Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, W. Pakusa, F. Reinhardt,
M. Voit

SS 2016

Algorithmic Model Theory — Assignment 10

Due: Friday, 1 July, 13:00

Remark: Graphs are undirected and finite for this exercise.

Exercise 1

Show that it is decidable, given a sentence FO(E), whether $\mu(\psi) = 0$ or $\mu(\psi) = 1$.

Exercise 2

A graph $G = (V, E)$ is k-connected if $|V| \geq k$ and if the removal of any set of at most $k - 1$ vertices does not disconnect the graph. Equivalently, a graph G is k-connected if for each pair of distinct vertices v, w the graph contains at least k vertex-disjoint paths connecting v and w. We want to show, for all $k \geq 1$, that almost all graphs are k-connected.

(a) Let H_1 and H_2 be graphs such that H_1 is an induced subgraph of H_2. Show that almost all graphs G have the property that every isomorphism π_1 from H_1 onto an induced subgraph of G can be extended to an isomorphism π_2 from H_2 onto an induced subgraph of G.

(b) Use this to show that in almost all graphs G, each pair of vertices is connected by at least k different vertex-disjoint paths (for every fixed k).

Exercise 3

Let $\mathcal{K}$ be a class of graphs and let $\psi \in$ FO(E) be such that $\mu(\psi) = 1$. We say that $\mathcal{K}$ follows from ψ if for every graph $G \models \psi$ it holds that $G \in \mathcal{K}$. For instance, the class of connected graphs follows from the sentencce $\forall x \forall y(\neg Exy \rightarrow \exists z(Exz \wedge Eyz))$ with asympotic propability 1. Of course, each such class $\mathcal{K}$ itself has asympotic propability 1.

We want to show that the class $\mathcal{R}$ of all rigid graphs does not follow from any $\psi \in$ FO(E) with $\mu(\psi) = 1$. Recall that a graph G is rigid if it has no non-trivial automorphisms. However, it is known, but not so easy to prove, that almost all graphs are rigid. This shows that there are interesting properties of graphs which hold for almost all graphs, but which do not follow from any first-order definable property of almost all graphs (another example is hamiltonicity).

(a) Explain why it suffices to show that for every finite set $T_0 \subseteq T$ there exists a non-rigid graph G, i.e. a graph with non-trivial automorphisms, such that $G \models T_0$.

(b) Consider the class $\mathcal{K}$ of graphs with vertex set $V_\ell = \{-\ell, \dots, -1, 1, \dots, \ell\}$, for all $\ell \geq 1$, and with the property that there is an edge between i and j if, and only if, there is an edge between $-i$ and $-j$ for all $i, j \in V_\ell$. All graphs in $\mathcal{K}$ are non-rigid (why?).

(c) Show that every extension axiom $\sigma \in T$ has asympotic propability 1 on the class $\mathcal{K}$ (in particular, each extension axiom has a model in $\mathcal{K}$). To prove this, it can be helpful to observe that a random graph in $\mathcal{K}$ results by tossing a fair coin for every possible edge pair $\{i, j\}, \{-i, -j\}$. Put everything together to prove the claim.