Lehr- und Forschungsgebiet
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SS 2008

Algorithmic Model Theory — Assignment 4

Due: Tuesday, 20 May, 12:00

Exercise 1

ε-FO is the extension of FO by Hilbert's *choice operator* (also known as ε-operator). The syntax of FO is given by the usual rules together with an additional ε-rule: If ψ is a formula, and x is a variable, then $\varepsilon x \psi$ is a term (read "an x such that ψ"). The interpretation of ε in a structure with universe A is given by an arbitrary choice function $F : \mathcal{P}(A) \to A$ such that $F(X) \in X$ for all $X \neq \emptyset$.

(a) Show that the quantifiers $\exists$ and $\forall$ can be expressed with the ε-operator.

(b) Construct an infinity axiom φ in ε-FO^2 over the empty vocabulary, i.e., φ contains only the ε-operator, two variables x and y, and equality, but neither relation nor function symbols.

Exercise 2

Show that the class $[\exists^*\forall, (0), (1)]_=$ has the finite model property.
Hint: Consider the Skolem normal-form of such sentences φ, and try to prune a possibly infinite model of φ by considering equivalence relations between elements of the structure relating those elements that satisfy the same atomic formulae in one free variable in which the function is applied only a bounded number of times.

Exercise 3

(a) Show that the problem whether a sentence of length n given in prenex normal form with q universal quantifiers has a model with at most s elements can be decided nondeterministically in time $p(s^q n)$ for some polynomial p.

(b) Conclude, using the arguments from Exercise 2 of Assignment 1, that $\mathrm{Sat}[\exists^*\forall^*, \mathrm{all}, (0)]_= \in$ NEXPTIME.

(c) Show that $\mathrm{Sat}[\exists^*\forall^*, \mathrm{all}, (0)]_=$ is even NEXPTIME-complete by proving the hardness via a reduction from $\mathrm{Domino}(\mathfrak{D}, 2^n)$ to $\mathrm{Sat}[\exists^2\forall^*, \mathrm{all}, (0)]_=$.

Hint: Use sentences of the form $\exists 0 \exists 1 \forall \bar{x} \forall \bar{y} \ldots (0 \neq 1 \wedge \varphi)$ where tuples $\bar{x} = x_0 \ldots x_{n-1}$ represent coordinates and φ describes a correct tiling using appropriate relations.