Infinitary Logic with Counting

$C^k_{\infty\omega}$: extension of $L^k_{\infty\omega}$ by

counting quantifiers $\exists^{\geq m} x$, $m \geq 1$

$\exists x_1 \ldots \exists x_m \left(\bigwedge_{i \neq j} x_i \neq x_j \wedge \ldots \right)$

$$C^\omega_{\infty\omega} = \bigcup_{k \geq 1} C^k_{\infty\omega}$$

<u>Recall:</u> $\quad \mathrm{IFP} \leq L^\omega_{\infty\omega}$

We want to establish the analogous result for the setting with counting:

$$\mathrm{FPC} \leq C^\omega_{\infty\omega}$$

<u>Lemma:</u> For every formula $\varphi(\bar{x}, \bar{\mu}) \in FPC$, $\bar{x} = x_1 \dots x_k$, $\bar{\mu} = \mu_1 \dots \mu_\ell$, there ex. $s \geq 1$ (which only depends on the number of variables in φ (including $\bar{x}, \bar{\mu}$)), such that for every $n \geq 1$, every $\bar{m} \in [n+1]^\ell$ there ex. a $C^s_{\infty\omega}$-formula $\varphi^*_{n,\bar{m}}(\bar{x})$ such that for every structure $\mathfrak{A}$ of size n, and every $\bar{a} \in A^k$:

$$\mathfrak{A} \models \varphi(\bar{a}, \bar{m}) \iff \mathfrak{A} \models \varphi^*_{n,\bar{m}}(\bar{a})$$

<u>Pf:</u> We give an inductive translation

$$\varphi(\bar{x}, \bar{\mu}) \longmapsto \varphi^*_{n,\bar{m}}(\bar{x}) \in C^s_{\infty\omega}$$

for all $n, \bar{m}$ as above and for a an $s \geq 1$ which only depends on the number of variables in $\varphi(\bar{x}, \bar{\mu})$.

Without loss of generality we can assume that counting terms only occur in the form $\#_y\, \varphi(y) = \mu_i$

$$\left(\begin{array}{lll} t_1 = t_2 & \to & \exists_\mu \left(\mu = t_1 \wedge \mu = t_2\right) \\ t_1 \leq t_2 & \to & \exists_\mu \left(t_1 + \mu = t_2 \right) \end{array} \right)$$

Atomic propositions:

$$R\,\bar{y} \longmapsto R\,\bar{y}$$

$$\mu_i < \mu_j \longmapsto \begin{cases} \text{true}, & m_i < m_j \\ \text{false}, & m_i \geq m_j \end{cases}$$

Boolean connectives:

Clear by induction hypothesis.

FO - quantifier:

$$\exists_y\, \varphi \longmapsto \exists_y\, \varphi^*$$

$$\exists_\mu\, \varphi(\bar{x}, \bar{\mu}, \mu) \longmapsto \bigvee_{m \in [n+1]} \varphi^*_{n,\bar{m},m}(\bar{x})$$

Counting quantifiers:

$$\#_y\, \varphi(\bar{x}, \bar{\mu}, y) = \mu_i \quad \longmapsto$$

$$\exists^{\geq n_i} y\ \varphi^{*}_{n,\bar{m}}(\bar{x}, y) \wedge \neg \exists^{\geq n_i + 1} y\ \varphi^{*}_{n,\bar{m}}(\bar{x}, y) \qquad (*)$$

Fixed points

Over structures of size n, an FPC-induction can be expressed in FOC:

$$\psi = [\mathrm{ifp}\ R\bar{x}\bar{\mu}.\ \varphi(R, \bar{x}, \bar{\mu})](\bar{x}, \bar{\mu})$$

$$\varphi^{0}(\bar{x}, \bar{\mu}) := \emptyset$$

$$\varphi^{i+1}(\bar{x}, \bar{\mu}) := \varphi^{i}(\bar{x}, \bar{\mu}) \vee \varphi(R\bar{u}\bar{v} / \varphi^{i}(\bar{u}, \bar{v}), \bar{x}, \bar{\mu})$$

$$\psi \equiv \varphi^{n^{ar(R)}}(\bar{x}, \bar{\mu}) \quad \text{over structures of size } n$$

This translation does not increase the number of variables ?

<u>Thm</u> FPC $\leq C^{\omega}_{\infty\omega}$

Pf: Let ψ be an FPC-sentence.

By the above lemma we find $s \geq 1$ and sentences $\psi^*_n \in C^s_{\infty\omega}$ for $n \geq 1$ such that for every $\mathfrak{A}$ of size n:

$$\mathfrak{A} \models \psi \iff \mathfrak{A} \models \psi^*_n .$$

Hence

$$\psi \equiv \underbrace{\bigvee_{n \geq 1} \left[\exists^{\geq n} x\, (x=x) \wedge \neg \exists^{\geq n+1} x\, (x=x) \wedge \psi^*_n \right]}_{\in C^s_{\infty\omega}}$$

(over finite structures).

$\boxed{\equiv}$