structure $\mathcal{K} = (K, \{E_j\}, \{P_j\})$ is defined by the formula $\psi(u,v) =$

$$\Big[\mathrm{gfp}\, Bxy. \Big(\bigwedge_i (P_ix \leftrightarrow P_iy) \land \\ \bigwedge_j (\forall x'(E_j(x,x') \to \exists y'(E_j(y,y') \land B(x',y'))) \land \\ \forall y'(E_j(y,y') \to \exists x'(E_j(x,x') \land B(x',y')))) \Big) \Big](u,v)\,,$$

i.e. $u^{\mathcal{K}}$ and $v^{\mathcal{K}}$ are bisimilar if and only if $\mathcal{K}, u^{\mathcal{K}}, v^{\mathcal{K}} \models \psi(u,v)$.

The duality between the least and greatest fixed points implies that for any formula ψ,

$$[\mathrm{gfp}\, R\overline{x} \,.\, \psi](\overline{t}) \equiv \neg[\mathrm{lfp}\, R\overline{x} \,.\, \neg\psi[R/\neg R]](\overline{t}),$$

where $\psi[R/\neg R]$ is the formula obtained from ψ by replacing all occurrences of R-atoms by their negations. (As R occurs only positively in ψ, the same is true for $\neg\psi[R/\neg R]$.) Because of this duality, greatest fixed points are often omitted in the definition of LFP. On the other hand, it is sometimes convenient to keep the greatest fixed points, and to use the duality (and de Morgan's laws) to translate LFP-formulae to *negation normal form*, i.e. to push negations all the way to the atoms.

3.3.1 Capturing Polynomial Time

From the fact that first-order operations are polynomial-time computable and from Lemma 3.6, we can conclude that every LFP-definable property of finite structures is computable in polynomial time.

Proposition 3.13. Let ψ be a sentence in LFP. It is decidable in polynomial time whether a given finite structure $\mathfrak{A}$ is a model of ψ. In short, LFP $\subseteq$ PTIME.

Obviously LFP, is a fragment of second-order logic. Indeed, by the Knaster-Tarski Theorem,

$$[\mathrm{lfp}\, R\overline{x}\,.\,\psi(R,\overline{x})](\overline{y}) \equiv \forall R((\forall\overline{x}(\psi(R,\overline{x}) \to R\overline{x})) \to R\overline{y}).$$

We next relate LFP to SO-HORN.

Theorem 3.14. Every formula $\psi \in$ SO-HORN is equivalent to some formula $\psi^* \in$ LFP.

Proof. By Theorem 2.9, we can assume that $\psi = (\exists R_1)\cdots(\exists R_m)\varphi \in \Sigma^1_1$-HORN. By combining the predicates $R_1,\ldots,R_m$ into a single predicate R of larger arity and by renaming variables, it is easy to transform ψ into an equivalent formula

$$\psi' := \exists R\forall\overline{x}\forall\overline{y}\bigwedge_i C_i \wedge \bigwedge_j D_j,$$

where the C_i are clauses of the form $R\overline{x} \leftarrow \alpha_i(R,\overline{x},\overline{y})$ (with exactly the same head $R\overline{x}$ for every i) and the D_j are clauses of the form $0 \leftarrow \beta_j(R,\overline{x},\overline{y})$. The clauses C_i define, on every structure $\mathfrak{A}$, a monotone operator $F : R \mapsto \{\overline{x} : \bigvee_i \exists\overline{y}\alpha_i(\overline{x},\overline{y})\}$. Let R^ω be the least fixed point of this operator. Obviously $\mathfrak{A} \models \neg\psi$ if and only if $\mathfrak{A} \models \beta_i(R^\omega,\overline{a},\overline{b})$ for some i and some tuple $\overline{a}$, $\overline{b}$. But R^{ω} is defined by the fixed-point formula

$$\alpha^{\omega}(\overline{x}) := [\text{lfp}\, R\overline{x} \,.\, \bigvee_i \exists\overline{y}\alpha_i(\overline{x},\overline{y})](\overline{x}).$$

Hence, for $\beta := \exists\overline{x}\exists\overline{y}\bigvee_j \beta_j(\overline{x},\overline{y})$, ψ is equivalent to the formula $\psi^* := \neg\beta[R\overline{z}/\alpha^{\omega}(\overline{z})]$ obtained from $\neg\beta$ by substituting all occurrences of atoms $R\overline{z}$ by $\alpha^{\omega}(\overline{z})$. Clearly, this formula is in LFP. Q.E.D.

Hence SO-HORN $\leq$ LFP $\leq$ SO. As an immediate consequence of Theorems 2.12 and 3.14 we obtain the Immerman–Vardi Theorem.

Theorem 3.15 (Immerman and Vardi)**.** On ordered structures, least fixed-point logic captures polynomial time.

However, on unordered structures, SO-HORN is strictly weaker than LFP.

3.4 Infinitary First-Order Logic

Definition 3.16. Let $\kappa \in \text{Cn}^{\infty}$ be an infinite cardinal number and τ a signature. The *infinitary logic* $L_{\kappa\omega}(\tau)$ is inductively defined as follows.

- Each atomic formula in $\text{FO}(\tau)$ is in $L_{\kappa\omega}(\tau)$.
- If $\varphi \in L_{\kappa\omega}(\tau)$, then also $\neg\varphi$, $\exists x\varphi$, $\forall x\varphi \in L_{\kappa\omega}(\tau)$.
- If $\Phi \subseteq L_{\kappa\omega}(\tau)$ is a set of formulae with $|\Phi| < \kappa$, then $\bigvee\Phi$, $\bigwedge\Phi \in L_{\kappa\omega}(\tau)$.

Further, we write $L_{\infty\omega}(\tau)$ for $\bigcup_{\kappa\in\text{Cn}^{\infty}} L_{\kappa\omega}(\tau)$.

Note that the second parameter ω is always fixed as an index of our logics. This indicates that we only allow finite sequences of quantifiers.

The logic $L_{\omega\omega}(\tau)$ is precisely the logic $\text{FO}(\tau)$. The logic $L_{\aleph_1\omega}(\tau)$, in which disjunctions and conjunctions can be built over countable sets of formulae, is denoted by $L_{\omega_1\omega}$.

The semantics of the infinitary logic is defined in an obvious way. Clearly, we only have to treat the cases of $\bigwedge\Phi$ and $\bigvee\Phi$. Let $\overline{a} \subseteq A$ be an assignment of at most κ free variables, then

- $\mathfrak{A},\overline{a} \models \bigwedge\Phi$ if and only if $\mathfrak{A},\overline{a} \models \varphi$ for all $\varphi \in \Phi$.
- $\mathfrak{A},\overline{a} \models \bigvee\Phi$ if and only if there exists a $\varphi \in \Phi$ such that $\mathfrak{A},\overline{a} \models \varphi$.

In all other cases the semantics of infinitary logic coincides with that of first-order logic.

Example 3.17. Finiteness can be expressed in $L_{\omega_1\omega}$. Let

$$\varphi_{\geq n} := \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} (x_i \neq x_j)$$

and $\varphi_{\text{fin}} := \bigvee\{\neg\varphi_{\geq n} \mid n < \omega\} \in L_{\omega_1\omega}$. Then for each structure $\mathfrak{A}$, $\mathfrak{A} \models \varphi_{\text{fin}}$ if and only if $\mathfrak{A}$ is finite.

Remark 3.18. The Compactness Theorem does not hold for the logic $L_{\omega_1\omega}$. Consider for example the set of formulas $\varphi_{\text{fin}} \cup \{\varphi_{\geq n} \mid n < \omega\}$. It is unsatisfiable, but each of its finite subsets is satisfiable.

Theorem 3.19. Let $\kappa \in \text{Cn}^\infty$. For each formula $\varphi(\overline{x}) \in$ LFP there is a formula $\widehat{\varphi} \in L_{\kappa\omega}$ such that for all structures $\mathfrak{A}$ with $|\mathfrak{A}| < \kappa$ and all $\overline{a} \subseteq A$, we have $\mathfrak{A} \models \varphi(\overline{a})$ if and only if $\mathfrak{A} \models \widehat{\varphi}(\overline{a})$.

Proof. By using the duality between least and greatest fixed points we may assume that formulas in LFP only contain operators expressing least fixed points. We inductively define the translation from LFP to formulas of $L_{\infty\omega}$ as follows:

- for atomic formulas ψ we set $\widehat{\psi} = \psi$,
- $\widehat{\neg\psi} = \neg\widehat{\psi}$,
- $\widehat{\psi_1 \wedge \psi_2} = \widehat{\psi_1} \wedge \widehat{\psi_2}$, and $\widehat{\psi_1 \vee \psi_2} = \widehat{\psi_1} \vee \widehat{\psi_2}$.

For the case of $[\text{lfp}\, \widehat{R\overline{x}.\psi}](\overline{t})$, we build by transfinite induction a sequence of formulas $\psi^\alpha(\overline{x})$ for all ordinals $\alpha \leq \kappa$. These formulas intuitively correspond to the stages in the inductive evaluation of the least fixed-point. Accordingly, we start with the empty relation and set $\psi^0(\overline{x}) = \bot$. The induction proceeds as follows:

- $\psi^{\alpha+1}(\overline{x}) = \widehat{\psi}[R\overline{z}/\psi^\alpha(\overline{z})]$,
- for $\alpha = \bigcup_{\beta<\alpha} \beta$, let $\psi^\alpha(\overline{x}) = \bigvee_{\beta<\alpha}\{\psi^\beta(\overline{x}) \mid \beta < \alpha\}$.

Using induction on α and the definition of the semantics of $L_{\infty\omega}$, we see that the formulas ψ^α correspond exactly to the stages of fixed-point induction, i.e. $R^\alpha = \{\overline{x} \mid \psi^\alpha(\overline{x})\}$.

On structures $\mathfrak{A}$ with $|\mathfrak{A}| < \kappa$ we have $R^\kappa = R^\infty$ is the least fixed-point and which is thus defined by $\psi^\kappa(\overline{x})$. The claim follows. Q.E.D.

In general we can not drop the condition of bounded cardinality of the structures. In fact, the class of all well-orderings is definable in LFP by the following sentence:

$$\psi_{\mathrm{wo}} := \varphi_{\mathrm{lin}} \wedge \forall x[\mathrm{lfp}\, Wx(\forall y(y < x \rightarrow Wy))](x),$$

where φ_{lin} is a formula that expresses that $<$ is a linear order. One can show that this class is not definable in $L_{\infty\omega}$.

We also observe that the structure $(\omega, 0, S)$ is axiomatizable in $\mathrm{LFP}(0, S)$ up to isomorphism. To see this, note that $\{S^n(0) \mid n < \omega\}$ is the least fixed-point of the expression $x = 0 \vee \exists y(Ry \wedge Sy = x)$ (with respect to the variable R). Thus, $(\omega, 0, S)$ can be axiomatized by

$$\begin{aligned} &\forall x \forall y(Sx = Sy \rightarrow x = y) \wedge (\forall x Sx \neq 0) \wedge \\ &\qquad \forall x[\mathrm{lfp}\, Rx(x = 0 \vee \exists y(Ry \wedge Sy = x))](x). \end{aligned}$$

(The two first formulae in the conjunction are the first Peano axioms.) We conclude that the upward Löwenheim-Skolem theorem for LFP fails:
Remark 3.20. There exists a sentence $\varphi \in \mathrm{LFP}$ that has an infinite model and no uncountable model.

Next, we want to show that the Compactness Theorem does not hold for LFP either. For this we give an $\mathrm{LFP}(S)$-sentence φ such that φ has arbitrarily large finite models, but no infinite one.
Theorem 3.21. There is a sentence $\varphi \in \mathrm{LFP}(S)$ where S is a function symbol of arity one such that φ has arbitrarily large finite models, but no infinite one.

Proof. Define

$$\psi(x, z) := [\mathrm{lfp}\, Rx.(x = z \vee \exists y(Ry \wedge Sy = x))](x).$$

If $\mathfrak{A}$ is an S-structure then for all elements $a, b \in A$, we have $\mathfrak{A} \models \psi(b, a)$ if and only if there is some $n < \omega$ such that $(S^{\mathfrak{A}})^n(a) = b$. Now let

$$\varphi := \forall x \exists y(Sy = x) \wedge \exists x \forall y \psi(y, x).$$

For some S-structure $\mathfrak{A}$, we have $\mathfrak{A} \models \varphi$ if and only if $S^{\mathfrak{A}}$ is surjective and there is an $a \in A$ that generates the whole structure in the sense that $A = \{(S^{\mathfrak{A}})^n(a) \mid n < \omega\}$. For any $n < \omega$, the structure $\mathfrak{A} = (\{1, \ldots, n\}, S^{\mathfrak{A}})$ where $S^{\mathfrak{A}}(k) = k + 1$ for $k \in \{1, \ldots, n-1\}$ and $S^{\mathfrak{A}}(n) = 1$ is a model of φ. Thus, φ has arbitrarily large finite models.

On the other hand, φ has no infinite model. Let $\mathfrak{A} = (A, S^{\mathfrak{A}})$ be an S-structure with an infinite universe A such that there is an $a \in A$ with $A = \{(S^{\mathfrak{A}})^n(a) \mid n < \omega\}$, then $a \notin \mathrm{Img}(S^{\mathfrak{A}})$, so $S^{\mathfrak{A}}$ is not surjective. Indeed, assume that $a \in \mathrm{Img}(S^{\mathfrak{A}})$. Then $a = S^{\mathfrak{A}}(b)$ for some $b \in A$. Because $A = \{(S^{\mathfrak{A}})^n(a) \mid n < \omega\}$, it would follow $b = (S^{\mathfrak{A}})^n(a)$, so $(S^{\mathfrak{A}})^{n+1}(a) = a$. Then it would be $|\{(S^{\mathfrak{A}})^n(a) \mid n < \omega\}| \leq n$, in contradiction to the fact that A is infinite and $A = \{(S^{\mathfrak{A}})^n(a) \mid n < \omega\}$. It follows that $\mathfrak{A} \not\models \varphi$, and the statement is proven. Q.E.D.

Corollary 3.22. There exists an unsatisfiable set of sentences $\Phi \subseteq \mathrm{LFP}$ such that every finite subset of Φ is satisfiable, i.e. the Compactness Theorem fails for LFP.

Proof. According to Theorem 3.21 there is a sentence $\varphi \in \mathrm{LFP}(S)$ that has arbitrarily large finite models, but no infinite one. As before, consider the set of sentences $\Phi = \{\varphi\} \cup \{\exists x_1 \ldots \exists x_n \bigwedge_{i<j} x_i \neq x_j : n \in \omega\}$. Q.E.D.

We mention yet another property of LFP, that we do not prove here: the downward Löwenheim-Skolem theorem holds for LFP.

Theorem 3.23. Let $\varphi \in \mathrm{LFP}$ be a satisfiable sentence. Then φ has a countable model.

In particular, it follows that there is a sentence $\varphi \in L_{\infty\omega}(\tau)$ for some appropriate signature τ that is not equivalent to any sentence $\psi \in \mathrm{LFP}(\tau)$. For example, we can choose an uncountable set of constant symbols as τ and a conjunction of all sentences $c \neq d$ for pairwise distinct $c, d \in \tau$ as φ, which has no countable model.