\[ C^k_{\omega \omega} : \text{extension of } L^{k}_{\omega \omega} \text{ by counting quantifiers } \exists^m x, \ m \geq 1 \]

\[ C^\omega_{\omega \omega} = \bigcup_{k \geq 1} C^k_{\omega \omega} \]

\[ \exists x_1 \cdots \exists x_m (\forall x_i \neq x_j \land \cdots) \]

Recall: \[ \text{IFP} = L^\omega_{\omega \omega} \]

We want to establish the analogous result for the setting with counting:

\[ \text{FPC} \preceq C^\omega_{\omega \omega} \]
Lemma: For every formula $\varphi(x, \bar{\mu}) \in \text{FPC}$, there exists $s \geq 1$ (which only depends on the number of variables in $\varphi$ (including $x, \bar{\mu}$)), such that for every $n \geq 1$, every $\bar{m} \in [n+1]^k$, there exists a $C^s_{\text{oo}}$-formula $\varphi^*_{n, \bar{m}}(\bar{x})$ such that for every structure $\mathcal{O}^l$ of size $n$, and every $\bar{a} \in \mathbb{A}^k$:

$$\mathcal{O}^l \models \varphi(\bar{a}, \bar{m}) \iff \mathcal{O}^l \models \varphi^*_{n, \bar{m}}(\bar{a})$$

Proof: We give an inductive translation:

$$\varphi(x, \bar{\mu}) \rightarrow \varphi^*_{n, \bar{m}}(\bar{x}) \in C^s_{\text{oo}}$$

for all $n, \bar{m}$ as above and for any $s \geq 1$ which only depends on the number of variables in $\varphi(x, \bar{\mu})$. 

Without loss of generality we can assume that counting terms only occur in the form \( \#r \psi(r) = \mu_i \).

\[
\begin{align*}
\left( t_1 = t_2 \right) & \rightarrow \exists \mu \left( \mu = t_1 \land \mu = t_2 \right) \\
\left( t_1 \leq t_2 \right) & \rightarrow \exists \mu \left( t_1 + \mu = t_2 \right)
\end{align*}
\]

Atomic propositions:

\( RF \quad \rightarrow \quad RF \)

\( \mu_i < \mu_j \quad \rightarrow \quad \{ \text{true} \quad , \quad \mu_i < \mu_j \} \quad \text{false} \quad , \quad \mu_i \geq \mu_j \)

Boolean connectives:

Clear by induction hypothesis.

\( \text{FO - quantifiers:} \)

\( \exists x \psi \quad \rightarrow \quad \exists x \psi^* \)

\( \exists \mu \psi(\overline{x}, \overline{\mu}, \mu) \quad \rightarrow \quad \bigvee_{\mu \in \mathbb{N}^n} \psi^*_{\mu, \overline{\mu}, \overline{\mu}}(\overline{x}) \)
Counting quantifiers:

\[ \#_r \psi(\bar{x}, \bar{\mu}, r) = n_i \quad \rightarrow \]

\[ \exists n_i \forall \bar{y} \in \psi_{n, \bar{\mu}}(\bar{x}, r) \land \exists n_i+1 \psi_{n, \bar{\mu}}(\bar{x}, r) \]

Fixed points

Over structures of size \( n \), an FPC induction can be expressed in FOC:

\[ \psi = \left[ \text{if } R \bar{x} \bar{\mu}, \psi(R, \bar{x}, \bar{\mu}) \right](\bar{x}, \bar{\mu}) \]

\[ \psi^0(\bar{x}, \bar{\mu}) := \emptyset \]

\[ \psi^{i+1}(\bar{x}, \bar{\mu}) := \psi^i(\bar{x}, \bar{\mu}) \lor \psi(R\bar{\mu} \bar{\bar{\mu}} / \psi^i(\bar{\omega}, \bar{\bar{\omega}}), \bar{x}, \bar{\mu}) \]

\[ \psi = \psi^n(\bar{x}, \bar{\mu}) \text{ over structures of size } n \]

This translation does not increase the number of variables.
Thm. \( FPC < C^{\omega}_{\omega_0} \)

Proof. Let \( \psi \) be an FPC-sentence. By the above lemma we find \( S \models 1 \) and sentences \( \psi^*_n \in C^{\omega}_{\omega_0} \) for \( n \geq 1 \) such that for every \( \Omega \) of size \( n \):

\[
\Omega \models \psi \implies \Omega \models \psi^*_n
\]

Hence

\[
\psi = \bigvee \left[ \exists x^{2^n} (x=x) \land \exists y^{2^m} (x=x) \land \psi^*_n \right]_{n \geq 1}
\]

\( e \in C^{\omega}_{\omega_0} \) (our finte structure).