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WS 2013/14

Algorithmic Model Theory — Assignment 9

Due: Monday, 6 January, 12:00

Exercise 1

Let $\mathcal{K}$ be a class of (finite) τ-structures with the following property. For every $m \in \mathbb{N}$, there exists a structure $\mathfrak{A} \in \mathcal{K}$ such that for all m-tuples $\overline{a}$ in $\mathfrak{A}$ there exists a non-trivial automorphism of $(\mathfrak{A}, \overline{a})$. Show that $\mathcal{K}$ does not admit definable orders (even with parameters) in any logic which is isomorphism-invariant.

Exercise 2

(a) Construct formulae of the multidimensional μ-calculus that define the following classes $\mathcal{C}_i$ of rooted transition systems.

$\mathcal{C}_1 = \{(\mathcal{G}, v)$: from v a terminal vertex is reachable that satisfies precisely the same predicates$\}$

$\mathcal{C}_2 = \{(\mathcal{G}, v)$: there are two infinite paths π, σ starting from v such that for all positions $i > 0$ and all predicates P it holds $(\mathcal{G}, \pi[i]) \models P$ if, and only if, $(\mathcal{G}, \sigma[i]) \not\models P\}$

(b) Show that for $\mathcal{K}_1, \bar{v}$ and $\mathcal{K}_2, \bar{w}$ with $\mathcal{K}_1, v_i \sim \mathcal{K}_2, w_i$ for $1 \leq i \leq k$ it holds that $\mathcal{K}_1^k, \bar{v} \sim \mathcal{K}_2^k, \bar{w}$. Conclude, using the bisimulation invariance of L_μ, that the multidimensional μ-calculus is bisimulation invariant as well.

Exercise 3

Let $\mathfrak{A}$ be a finite τ-structure. We make the following convention: we interpret numerical tuples $\bar{\nu} = (\nu_{k-1}, \dots, \nu_1, \nu_0) \in \{0, \dots, |A|-1\}^k$ as numbers in $|A|$-adic representation, i.e. we associate the value $\sum_{i=0}^{k-1} \nu_i |A|^i$ to each tuple $\bar{\nu} \in \{0, \dots, |A|-1\}^k$.

Show that the expressive power of FPC does not increase if we allow counting quantifiers of higher arity, i.e. formulas $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1}) \leq (\nu_{k-1}, \dots, \nu_0)$ where in a structure $\mathfrak{A}$ the value of $\#_{x_0 x_1 \cdots x_{k-1}} \varphi(x_0, \dots, x_{k-1})$ is the number of tuples $\bar{a}$ such that $\mathfrak{A} \models \varphi(\bar{a})$ (with respect to the encoding introduced above). For simplicity, only consider the case $k = 2$.

Exercise 4

Recall the method of colour refinement that was presented in the lecture (see Example (6.5) from the lecture notes). Give an explicit definition of the stable colouring in FPC as a numerical term $\eta(x)$ of the form

$$\eta(x) = \#_z \left([\mathsf{ifp}\ u \prec v \,.\, \varphi(\prec, u, v)]\,(z, x) \right),$$

such that for any (finite, undirected) graph $\mathcal{G} = (V, E)$ the number $\eta(v)^{\mathcal{G}}$ is the colour of the vertex $v \in V$ in the stable colouring of the graph G.