Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
Prof. Dr. E. Grädel, D. Berwanger, T. Ganzow

SS 2008

Algorithmic Model Theory — Assignment 6

Due: Tuesday, 3 June, 12:00

Exercise 1

Prove the following claim:

Let σ be finite, and T' be a finitely axiomatisable σ-theory, i.e., there exists a $\psi \in \mathrm{FO}(\sigma)$ such that $T' = \{\psi\}^{\models}$. Furthermore, let T be a τ-theory and I be a (τ, σ)-interpretation interpreting at least one model of T' in a model of T. If T_0 is a subtheory of T, then

$$X := \{\varphi \in \mathrm{FO}(\sigma) \ : \ (\alpha_I \wedge I(\psi)) \to I(\varphi) \in T_0\}$$

where α_I denotes the correctness conditions of I is closed under consequences.

Exercise 2

Let $T \subseteq \mathrm{FO}(\sigma)$ and $T' \subseteq \mathrm{FO}(\tau)$ be theories. The theory T' is called an *inessential extension* of T if

(a) $\tau = \sigma \cup \{c_1, \ldots, c_k\}$ with finitely many constant symbols $c_1, \ldots, c_k$, and

(b) $T' = T^{\models_\tau}$,

i.e., $\mathrm{Mod}(T') = \{(\mathfrak{A}, a_1, \ldots, a_k) \ : \ \mathfrak{A} \models T, a_1, \ldots, a_k \in A\}$.

Lemma 1. *If T' is an inessential extension of T and T' is (essentially) undecidable, then T is also (essentially) undecidable.*

Let $\mathfrak{G} = (\mathrm{Aut}(\mathbb{Z}), \circ)$ be the automorphism group of $\mathbb{Z}$, and let $\mathfrak{G}^+ = (\mathfrak{G}, s)$ be $\mathfrak{G}$ extended by the constant element $s : z \mapsto z + 1$.

Lemma 2. *$\mathfrak{Z} = (\mathbb{Z}, +, |, 1)$ is interpretable in $\mathfrak{G}^+$.*

(a) Prove Lemma 1.

(b)* Prove Lemma 2.

Hint: Complete the definition of the following interpretation $I = (\partial, \varphi_+, \varphi_|, \varphi_1)$ where $\partial(x) = (x \circ s = s \circ x)$ and $\varphi_|(x, y) = \forall z(z \circ x = x \circ z \to z \circ y = y \circ z)$. Show that $\pi : \mathfrak{Z} \to I(\mathfrak{G}^*)$ with $\pi(c) = (z \mapsto z + c)$ is an isomorphism.

(c) Conclude that group theory is undecidable.