Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, W. Pakusa, F. Reinhardt,
M. Voit

SS 2016

Algorithmic Model Theory — Assignment 4

Due: Friday, 13 May, 13:00

Exercise 1

We consider the class X of all FO-sentences of the form

$$\exists x_1 \cdots \exists x_r \forall y\, \eta(\bar{x}, y), \eta \in \mathrm{FO}(\{f\}),$$

where η is quantifier-free and f is a unary function symbol. Prove that X has the finite model property.

Hint: Consider the Skolem normal form of such sentences φ and try to prune a possibly infinite model of φ by using the fact that in all terms that appear in φ the number of iterations of f is bounded.

Exercise 2

For $n \geq 2$ we consider the directed path $\mathcal{P}_n$ of length n, i.e. the $\{E\}$-structure

$$\mathcal{P}_n = (\{0, \dots, n-1\}, \{(i, i+1) : 0 \leq i < n-1\}).$$

Construct for every $n \geq 2$ a sentence $\varphi_n \in \mathrm{FO}^2$ such that for every $\{E\}$-structure $\mathfrak{A}$ it holds that $\mathfrak{A} \models \varphi_n$ if, and only if, $\mathfrak{A} \cong \mathcal{P}_n$.

Exercise 3

ε-FO is the extension of FO by Hilbert's *choice operator* (also known as ε-operator). The syntax of ε-FO is given by the usual rules together with an additional ε-rule: If ψ is a formula, and x is a variable, then $\varepsilon x \psi$ is a term (read "an x such that ψ").

An interpretation for an ε-FO formula is given by an FO-interpretation $(\mathfrak{A}, \mathcal{I})$ together with a choice function on the universe of $\mathfrak{A}$, i.e. a mapping $F : \mathcal{P}(A) \to A$ such that $F(X) \in X$ for all $X \neq \emptyset$. The value of a term $\varepsilon x \psi$ is defined as $F(\{a \in A : (\mathfrak{A}, \mathcal{I}) \models \psi(a)\})$.

(a) Show that the quantifiers $\exists$ and $\forall$ can be expressed with the ε-operator.

(b) Construct an infinity axiom φ in ε-FO^2 over the empty vocabulary, i.e., φ contains only the ε-operator, two variables x and y, and equality, but neither relation nor function symbols.