Lehr- und Forschungsgebiet
Mathematische Grundlagen der Informatik
RWTH Aachen
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SS 2008

Algorithmic Model Theory — Assignment 1

Due: Tuesday, 22 April, 12:00

Note: – You may work on the exercises in groups of up to three students.

– Hand in your solutions at the end of the lecture or put them into the box at the institute.

Exercise 1

(a) Show that any two disjoint co-recursively enumerable languages A and B are recursively separable, i.e. there exists a recursive set C such that $A \subseteq C$ and $B \cap C = \emptyset$.

(b) Given a recursively enumerable language L, let $\text{code}(L) = \{\rho(M) \ : \ L(M) = L\}$. Show that if L_1 and L_2 are recursively enumerable languages and $L_1 \subsetneq L_2$, then $\text{code}(L_1)$ is recursively inseparable from $\text{code}(L_2)$.

Hint: Use a reduction from a suitable pair of recursively inseparable sets and recall the proof of Rice's theorem.

Exercise 2

Let X be the set of relational FO-sentences of the form $\exists x_1 \dots \exists x_r \forall y_1 \dots \forall y_s \, \varphi$ where $r, s \in \mathbb{N}$ and φ is quantifier-free. Show that $\text{Sat}(X)$ is decidable.
Hint: Show that each satisfiable sentence in X has a model with at most r elements.

Exercise 3

A k-register machine resembles a Turing machine, but instead of a tape it uses k registers each of which stores a natural number. In each step, the next state is determined by the current state and the set of registers currently holding a zero, and the machine can increment or decrement the contents of the registers (if they are not already zero). Formally, the transition function δ can be described as follows:

$$\delta : Q \times \{0, +\}^k \to Q \times \{-1, 0, +1\}^k$$

such that $\delta(q, 0{+}{+}0) = (q', +1\,-1\,0\,-1)$ means that if the (4-)register machine M is currently in state q, register 1 and 4 contain a zero and register 2 and 3 both contain a positive number, then M changes to state q', increments the first register, and decrements the second and fourth register (however, the fourth register still holds a zero afterwards).

It is well known that the halting problem for 2-register machines is undecidable. Reduce this halting problem to a validity problem for FO-formulae as in the proof of Trakhtenbrot's theorem.
Hint: The configurations of 2-register machines can be represented as tuples in $Q \times \mathbb{N} \times \mathbb{N}$. Consider an appropriate structure that contains a binary relation C_q for each state q of the register machine such that $C_q = \{(m, n) \in \mathbb{N} \times \mathbb{N} \ : \ (q_0, 0, 0) \vdash_M^* (q, m, n)\}$.