Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
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Algorithmic Model Theory — Assignment 6

Due: Monday, 28 November, 12:00

Exercise 1

We say that an $\mathrm{FO}(\tau \cup \{<\})$-sentence φ is *order-invariant* if for all *finite* τ-structures $\mathfrak{A}$ and linear orderings $<, <'$ on A we have

$$(\mathfrak{A}, <) \models \varphi \Leftrightarrow (\mathfrak{A}, <') \models \varphi.$$

Show that it is undecidable whether a given $\mathrm{FO}(\tau \cup \{<\})$-sentence φ is order-invariant.
Hint: Show that $\mathsf{Fin\text{-}sat}(\mathrm{FO})$ is reducible to this problem.

Exercise 2

Recall the encoding of ordered structures presented in the lecture. Let $\tau = \{P, R\}$ be a signature consisting of a unary predicate P and a binary predicate R. Construct formulae $\beta_0(\bar{x})$ and $\beta_1(\bar{x})$ defining the $\bar{x}$-th symbol of the encoding of an ordered τ-structure.

Exercise 3

(a) Show that the following classes of (undirected) graphs are in NP by explicitly constructing Σ_1^1-sentences defining them.

  (i) The class of regular graphs (i.e. every node has the same number of neighbours),

  (ii) the class of Hamiltonian graphs, and

  (iii) the class of graphs that admit a perfect matching.

(b) Let $k \geq 1$. An (undirected) graph $G = (V, E)$ has connectivity k if $|G| > k$ and

  - for all $S \subseteq V, |S| < k$ the graph $G \setminus S$ is connected, and
  - there exists a set $S \subseteq V, |S| = k$ such that $G \setminus S$ is not connected.

  Construct a Σ_1^1-sentence defining the class of (undirected) graphs with connectivity k.

Exercise 4

The *spectrum* of a sentence $\varphi \in \mathrm{FO}(\tau)$ is defined as

$$\mathrm{spec}(\varphi) := \{n \in \mathbb{N} : \text{there exists } \mathfrak{A} \models \varphi \text{ with } |A| = n\}.$$

Show that a set $S \subseteq \mathbb{N}$ is a spectrum of an FO-sentence if and only if $S \in \mathrm{NE}\textsc{xptime}$.
Hint: Use Fagin's Theorem.