Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
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Algorithmic Model Theory — Assignment 9

Due: Monday, 19 December, 12:00

Exercise 1

Show that the following classes are $L_{\omega_1\omega}$ definable over the appropriate signatures.

(a) torsion Abelian groups (This means all elements of the group have finite order);

(b) finitely generated fields (The whole field can be generated by a finite set through applications of addition and multiplication);

(c) linear orders isomorphic to $(\mathbb{Z}, <)$;

(d) connected graphs;

(e) acyclic directed graphs.

Exercise 2

(a) Prove that LFP contains a formula that has finite models of unbounded cardinality, but no infinite model. Conclude that the compactness theorem does not hold for LFP.

(b) Let $\mathcal{K}$ be a class of finite structures. We say that $\mathcal{K}$ is *fixed-point bounded* if for any first-order formula $\varphi(X, \bar{x})$ (positive in X) there is a constant m_φ such that for all structures $\mathfrak{A} \in \mathcal{K}$ we have $(F_\varphi^{\mathfrak{A}})^{m_\varphi} = (F_\varphi^{\mathfrak{A}})^{m_\varphi+1}$ (i.e. the inductive construction for the least fixed-point of the monotone operator defined by φ terminates after at most m_φ steps). Show that LFP $\equiv$ FO over fixed-point bounded structures $\mathcal{K}$.

Exercise 3

(a) Show that every class of finite structures can be defined in $L_{\infty\omega}$.

(b) Construct a satisfiable sentence $\varphi \in L_{\omega_1\omega}$ over a *countable* signature τ such that every model of φ is uncountable.

Exercise 4

In the lecture it was shown that (over finite structures) every LFP-formula is equivalent to a formula in $L_{\infty\omega}$. Show that this can be improved to $L^{\omega}_{\infty\omega}$, i.e. show that every formula $\varphi \in$ LFP can be translated into a formula $\varphi^\star \in L_{\infty\omega}$ which is equivalent to φ (on finite structures) and which uses only a finite number of variables.