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WS 2013/14

Algorithmic Model Theory — Assignment 8

Due: Monday, 16 December, 12:00

Exercise 1

Show that the following classes are $\mathrm{L}_{\omega_1\omega}$ definable over the appropriate signatures.

(a) torsion Abelian groups (This means all elements of the group have finite order);

(b) finitely generated fields (The whole field can be generated by a finite set through applications of addition and multiplication);

(c) linear orders isomorphic to $(\mathbb{Z}, <)$;

(d) connected graphs;

(e) acyclic directed graphs.

Exercise 2

(a) Show that every model class of finite τ-structures can be defined in $\mathrm{L}_{\infty\omega}$.

(b) Let $\mathcal{K}$ be a model class of finite structures. We say that $\mathcal{K}$ is *fixed-point bounded* if for any first-order formula $\varphi(X, \bar{x})$ (positive in X) there is a constant m_φ such that for all structures $\mathfrak{A} \in \mathcal{K}$ we have $(F_\varphi^{\mathfrak{A}})^{m_\varphi} = (F_\varphi^{\mathfrak{A}})^{m_\varphi+1}$ (i.e. the inductive construction for the least fixed-point of the monotone operator defined by φ terminates after at most m_φ steps). Show that LFP $\equiv$ FO over fixed-point bounded structures $\mathcal{K}$.

Exercise 3

In the lecture it was shown that (over finite structures) every LFP-formula is equivalent to a formula in $\mathrm{L}_{\infty\omega}$. Show that this can be improved to $\mathrm{L}^{\omega}_{\infty\omega}$, i.e. show that every formula $\varphi \in$ LFP can be translated into a formula $\varphi^\star \in \mathrm{L}_{\infty\omega}$ which is equivalent to φ (on finite structures) and which uses only a finite number of variables.

Exercise 4

Construct LFP-formulas which define in a rooted tree $\mathcal{T} = (V, E, r)$, where r denotes its root, the following relations.

(a) $R_1 = \{(x, y) : \text{the subtrees rooted in } x \text{ and } y \text{ have the same height}\}$

(b) $R_2 = \{(x, y) : \text{the nodes } x \text{ and } y \text{ are on the same level of the tree}\}$

(c) $R_3 = \{x : \text{the subtree rooted in } x \text{ possesses a perfect matching}\}$.

Hint: Use the inflationary stage comparison relations $\prec_\varphi^{\mathrm{inf}}$ which were presented in the lecture.