Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
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Algorithmic Model Theory — Assignment 7

Due: Monday, 05 December, 12:00

Exercise 1

To justify the definition of SO-HORN, show that the admission of arbitrary first-order prefixes would make the restriction to Horn clauses pointless. This extension of SO-HORN has the full power of second-order logic.

Exercise 2*

weak-SO-HORN is the set of all formulae of the form

$$QR_1 \dots QR_k \forall x_1, \dots \forall x_l \bigwedge_{1 \le i \le r} C_i.$$

The C_i are of the form $B_1 \wedge \dots \wedge B_n \to H$ where the B_i are either atoms or negated atoms with the restriction that the relations $R_1, \dots, R_k$ may only occur positively. That means weak-SO-HORN differs from SO-HORN in that fact that only atomic or negated atomic first order formulae are allowed in the clauses.

(a) Show that on ordered structures weak-SO-HORN is strictly less expressive than SO-HORN.

Hint: Show that for every weak-SO-HORN sentence ψ the class $\{\mathfrak{A} : \mathfrak{A} \models \psi\}$ is closed under substructures.

(b) Show that, however, on ordered structures with the additional successor relation and constants $0, e$ for the first and last element in the order weak-SO-HORN and SO-HORN are equally expressive.

Hint: show that on this domain weak-SO-HORN captures PTIME

Exercise 3

The problem GEN can be represented as the set of structures (A, S, f, a) in the vocabulary of a unary predicate S, a binary function f, and a constant a, such that a is contained in the closure of S under f.

(i) Give an weak-SO-HORN sentence that defines the *complement* of GEN.

(ii) Show that GEN itself is not definable in weak-SO-HORN.

Hint: Use exercise 2(a)

Exercise 4

An operator $F : \mathcal{P}(A) \to \mathcal{P}(A)$ is called *inflationary* if $F(X) \supseteq X$ for all $X \subseteq A$.
Give examples for operators $F : \mathcal{P}(A) \to \mathcal{P}(A)$ with the following properties:

(i) F has a fixed point but no least one.

(ii) F has a least fixed point but is not monotone.

(iii) F is monotone but not inflationary.

(iv) F is inflationary but not monotone.